BINARY MAPPING OF MONOTONIC SEQUENCES – THE ARONSON AND THE CELLULAR AUTOMATON FUNCTIONS

(Manuscript: 12th of September 2004, updated Feb. 2008)

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Abstract. The n'th binary digit of the real number ξ in $(0,1)$ is 1 if and only if n is in the monotonic sequence $\mathbf{S}$, then ξ is the *binary map* of $\mathbf{S}$. This mapping is an isomorphism between the set of positive monotonic sequences and a subset of the real numbers in the interval $(0,1)$. Due to this mapping, any automorphism over the set of monotonic sequences corresponds to some kind of a function $(0,1) \to (0,1)$. For example, Cloitre et al. in 2003 introduced the Aronson transform $\mathbf{B}$ of a monotonic integer sequence $\mathbf{A}$ as "*n is in $\mathbf{B}$ if and only if b_n is in $\mathbf{A}$*", which is an automorphism. Hence, the *binary mapping* transforms it to a function – the *Aronson function* –, which exhibits a delicate fractal structure. The author shows that the analogous function related to the inverse Aronson transformation is the inverse of the Aronson function. Due to the binary mapping, it is possible to introduce a simple arithmetics among the monotonic sequences, increasing vastly the amount of definable sequences. The binary mapping and its inverse makes also possible to define functions – as well as sequence transformations – by using cellular automata.

Mathematics Subject Classification 2000, main: 11B83, 11B85, 11B99, 11Y55; additional: 20M14, 20M15, 20K20, 37B15.

Key words: integer sequences, sequence transformation, bijective mapping, binary expansion, semigroup, Abelian group, fractal, everywhere discontinuous function, cellular automata

1. Notations, definitions

Throughout this paper:

- *sequence* means an infinite integer sequence with initial index of 1;
- *monotonic sequence* means a strictly monotonically increasing sequence;
- any lower case Latin symbol stands for an integer,
- any lower case Greek symbol stands for a real number.

Bold capitals, e.g., $\mathbf{X}$, are used to denote an infinite integer sequence. The corresponding lower case symbol with lower index, x_i, denotes the ith member of $\mathbf{X}$.

N denotes the sequence of (positive) natural numbers and **P** denotes the sequence of primes.

Double lined capitals, like $\mathbb{S}$, denote sets.

2. Mapping the monotonic sequences into the interval $(0, 1)$

In general, a *positive monotonic sequence* **A** is an infinite ordered subset of **N**. Thus, one can identify the selection of numbers from **N** by an infinite sequence of 1s and 0s, so that if i (the ith member of **N**) is member of **A**, then the ith element is 1, otherwise 0. This infinite sequence of 1s and 0s can also be regarded as the fractional part of the binary representation of a real number in the interval of $(0, 1)$. In other words, we create a real number in $(0, 1)$ from the *characteristic function* [2] of the monotonic sequence.

Definition 1. *Binary mapping.* Let $\mathbb{S}_{\mathrm{m}}$ be the set of positive monotonic integer sequences and $\mathbb{R}_{(0,1)}$ the open interval of real numbers $(0, 1)$ then there exists the injective mapping $\mathcal{M} : \mathbb{S}_{\mathrm{m}} \to \mathbb{R}_{(0,1)}$, which is defined so that the real number $\mathcal{M}(\mathbf{A}) = \rho$ – the picture of **A** – is such that in its binary representation the ith binary digit of the fractional part is equal to $[i \in \mathbf{A}]$. Here, [*statement*] is the Iverson bracket [2], i.e., 1 if *statement* is true, 0 otherwise.

Remark 1. Clearly, the *binary mapping* $\mathcal{M}(\mathbf{A})$ always exists and is well defined if $\mathbf{A} \in \mathbb{S}_{\mathrm{m}}$. It is also obvious that $\mathcal{M}^{-1}(\rho)$ for some $\rho \in (0, 1)$ is not necessarily element of $\mathbb{S}_{\mathrm{m}}$, since ρ may have only a finite number of non-zero binary digits. Thus, the image of $\mathbb{S}_{\mathrm{m}}$ is $\mathbb{R}_{\mathrm{m}(0,1)} \subset \mathbb{R}_{(0,1)}$, which consists of the real numbers with infinite number of non-zero binary digits: $\mathbb{R}_{\mathrm{m}(0,1)} = \{\rho \in (0, 1) \mid \rho \neq a/2^b\}$, where $\{a, b\} \in \mathbf{N}$.

Remark 2. It is also clear, that it is not necessary to restrict Definition 1 to monotonic sequences, it is readily applicable to any *monotonic subset of natural numbers*, finite or infinite. Thus, the mapping $\mathcal{M}^* : \mathbb{M} \leftrightarrow \mathbb{R}_{(0,1)}$ with analogous definition – where $\mathbb{M}$ is the *set of monotonic subsets* of naturals (both finite and infinite) – is a bijective mapping. Note that $\mathbb{S}_{\mathrm{m}} \in \mathbb{M}$. Clearly, $\mathbb{M}$ is the image set of $\mathcal{M}^{*(-1)}$, if its argument covers $\mathbb{R}_{(0,1)}$.

Remark. For brevity, we will use the $\mathbb{R} = \mathbb{R}_{(0,1)}$ and the $\mathbb{R}_{\mathrm{m}} = \mathbb{R}_{\mathrm{m}(0,1)}$ shorthand.

Example 1. One case of such a mapping is quite well known and studied without calling it "binary mapping": the sequence A000069 in [3] is the so called "odious numbers" (the positive integers having odd number 1 digits in their binary expansion). Its binary map (as defined in Definition 1) is eventually 2-times the so called "Thue-Morse constant" (see e.g., [2], or A014571 in [3]):

$$0.41245403364010759778336136825...$$

The factor 2 only comes from the difference in the indexing convention of the first term of a sequence. The odious sequence **O** is related to the famous Thue-Morse (or Prouhet-Thue-Morse) sequence **T** so that $t_i = [i \in \mathbf{O}]$. The Thue-Morse sequence (A010060 in [3]) – as well as the several related sequences – have been quite extensively studied recently, e.g., by J. P-. Allouche and J. Shallit [4, 5] or by R. Astudillo [6].

Example 2. When Liouville – in connection with the Liouville theorem – introduced the Liouville number $\lambda = 0.11000100...$, having a fractional part such that in its *decimal representation* it contains 1 at the r'th decimal place if and only if $r = m!$ for some positive m, he applied a similar mapping as $\mathcal{M}$ (see A12245 in [3]). Thus, if $\mathbf{F}$ is the sequence of factorials (A00142 in [3]), then the $\mathcal{M}(\mathbf{F}) = \lambda_2 = 0.11000100..._2$ *binary number* is the *binary analogue of* λ, exhibiting similar transcendental features (see A92874 in [3]).

Definition 2. Let us call $\overline{\mathbf{A}}$ the *complement sequence* of $\mathbf{A} \in \mathbb{S}_{\mathrm{m}}$, if $\mathbf{A} \cup \overline{\mathbf{A}} = \mathbf{N}$ and $\mathbf{A} \cap \overline{\mathbf{A}} = \varnothing$.

