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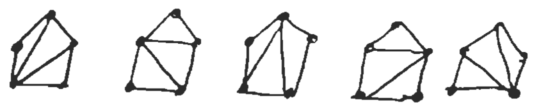
CATALAN NUMBERS

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$$

1. number of ways of dividing an (n+2)-gon into n triangles by n-1 diagonals.



2. number of ways of parenthesizing a string of n+1 letters with a binary multiplication

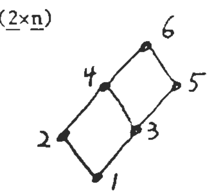
$(x^2\cdot x)x \quad x(x^2\cdot x) \quad (x\cdot x^2)x \quad x(x\cdot x^2) \quad x^2\cdot x^2$.

3. number of plane binary trees with n vertices

4. number of sequences of n 1's and n -1's such that each partial sum is ≥ 0

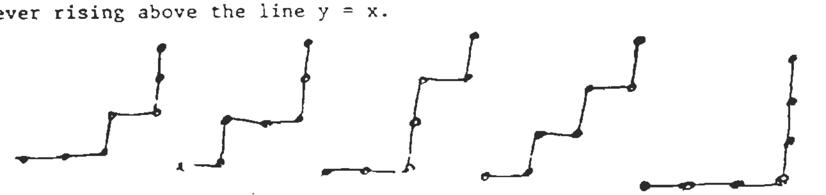
111---   11-1--   11--1-   1-11--   1-1-1-

5. $e(\underline{2}\times\underline{n})$



123456
132456
123546
132546
135246

6. number of paths from (0,0) to (n+1,n+1) with steps (0,1) or (1,0), never rising above the line y = x.



7. number of order ideals of $S(\underline{n-1})$, the poset of intervals of $\underline{n-1}$

$\underline{2}$ $S(\underline{2})$

ϕ, a, b, ab, abc

8. number of non-isomorphic n-element posets with no induced subposet isomorphic to  or .

9. number of permutations $a_1 \cdots a_n$ of [n] with longest increasing subsequence of length ≤ 2

132   213   231   312   321

10. number of sequences $1 \le a_1 \le \cdots \le a_n$, where $a_i \in \mathbb{P}$ and $a_i \le i$

111   112   113   122   123

11. number of plane (rooted) trees with n+1 vertices

12. number of partitions $\pi = \{B_1,\ldots,B_k\}$ of [n] such that if $a < b < c < d$ and $a,c \in B_i$ and $b,d \in B_j$, then $i = j$

123   12-3   13-2   1-23   1-2-3

13. The partitions in #12 form a subposet P_n of Π_n, with Möbius function μ. Then

$$(-1)^{n-1}\mu(\hat{0},\hat{1}) = C_{n-1}.$$

14. number of 2-sided ideals in the algebra of all (n-1)x(n-1) upper triangular matrices over a field.

Stanley, Interp. of Catalan Nos → 108

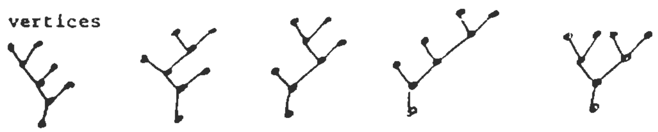
15. There is a unique sequence $a_0, a_1, a_2, \cdots$ such that for all n,

$$\det\begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ & & \vdots & \\ a_n & a_{n+1} & \cdots & a_{2n} \end{bmatrix} = \det\begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ & & \vdots & \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} = 1$$

Then $a_n = C_n$.

16. number of ways $2n$ points on the circumference of a circle can be joined in pairs by n non-intersecting chords

17. number of planted (i.e., root has degree 1) trivalent plane trees with $2n+2$ vertices



18. number of n-tuples $(a_1,\ldots,a_n) \in \mathbb{P}^n$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbors

14321 13521 13231 12531 12341

19. number of permutations $a_1\cdots a_n$ of $[n]$ with no subsequence $a_ia_ja_k$ $(i<j<k)$ such that $a_j < a_k < a_i$

123 132 213 231 321

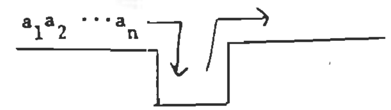
20. number of permutations $a_1a_2\cdots a_{2n}$ of the multiset $\{1^2,2^2,\ldots,n^2\}$ such that:

(a) the first occurrence of $1,2,\ldots,n$ appear in increasing order, and

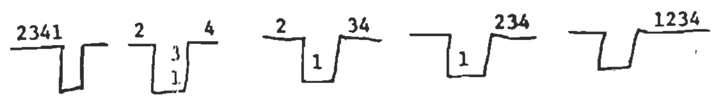
(b) there is no subsequence of the form $\alpha\beta\alpha\beta$

112233 112332 122331 123321 122133

21. number of permutations $a_1a_2 \cdots a_n$ of $[n]$ which can be put in increasing order on a single stack:

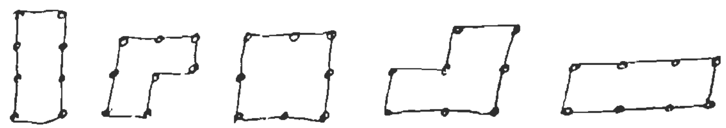


E.g.,

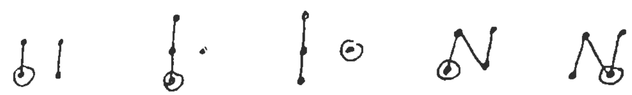


123 132 213 231 321

22. number of (unordered) pairs of lattice paths with $n+1$ steps starting from $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same place, and only intersecting at the beginning and end.



23. number of non-isomorphic $(n+1)$-element posets which are the union of two chains and which cannot be written as a (non-trivial) ordinal sum, rooted at a minimal element



24. number of sequences $(a_1,\ldots,a_n) \in \mathbb{P}^n$ such that the matrix

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & a_3 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & a_4 & \cdots & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_n \end{bmatrix}$$

is positive definite with determinant 1

131 122 221 213 312