Definition 3. If both $\overline{\mathbf{A}}$ and $\mathbf{A} \in \mathbb{S}_{\mathrm{m}}$, then $\mathbf{A}$ is called *non-trivial.* The set of non-trivial monotonic sequences is denoted by $\mathbb{S}^*_{\mathrm{m}}$, or shortly by $\mathbb{S}^*$. The image of $\mathbb{S}^*$in $\mathbb{R}$ by the mapping $\mathcal{M}$, we will denote by $\mathbb{R}^*$.

Theorem 1. *The sets $\mathbb{S}^*$ and $\mathbb{R}^*$ are isomorphic, and the cardinality of the set of positive monotonic non-trivial sequences, $\mathbb{S}^*$ is $\aleph_1$, i.e., continuum.*

Proof. The mapping $\mathcal{M} : \mathbb{S}^* \to \mathbb{R}^*$ is an isomorphism, which clearly follows from Definitions 1 and 3.

$\mathbb{R}^*$ can be obtained from $\mathbb{R}_{(0,1)}$ by removing the set $\mathbb{Q} = \{\xi \in (0,1) \,|\, \xi = a/2^b\}$ for every possible pairs of $\{a, b \,|\, a < 2^b\} \in \mathbf{N}$, and also by removing the set of binary complements of the elements of $\mathbb{Q}$, which is denoted by $\overline{\mathbb{Q}}$, the image of the trivial monotonic sequences. Since both $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ are subsets of the rationals, therefore the cardinality of $\mathbb{Q} \cup \overline{\mathbb{Q}}$ is $\leq \aleph_0$. Hence, both $\mathbb{R}^*$and $\mathbb{S}^*$ have continuum, $\aleph_1$ cardinality. □

Remark 3. The set $\mathbb{R}^*$, under the multiplication forms a *semigroup.* It is left to the reader to see that if $\{\alpha, \beta\} \in \mathbb{R}^*$ then $\alpha \cdot \beta \in \mathbb{R}^*$. Due to the isomorphism, this also means that the set of positive monotonic sequences also forms a semigroup under the operation which is defined as $\mathbf{A} \otimes \mathbf{B} = \mathcal{M}^{-1}(\mathcal{M}(\mathbf{A}) \cdot \mathcal{M}(\mathbf{B}))$. Due to this, it is possible to define powers of monotonic sequences (Note, that in this sense $\mathbf{A}^2$ is totally different from the notion of the "square" of a sequence as introduced by Cloitre et al.[1])

Definition 4. The set $\widehat{\mathbb{R}} = \mathbb{R}_{(0,1)} \cup \omega$, where $\omega = 0.1111..._2$, the binary number with infinite number of all "1" digits, forms an *Abelian group* under the composition for $\{\alpha, \beta\} \in \widehat{\mathbb{R}}$:

$$\alpha \oplus \beta = \begin{cases} \text{if } \alpha + \beta \leq \omega; \ (\alpha + \beta), \\ \text{if } \alpha + \beta > \omega; \ (\alpha + \beta) - \omega \end{cases}$$

i.e., basically the fractional part of the "normal" sum of the two elements.

Lemma 1. *The set $\widehat{\mathbb{R}}$ according to Definition 4 forms an Abelian group.*

Proof. a) The *addition* is associative and commutative: it follows from the definition.

b) There exists a *zero* element $\theta \in \widehat{\mathbb{R}}$, and for every $\alpha \in \widehat{\mathbb{R}}$ there is a unique element $\beta \in \widehat{\mathbb{R}}$, such that $\alpha \oplus \beta = \theta$: if we chose $\theta = \omega$, then clearly the *negative* of an element α is its binary complement.

Let define deliberately that the negative of ω is – by definition – itself. $\square$

Remark 4. Due to the fact that $\mathbb{R}^*$and $\mathbb{S}^*$ are isomorphic and $\mathbb{R}^* \in \widehat{\mathbb{R}}$, the set $\widehat{\mathbb{S}} = \mathcal{M}^{-1}(\widehat{\mathbb{R}})$ is an Abelian group under the operation $\mathbf{A} \oplus \mathbf{B} = \mathcal{M}^{-1}(\mathcal{M}(\mathbf{A}) \oplus \mathcal{M}(\mathbf{B}))$ for $\{\mathbf{A}, \mathbf{B}\} \in \hat{\mathbb{S}}$. Note however, that $\widehat{\mathbb{S}}$ contains not only the true monotonic sequences, but also the *trivial* ones, as well as the *finite monotonic ordered sets* of integers. It is also easy to show that from $\{\mathbf{A}, \mathbf{B}\} \in \mathbb{S}^*$, it does not necessarily follow that $\mathbf{A} \oplus \mathbf{B} \in \mathbb{S}^*$.

Remark 5. The author tried to find a way to unify the semigroup and the Abelian group structure into a consistent field or ring structure for the true monotonic sequences, but failed to find a suitable definition.

In Section 4 we will show some sequences resulted from the arithmetics utilizing the semigroup and Abelian group compositions on sequences, such as sums, differences and powers of some basic sequences.

3. The Aronson transformation and the Aronson function

Cloitre et al. in 2003 studied [1] the numerical analogues of the "Aronson sequence" [7, 8], which is playfully defined by the infinite English sentence: "*T is the first, fourth, eleventh, ... letter of this sentence.*" As an important generalization of this logic, they introduced the Aronson transformation of monotonic sequences.

Definition 5. *Aronson transform.* As it was introduced by Cloitre et al. [1], if $\mathbf{A} \in \mathbb{S}^*$ then $\mathbf{B} = \mathcal{A}(\mathbf{A})$, the *Aronson transform* of $\mathbf{A}$, is defined so that the condition$n \in \mathbf{B} \iff b_n \in \mathbf{A}$ (referred later on as *the condition*) must be satisfied, and for any $i \in \mathbf{N}$, $b_{i+1}$is *the smallest positive number* $k > b_i$, consistent with *the condition.*

Theorem 2. *If* $\mathbf{A} \in \mathbb{S}^*$ *then* $\mathbf{B} = \mathcal{A}(\mathbf{A})$ *uniquely exists and* $\mathbf{B} \in \mathbb{S}^*$*, i.e.,* $\mathbf{B}$ *is a non-trivial monotonic sequence.*

Proof. The proof below is basically identical to the proof given by Cloitre et al. in [1], apart from a minor correction – at Case (ii) – and from the addition of the non-triviality of $\mathbf{B}$.

The proof of the first part goes by induction.

Let us first determine b_1, as follows:

Case (a), if $a_1 = 1$ then $b_1 = 1$, since this satisfies "*the condition*" in both directions and 1 is the least such positive number.

Case (b), if $a_1 = 2$ then $b_1 = a_k + 1$, where k is the largest index such that $a_k = a_1 + k - 1$, since it is easy to see that if $b_1 = 1$ then according to the condition $b_1 \in A$ should have been true. If $b_1 \in \{a_1, ..., a_k\}$ then the reverse condition is not satisfied, since $1 \notin \mathbf{B}$. Whereas, if the $b_1 = a_k + 1$ selection is taken, then the condition $b_{a_k+1} \in \mathbf{A}$ can easily be satisfied later on.

Case (c), if $a_1 > 2$ then $b_1 = 2$ is the proper choice, since the forward part of the condition can be satisfied easily by choosing $b_2 = a_1$, while the reverse part of the condition is automatically satisfied in negative sense.

Now the induction:

Assuming that $S = \{b_1, ..., b_k\}$ non-empty ordered set – containing the first $k \geq 1$ elements of $\mathbf{B}$ – is known , then b_{k+1} is uniquely determined, as follows:

Case (i), $b_k = k$. If $k+1 \in \mathbf{A}$ then $b_{k+1} = k+1$, otherwise b_{k+1} is the least number $x > k+1$, such that $x \notin \mathbf{A}$ (due to the reverse condition).

Case (ii), $b_k > k$. Then, due to the forward condition, if $k+1 \in S$ then b_{k+1} is the least member a_j of $\mathbf{A}$, such that $a_j > b_k$. Whereas, if $k+1 \notin S$ then $b_{k+1} = x > b_k$, so that x is the least such number, satisfying that $x \notin \mathbf{A}$.

To see the non-triviality of $\mathbf{B}$, let us assume that $\mathbf{B}$ is trivial, i.e., $\exists m$ such that, $\forall i > m$, then $b_i = b_m + i - m$. This means – in other words – that every number $n \geq b_m$ is member of $\mathbf{B}$. From the condition, it is required then that $b_n \in \mathbf{A}$. It is clear that $b_n \geq n$, therefore $\forall\, \ell > b_m - m$, $b_\ell \in \mathbf{A}$, from what it follows that $\mathbf{A}$ must also be trivial, which is contradicting to the initial assumptions. □

Remark 6. Note that, unless the non-triviality of $\mathbf{A}$ is assumed, one *can not fulfill* the second sub-case in Case (ii) of the proof, with the exception of $\mathbf{A} = \mathbf{N}$, the sequence of naturals. In fact, as it easily follows from the proof, that the Aronson transformation can be applied for the elements of $\mathbb{Q} = \mathbb{S}^* \cup \mathbf{N}$, and the only "eigen-sequence" of $\mathcal{A}$ is $\mathbf{N}$, i.e., $\mathcal{A}(\mathbf{N}) = \mathbf{N}$. Thus, the Aronson transformation $\mathcal{A} : \mathbb{Q} \to \mathbb{Q}$ is an automorphism.

Definition 6. The *inverse of Aronson transform* of sequence $\mathbf{A}$ is defined as $\mathcal{A}^{-1}(\mathcal{A}(\mathbf{A})) = \mathbf{A}$. Cloitre et al. [1] have provided a constructive proof on the existence and uniqueness of the transformation of $\mathcal{A}^{-1} : \mathbb{Q} \to \mathbb{Q}$, alike $\mathcal{A}$ itself.

Now, everything is together to define the Aronson function:

Definition 7. *The Aronson function.* The function arf : $\mathbb{R}^* \to \mathbb{R}^*$ is defined as $\mathrm{arf}(\rho) = \mathcal{M}(\mathcal{A}(\mathcal{M}^{-1}(\rho)))$, where $\rho \in \mathbb{R}^*$.

In Figure 1, one can see the graph of the Aronson function. The author calculated the function at 4095 points and to avoid the arguments with finite (and short) binary representation, the argument points were chosen as $\{\xi_k = k/4096 + 10^{-7} \cdot \pi \,|\, k = 1, 2, ..., 4095\}$. The calculation was carried out by the PARI/GP code [11], using 50 decimal digits accuracy. Note, that the characteristic shape of the function is insensitive to the applied arbitrary shift $(10^{-7} \cdot \pi)$. The delicate fractal structure is striking, however. The fractal character is further illustrated in Figure 2, which presents the zoomed section of the function in the range of $(0.25, 0.3125) = (1/2^2, 5/2^4)$ which looks much like a featureless straight line in Fig. 1.

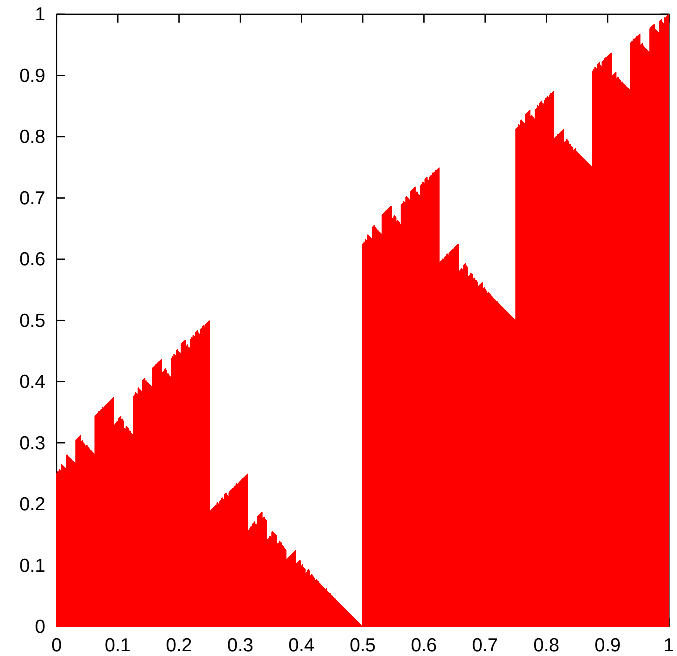

Figure 1. The Aronson function. The fractal structure – due to the infinite number of discontinuities – is well visible .

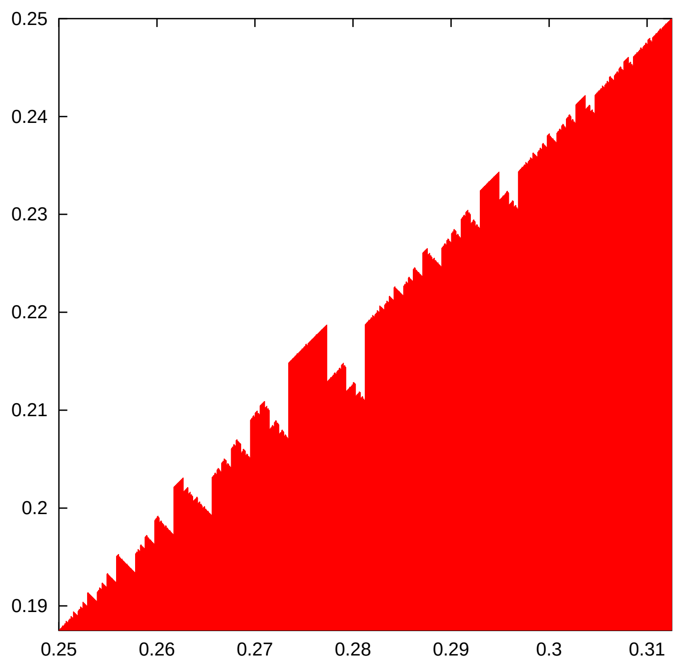

Figure 2. The Aronson function in the range of $(1/4, 5/16)$. Seemingly, this section is quite similar to the section of $(0, 1/2^2)$ in Fig. 1, though this section seems to be even more complex.

Remark 7. By analyzing the Aronson function just by the naked eye, it looks like consisting of characteristic sections in the intervals: $(0, 1/2), (1/2, 3/4), (3/4, 7/8), \ldots$. The general formula for the boundaries of the kth section is

$$\left(\frac{2^{k-1}-1}{2^{k-1}}, \frac{2^k-1}{2^k}\right). \tag{1}$$

The shape of the function looks identical within these sections, apart from being scaled to fit into the sections, i.e., scaled down by a factor of two at each subsequent intervals. This observation can easily be demonstrated to be true, since e.g., the sequences being isomorphic with the numbers in the second section, (1/2, 3/4) are almost the same as those corresponding to the first section (0, 1/2) with the only difference that all the sequences corresponding to the second section contain the number "1", while none does it have in the first section (since no number in section one contains the first fractional binary digit). Similarly, the sequences corresponding to the numbers in the kth section start with the first $k-1$ consecutive natural numbers, since the corresponding numbers start with $k-1$ "1" digits. Thus, if a number in the kth section is at $\delta/2^k$ distance from the section's starting point for some $\delta \in \mathbb{R}^*$, then the binary pattern of that number starts with $k-1$ digits of "1" , and then continues with the same binary digit pattern as that of δ. In the domain of sequences this means that the corresponding sequence in the section k starts with the first $k-1$ consecutive naturals and then continues similarly to the sequence corresponding to the number δ, with the only difference that the value of $k-1$ is added to the every member of the original sequence. In mathematical terms: if $\mathbf{A} = \mathcal{M}^{-1}(\delta)$ for some $\delta \in \mathbb{R}^*$, and in the kth section $\mathbf{A}^{(k)} = \mathcal{M}^{-1}\left((2^{k-1}-1)/2^{k-1} + \delta/2^k\right)$, then the term with index $i > k$ of $\mathbf{A}^{(k)}$ is

$$a_i^{(k)} = a_{i-k+1} + k - 1,$$

where a_{i-k+1} is the $(i-k+1)$-th term of $\mathbf{A}$. Hence, due to the features of the Aronson transformation (cf. with the proof of Theorem 2), the transform of $\mathbf{A}^{(k)}$will also start with $k-1$ binary digits of "1" and then will continue analogously to the transform of $\mathbf{A}$.

It is also notable that the co-domain corresponding to a section – as defined in equation (1) – in the domain of the Aronson function is identical to the domain section itself. By scrutinizing the constructive proof of the Theorem 2, this behavior can be understood.

Clearly, by using the inverse Aronson transformation $\mathcal{A}^{-1}$ and the $\mathcal{M}$ binary mapping analogously to Definition 7, a different function can be obtained:

Definition 8. The *Aronson inverse function.* The function $\mathrm{arfi} : \mathbb{R}^* \to \mathbb{R}^*$ is defined as $\mathrm{arfi}(\rho) = \mathcal{M}(\mathcal{A}^{-1}(\mathcal{M}^{-1}(\rho)))$, where $\rho \in \mathbb{R}^*$.

Theorem 3. *The inverse of the Aronson function is the Aronson inverse function:* $\mathrm{arfi}(\rho) = \mathrm{arf}^{-1}(\rho)$.

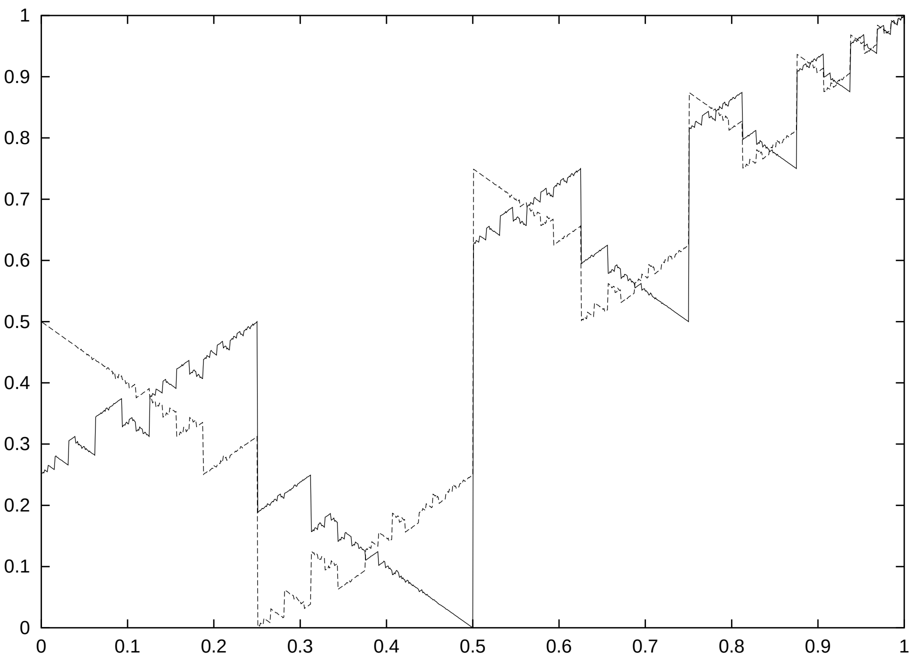

FIGURE 3. The Aronson function (arf) (continuous line) and the Aronson inverse function (arf^{-1}) (dashed line)
(Note that the vertical lines are not parts of the functions!)

Proof. The claim follows from Theorem 2, since $\mathcal{A}(\mathbf{A})$ is an automorphism in $\mathbb{S}^*$, therefore $\text{arf}(\rho) = \mathcal{M}(\mathcal{A}(\mathcal{M}^{-1}(\rho)))$ is automorphism in $\mathbb{R}^*$ and $\mathcal{M}$ is isomorphism between $\mathbb{S}^*$ and $\mathbb{R}^*$.

More explicitely: in general, for $f : \mathbb{X} \to \mathbb{X}$ automorphism the inverse $f^{-1}(x) = \{x^* | \, f(x^*) = x\}$ exists if $x^* \in \mathbb{X}$ is unique and exists for every $x \in \mathbb{X}$. Thus, $\text{arf}^{-1}(\rho) = \{\rho^* | \, \text{arf}(\rho^*) = \rho\}$. Let denote the sequence $\mathcal{M}^{-1}(\rho) = \mathbf{B}$ and let assume that $\rho^* = \mathcal{M}(\mathcal{A}^{-1}(\mathbf{B}))$. Then

$$\text{arf}(\rho^*) = \mathcal{M}(\mathcal{A}(\mathcal{M}^{-1}(\rho^*))) = \mathcal{M}(\mathcal{A}(\mathcal{M}^{-1}(\mathcal{M}(\mathcal{A}^{-1}(\mathbf{B}))))) = \mathcal{M}(\mathbf{B}) = \rho\,.$$

□

Another consequence of the fact that the Aronson transformation $\mathcal{A}(\mathbf{A})$ is automorphism in $\mathbb{S}^*$, is that the transformation can be iterated, i.e., $\mathcal{A}(\mathcal{A}(\mathbf{A}))$ is also in $\mathbb{S}^*$. Without going deeply into the iterative behavior of the Aronson transformation, we present in Figure 4 the function related to the second iteration of the transformation: $\text{arf2}(\rho) = \mathcal{M}(\mathcal{A}(\mathcal{A}(\mathcal{M}^{-1}(\rho))))$.

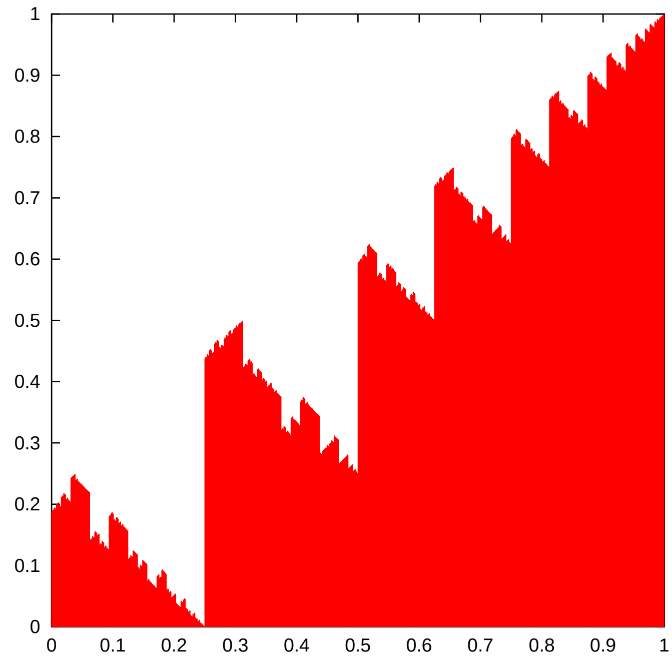

FIGURE 4. The shape of the function arf2(ρ), related to the second iteration of the Aronson transform

It is notable that the scaled repetitive part of the function arf2(ρ) is also in the range $(0, 0.5)$ but it is subdivided into two subsections which look more similar to each other than in the case of the arf(ρ) function.

By looking at the graphs on Figures 1 through 4, their fractal-like character seems plausible. Note, however that the graphs of the functions themselves are not fractals, since they do not form compact sets due to the infinite number of discontinuities. It is easy to see that by using the function points themselves, then the *Hausdorff dimension* of the points can be calculated and it turns out to be exactly 1.0. However, if one calculates the points of the functions at some resolution then connects the discontinuous points by straight line sections and then repeats this process until the limit of infinitesimally fine resolution, then one gets true fractals. In other words, this limit curve is the boundary curve of the compact 2-dimensional set of the points under the discontinuous set of points of the function‘s graph. In the table below we give relatively rough estimates of the fractal dimension of these curves, by using the “*box counting*” method:

arf	1.104
arf_{sec}	1.134
arf^{-1}	1.133
arf2	1.130

where arf_{sec} corresponds to the section of the arf function presented in Figure 2, which – due to the finer resolution – provided a somewhat higher value then the whole function. It can be seen that the fractal dimension of these curves is rather similar, probably identical.

Remark. Without proving the Lebesgue integrability of $\mathrm{arf}, \mathrm{arf}^{-1}$ and arf2 functions, computational evidence supports that their Lebesgue integral for the open interval of $(0, 1)$ is equal to $1/2$.

4. Arithmetics with monotonic sequences

We have seen in Section 2 that the positive monotonic sequences are isomorphic to a subset of the real interval $(0, 1)$, namely to $\mathbb{R}^*$. Thus, for example the sequence of primes corresponds to the real number $\mu = \mathcal{M}(\mathbf{P})$ (see A092856 in [3]):

0.414682509851111660248109622154307708365774238137916977868245414488640 96...

what is quite close to $\sqrt{2} - 1$ but it is not identical. Also note that the real number above is identical to the real number created from the characteristic function of the sequence of primes (A010051 in [3]) as binary fractional digits.

Conversely, every irrational number in $(0, 1)$ (and also many rationals) corresponds to some monotonic integer sequence, e.g., $\mathcal{M}^{-1}(\sqrt{2} - 1)$ yields the sequence (A092855 in [3]):

2, 3, 5, 7, 13, 16, 17, 18, 19, 22, 23, 26, 27, 30, 31, 32, 33, 34, 35, 36, 39, 40, 41, 43, 44, 45, 46, 49, 50, 53, 56, 61, 65, 67, 68, 71, 73, 74, 75, 76, 77, 79, 80, 84, 87, 88, 90, 91, 94, 95, 97, 98, 99, 101, 103, 105, 108, 110, 112, 114, 115, 116, 117, 118, 120, 123, 124, 125, 126, 127, 131, 132, 133, 135, 137, 138, 140, 141, 142, 143, 145, 146, 152, 154, 155, 156, 158, 160, 164, 167, 170, 171, 172, 174, 175, 176, 178, 180, 185, 188, 189, 192, 193, 194, 196, 197, 199, 203, 205, 206, 207, 208, 210, 212, 213, 216, 221, 223, 224, 230, 231, 234, 235, 238, 239, 240, 243, 244, 247, 251, 253, 255... .

This sequence, alike a completely different sequence, the Aronson transform of the "evil" sequence (A092863 in [3]), looks quite reach in primes. This peculiarity is deceiving, however in both cases, since by checking further the sequences, the excess prime content vanishes.

Just one more example of mapping a mathematical constant into an integer sequence is the $\mathcal{M}^{-1}(1/\sqrt{2\pi})$ (A092857 in [3]):

2, 3, 6, 7, 11, 16, 20, 22, 25, 26, 29, 30, 31, 32, 34, 36, 41, 42, 44, 45, 48, 50, 55, 59, 60, 62, 67, 68, 69, 70, 71, 72, 75, 77, 78, 81, 82, 83, 84, 88, 90, 99, 101, 102, 103, 105, 107, 109, 110, 111, 115, 116, 117, 121, 123, 124, 125, 126, 127, 128,... .

Obviously, a sequence mapped from some mathematical constant may not be very interesting, unless somebody finds some known real constant γ, such that $\mathcal{M}^{-1}(\gamma)$ is an otherwise known sequence. Though the author made some efforts in this direction, they were unsuccessful, so far.

As it was mentioned in Section 2, an arithmetics of the sequences can be introduced corresponding to the arithmetics in its isomorphic map within the interval $(0,1)$. Recalling: $\mathbf{A} \oplus \mathbf{B} = \mathcal{M}^{-1}(\mathcal{M}(\mathbf{A}) \oplus \mathcal{M}(\mathbf{B}))$ and also $\mathbf{A} \otimes \mathbf{B} = \mathcal{M}^{-1}(\mathcal{M}(\mathbf{A}) \cdot \mathcal{M}(\mathbf{B}))$.

Note that for the "addition", as it is defined in Definition 4, some restrictions apply to make sure that the resulting set of integers is a "non-trivial" integer sequence (cf. Def. 3). It is easy to see, that for two non-trivial sequences $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \oplus \mathbf{B} \in \mathbb{S}^*$, unless there is an index i in $\mathbf{A}$ and an index j in $\overline{\mathbf{B}}$, such that $\forall\, k > 0$, $a_{i+k} = \bar{b}_{j+k}$.

Thus, one can create e.g., the "sum" of the sequence of primes $\mathbf{P}$ and the sequence of triangular numbers (A00217 in [3]) $\mathbf{T}$, where $t_i = i(i+1)/2$ (A092858 in [3]):

$\mathbf{P} \oplus \mathbf{T}$ =[5, 6, 7, 10, 11, 13, 15, 17, 19, 21, 23, 28, 29, 31, 36,...].

Similarly, the *difference* between $\mathbf{T}$ and $\mathbf{P}$ can also be given (A092859 in [3]):

$\mathbf{T} \oplus \overline{\mathbf{P}}$ =[3, 4, 5, 7, 12, 13, 16, 18, 19, 22, 23, 30, 31, 38, 39, 40,...].

It is a direct consequence of having a defined addition that we can also multiply the monotonic sequences by natural numbers. Obviously, $2\mathbf{P} = \mathbf{P} \oplus \mathbf{P} = \mathbf{X}$ and then $x_i = p_i - 1$. The sequence $3\mathbf{P}$ (A092860 in [3]) is not so simple:

$3\mathbf{P}$ =[3, 4, 5, 6, 7, 10, 11, 12, 13, 16, 17, 18, 19, 22, 23, 28, 29, 30, 31, 36, 37, 40, 41, 42, 43, 46, 47, 52, 53, 58, 59, 60, 61, 66, 67, 70, 71, 72, 73, 78, 79, 82, 83, 88, 89, 96, 97,...],

in which sequence due to the truncation (fractional part of the sum) in Definition 1, one term is eliminated. Due to this effect, the inverse of integer multiplication in general, does not exist.

As we have seen, the positive monotonic sequences form a semigroup over the "multiplication" as defined in Remark 3. The sequence A092861 in [3] illustrates this:

$\mathbf{P} \otimes \mathbf{E}$ =[4, 7, 9, 12, 14, 15, 18, 19, 21, 25, 26, 33, 35, 36, 37, 40, 41, 42, 44, 47, 48, 50, 54, 55, 58, 59, 60, 64, 65, 66, 69, 72, 77, 78, 79, 80, 84, 86, 87, 88, 89, 90, 91, 97, 99, 100, 105, 106, 107, 108, 110, 111,...],

where $\mathbf{E}$ is the sequence of "evil numbers" (A001969 in [3]), containing the numbers having even number of non-zero binary digits.

By having a well defined multiplication among the monotonic sequences, we also can create the integer powers of the sequences. Thus, for example, the square of the prime sequence, $\mathbf{P}^2 = \mathcal{M}^{-1}(\mathcal{M}^2(\mathbf{P}))$ (A092862 in [3]):

$\mathbf{P}^2$ =[3, 5, 6, 14, 16, 17, 19, 21, 22, 25, 27, 31, 32, 34, 36, 37, 41, 42, 44, 45, 48, 49, 52, 54, 57, 59, 60, 62, 64, 65, 69, 74, 75, 78, 81, 88, 90, 91, 92, 94, 97, 98, 100,...]

It is obvious, that this can be generalized for real exponents, as well. Thus, for example, we can have the π as exponent for $\mathbf{P}$ (A092863 in [3]):

$\mathbf{P}^\pi$ =[4, 7, 10, 16, 18, 20, 22, 23, 27, 28, 29, 31, 32, 33, 34, 35, 37, 38, 40, 42, 46, 51, 57, 60, 65, 66, 67, 68, 69, 70, 72, 73, 74, 77, 78, 80, 81, 82, 84, 85, 89, 91, 92, 93, 94, 95, 99, 101, 103, 107, 108, 110, 111, ...].

In general, it is easy to realize that for any function which is an automorphism in $(0, 1)$, the inverse binary mapping defines uniquely an automorphic transformation over the monotonic positive sequences.

5. Cellular Automaton Functions and Transformations

The cellular automata are known since the middle of the 20th century, though systematic studies appeared only at the beginning of the 21st century, e.g., by Ilachinski [10]. Also a very broad coverage of the topic – along with many other types of recursive mappings – can be found in the book of S. Wolfram [9]. In that book (see p. 53) the individual one-dimensional cellular automata – where the next state of a cell is determined by itself and by its two neighbors – are identified by 8-digit binary numbers (or an integer in the range of 0-255), where each of the digits determine the resulting state of the cell as function of the initial state of the 3 parent cells. For example the rule of the automaton identified by the number "30" is as follows:

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array} = 30_{10} \; .$$

This particular automaton – along with a few (13) others from among the possible 255 different kinds – exhibits a peculiar random-like behavior when it is applied recursively, starting from a single cell. These kinds of automata are categorized as "Class 3" and "Class 4" by S. Wolfram (p. 231 in [9]). Note that about the half of all possible such automata are "trivial", and many others produce regular, repetitive patterns.

Definition 9. *Cellular automaton mapping.* Concerning a generic definition of the cellular automata, we refer to for example Ilachinski's book [10], but the most readily accessible resource is Eric W. Weisstein's MathWorld [2]. Here we restrict ourselves to the "one-dimensional, two-states, first neighbors" automata, which are also referred as *elementary cellular automata* in [2]. Let $\mathbf{D}$ denote an infinite ordered set of discrete elements of two different kinds (states): one of the kinds is identified by "0" and the other by "1". The ordering of the set $\mathbf{D}$ is one-dimensional, hence every member d_i has two neighbors: d_{i-1} and d_{i+1}. Let $\mathbb{D}$ denote the set of such sets. The mapping $\mathrm{C} : \mathbb{D} \to \mathbb{D}$ is a (one-dimensional, first-neighbors) *cellular automaton maping* if the member d'_i of the image set $\mathbf{D'}$ is determined from the members d_{i-1}, d_i, d_{i+1} of $\mathbf{D}$ by applying the "rule" of the automaton, which uniquely defines the image value for every possible combination of the values d_{i-1}, d_i, d_{i+1}. Note that $i \in (-\infty, \infty)$. Notaion:$\mathbf{D'} = \mathrm{f_C}(\mathbf{D})$.

Remark 8. The definition above is applicable for "two-ways" infinite ordered sets of binary digits – because this corresponds to the usual definitions –, but it is easily adaptable to ordered sets which are only "one-way" infinite: i.e., for monotonic integer sequences. From among the three possible obvious ways, we have chosen the one where d'_i is determined by the set of d_{i-2}, d_{i-1}, d_i, along with postulating "0" values for the non-existent elements of d_{-1} and d_0.

According to Definition 1, we can map any monotonic sequence into an infinite sequence of binary 1's and 0's. This suggests to introduce a transformation by applying some cellular automaton such a way that after mapping the sequence into the form of a binary number, one applies the selected cellular automaton rule by shifting a 3-digit window over the binary digits of the number one-by-one, yielding the digits of the new number. Then, this new number is to be mapped back into the realm of integer sequences by using the *inverse binary mapping* (Definition 1).

Definition 10. *CA transformation of monotonic sequences.* Due to the injective character of binary mapping (as defined in Def. 1), any monotonic sequence can be transformed according to any simple (and also other) cellular automaton C by the

$$\mathcal{C}_{\mathrm{C}}(\mathbf{S}) = \mathcal{M}^{-1}(\mathrm{f}_{\mathrm{C}}(\mathcal{M}(\mathbf{S})))$$

formula, where $\mathrm{f}_{\mathrm{C}}(\mathcal{M}(\mathbf{S}))$ is the cellural automaton mapping by using the automaton C of $\mathcal{M}(\mathbf{S})$, according to Definition 9.

As an example for this, let us see the cellular automaton transform with code 110 of the sequence of primes (A093515 in [3]):

$\mathcal{C}_{110}(\mathbf{P})$ =[2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 17, 18, 19, 20, 23, 24, 29, 30, 31, 32, 37, 38, 41, 42, 43, 44, 47, 48, 53, 54, 59, 60, 61, 62, 67, 68, 71, 72, 73, 74, 79, 80, 83, 84, 89, 90, 97, 98, 101, 102, 103, 104, 107, 108, 109, 110, 113, 114, 127, 128, 131, 132, 137, 138, 139, 140, 149, 150, 151, 152, 157, 158, 163, 164, 167, 168, 173,...]

Note, that this sequence contains all primes in the range. This is a provable consequence of the actual form of the rule 110 and the fact that the gap between subsequent primes is at least 2. You can also see the sequences $\mathcal{C}_{30}(\mathbf{P})$, $\mathcal{C}_{45}(\mathbf{P})$,$\mathcal{C}_{73}(\mathbf{P})$,$\mathcal{C}_{89}(\mathbf{P})$, $\mathcal{C}_{90}(\mathbf{P})$, $\mathcal{C}_{137}(\mathbf{P})$, $\mathcal{C}_{225}(\mathbf{P})$ as A093510, A093511, A093512, A093513, A093514, A093516, A093517 in [3], respectively.

Definition 11. *Cellular automaton function – CA function.* The function $\mathrm{f}_{\mathrm{C}} : \mathbb{R}_{(0,1)} \to \mathbb{R}_{(0,1)}$ is defined according the following construction:

Let $\rho \in \mathbb{R}_{(0,1)}$ be an arbitrary number in the domain, then the binary digits of $\mathrm{f}_{\mathrm{C}}(\rho)$ are obtained by applying the rule of the one-dimensional cellular automaton identified by the code C on the binary digits of ρ. C is defined according to Definition 9 with the slight modification given in Remark 8. Thus, the left-most digit of the fractional part of $\mathrm{f}_{\mathrm{C}}(\rho)$ comes from two 0s and the first fractional digit of ρ by applying the rule of C, etc.

Remark 9. Clearly, – unlike we have seen with the Aronson function – the domain of the CA functions is the whole $\mathbb{R}_{(0,1)}$.

In Figure 5 we can see four examples of CA functions. First of all, one can notice the similar fractal character of these functions to the earlier defined Aronson function (cf. Def. 7 and Fig. 1). The main similarity between all of these functions is the infinite number discontinuous points and also the identical positions of the points of discontinuities (see eq. (1) in Remark 7). This character comes

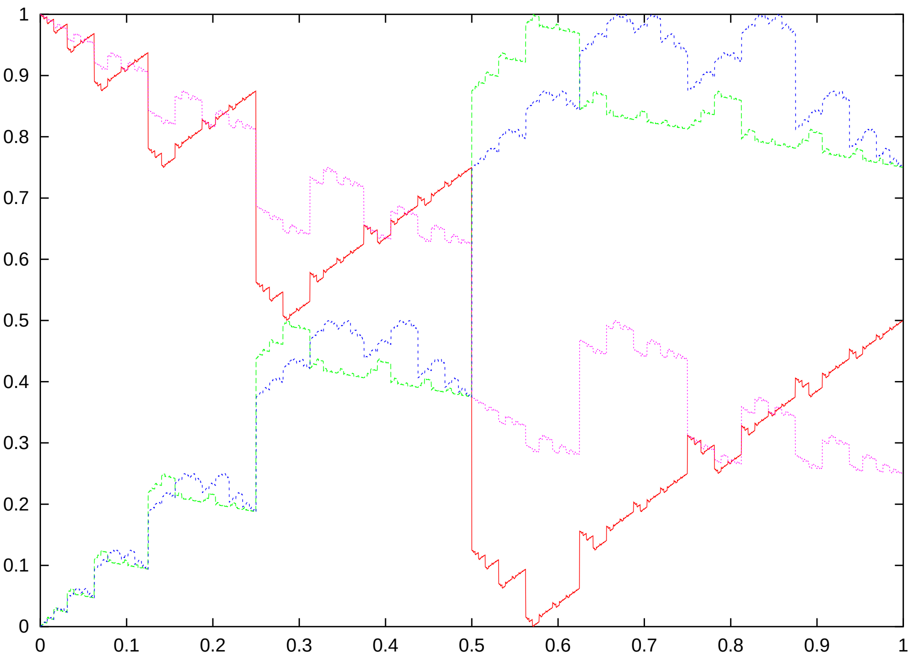

FIGURE 5. The graph of four different cellular automaton functions (code 30: green, code 45: purple, code 110: blue, and code 169: red)

quite obviously from the fact that all of these functions operate on the binary representation of the argument.

We have seen that the Aronson function has its inverse. The situation with the CA functions is not so easy, however. While for some of the CA rules the inverse can be defined, for many others (e.g., for some trivial ones) it is not possible. From among the four CA functions presented in Fig. 5 three probably have their inverses. The function $f_{110}(\rho)$, however certainly has no inverse, since it takes infinitely many times infinitely several different values (e.g., 0.5, 1.0, etc.). This is illustrated in Figure 6. Proving this statement goes beyond this paper, but it can be demonstrated easily in numerical ways that identical values occur at different – both at rational and irrational – arguments. Also note that from among the CA functions presented in Fig. 5, $f_{110}(\rho)$ is the only one in which the embedded CA rule is categorized by S. Wolfram as "Class 4", and he – together with M. Cook – has also proven the *universality* of that particular rule (see pages 675 and 1115 in [9]). One may suspect that the non-existence of an inverse of the corresponding CA function may be related to the *universality* of the embedded automaton (see also Rule110, as well as Universality in [2]) .

Similarly to the case of the arf function, we evaluated the *fractal dimension* of the selected CA functions (in the same sense as it is explained in Section 3).

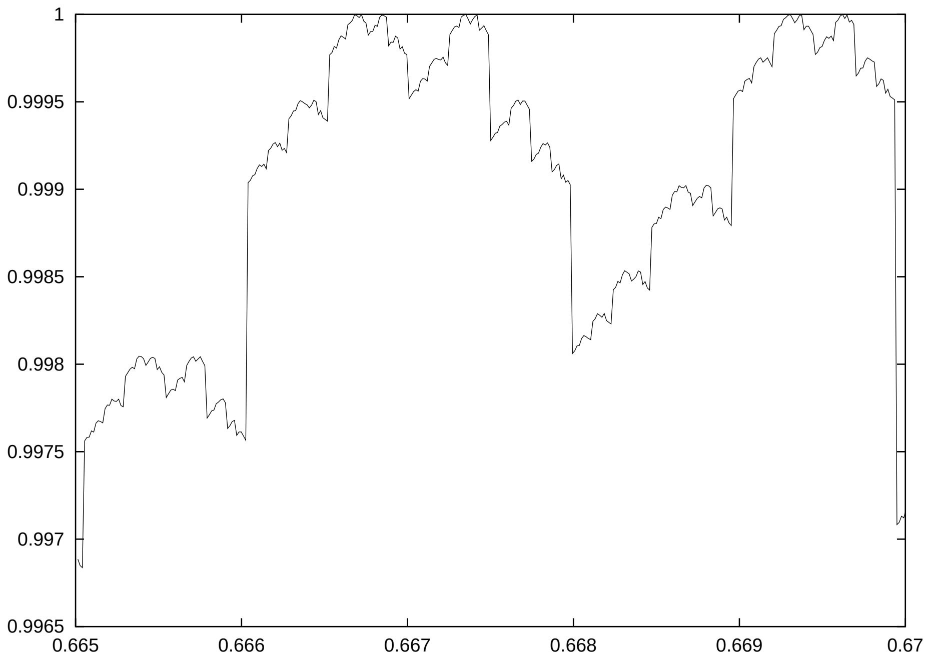

FIGURE 6. A zoomed section of the $f_{110}(\rho)$ function

The results of the rough evaluation by box counting method are as follows:

f_{30}	1.099
f_{45}	1.136
f_{110}	1.127
f_{169}	1.112

These values are also rather close to each other and also close to the values we have seen in the case of the arf function. If there are characteristic differences between the fractal dimensions, it can be evaluated only by some much more accurate methodology.

In case of the CA functions the *Lebesgue integrability* is quite obvious, and for the selected four functions their estimate is as follows:

$$\begin{aligned} \int_{(0,1)} f_{30}(\rho) &\approx 0.563... \\ \int_{(0,1)} f_{45}(\rho) &\approx 0.563... \\ \int_{(0,1)} f_{110}(\rho) &\approx 0.594... \\ \int_{(0,1)} f_{169}(\rho) &\approx 0.4998.... \end{aligned}$$

It is notable that the most simple of the selected CA rules (Rule169) yielded the lowest value (the author believes, it is exactly equal to 1/2) and the Rule110 – which is rated as Class 4 – gave the highest value.

6. Conclusion

The isomorphism – named as *binary mapping* – as defined in Definition 1 made possible to map any monotonically increasing integer sequence to a real number in $(0, 1)$. Hence, having an automorphism over the monotonic sequences, we obtain a function with domain and codomain in $(0, 1)$. Conversely, any automorphic function over $(0, 1)$ corresponds to an automorphic transformation over the set of monotonic sequences. The simple arithmetics defined between the sequences potentially yields an immense number of new sequences and hopefully some of them will exhibit important features. Perhaps, the isomorphism may help exploring or even proving some peculiarities of some transformations of the integer sequences, as well as some peculiarities of specific sequences.

The presented examples related to cellular automata provide new hints for continuing the exploration of this relatively newly approached realm.

Acknowledgment

The author thanks Benoit Cloitre for his preliminary review, encouragement and valuable ideas.

Appendix:

A set of PARI/GP [11] functions is given below, to realize the *binary mapping* and its inverse, the *Aronson transformation* and its inverse, as well as the *Aronson function* with its inverse:

```
{mt(v)= /*Returns the binary mapping of v */
local(a=0,p=1,q=0.0,l,m);l=matsize(v)[2];m=v[l];
for(i=1,m,p=isin(i,v,l,p);if(p>0,a+=2^(-i)));
q=0.+a;return(q)}
/****/
{mtinv(x)= /*Returns the inverse binary mapping of x */
local(z,q,v=[],r=[],l);
z=frac(x);v=binary(z)[2];l=matsize(v)[2];
for(i=1,l,if(v[i]==1,r=concat(r,i)));return(r)}
{farons(x)= /* The Aronson‘s function...*/
return(mt(arons(mtinv(x))))}
/****/
{frarons(x)= /* The inverse Aronson‘s function */
return(mt(rarons(mtinv(x))))}
/****/
{/* Returns the Aronson transform of v */
arons(v)=
local(x=[],pv=1,px=1,n=1,i=0,k,l);
l=matsize(v)[2];
/*The initial terms: */
if(n<v[pv],n+=1;while(n==v[pv],n+=1;pv+=1);x=concat(x,n);n+=1;i+=1,
while((n<l)&&(v[pv]==n),x=concat(x,n);n+=1;pv+=1;i+=1));
/*The induction:*/
while(abs(pv)<=l&&n<v[l],k=x[i];n=k; pv=isin(i+1,v,l,pv);
```

```
/* pv>0 if (i+1) is in v */
if(k==i,n+=1;if(pv<0,pv=abs(pv); while(pv>0,n+=1; pv=isin(n,v,l,pv))),
px=isin(i+1,x,i,px);  if(px>0,pv=-abs(pv);while(pv<0,n+=1;pv=isin(n,v,l,pv)),
      pv=abs(pv);while(pv>0,n+=1;pv=isin(n,v,l,pv))));
x=concat(x,n);i+=1);/*print(i);*/ return(x) }
/****/
{ /* Returns the inverse Aronson transform of v */
rarons(v)=
local(h=[],c=[],x=[],pv=1,px=1,n=1,i,j=1,l,m=0,f);
l=matsize(v)[2];h=vector(l);c=vector(l);for(i=1,l,h[i]=[];c[i]=[]);
while(n<=l,while((n<v[j])&&(n<=l),c[n]=concat(c[n],v[n]);n+=1);
if(n<=l,h[n]=concat(h[n],v[n]);j+=1;n+=1));
n=1;while(n<=l,f=matsize(h[n])[2];
  for(i=m+1,v[n]-1,if(f, c[n]=concat(c[n],i),
if((m<>n-1)||(i<>n), h[n]=concat(h[n],i), c[n]=concat(c[n],i)))); m=v[n];n+=1);
for(i=1,l,x=concat(x,h[i]));return(x) }
/****/
{isin(x,v,l,poi)=
/*If x integer is in v monotonic vector of length l,
the function returns a positive 'poi',
else a negative one. ("poi" is also used for acceleration,
the last returned value is recommended in the input) */
poi=abs(poi);if(poi==1&&x<v[1], return(-poi),
if(x<v[poi],while(x<v[poi]&&poi>1,poi-=1);
if(x<>v[poi],poi*=-1),
if(x>v[poi],while(x>v[poi]&&poi<l,poi+=1);
if(x<>v[poi],poi*=-1)));return(poi))}
```

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