THE PROBLEM OF THE n QUEENS

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1. Introduction. The problem of the n queens is to determine in how many ways it is possible to place n queens on an n×n chessboard so that no queen is attacking any other.

The original problem for n=8 was first proposed by Max Bezzel in the Berlin Schachzeitung, in 1848, and the twelve solutions, by Franz Nauck, were published in 1850 in the Leipzig Illustrierte Zeitung. Since then a considerable literature has grown up around the problem and solutions for n≤12 have been given in various places (Ahrens [1, vol 1,ch 9], Kraitchik [4, ch 10], Madachy [6, ch 2], Rouse Ball [7,ch 6]).

More recently interest has been shown in the problem as an exercise in computer programming and Dijkstra in [3, ch 1] uses it to introduce recursion as a programming technique. By developing the ideas of Dijkstra to search for a more restricted set of solutions it has been possible to extend the known results beyond those given by Kraitchik in [4, ch 10].

These results are given in Table I.

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2. Definitions. A queen is defined as a pair of integers (a,b), where $1 \leqslant a \leqslant n$ and $1 \leqslant b \leqslant n$.

A solution, ω, is a set of n queens.

$\{q_i \mid q_i = (i,b_i),\ i = 1,2,\cdots,n\}$, where for all i and j satisfying $1 \leqslant j < i \leqslant n$, then $|b_i - b_j| \neq i-j$.

The norm of ω is the base n number $b_1 b_2 \cdots b_n$.

wavy 3. Generating the solutions by computer. Dijkstra in [3, ch 1] discusses a method for generating all solutions using a recursive technique. Solutions are generated more quickly, however, if the solutions are collected into classes, the members of which are equivalent under a group of transformations. The choice of the group requires that the equivalence class of any solution can be determined easily together with the number of solutions belonging to that class. A class can
wavy be identified by a characteristic solution which is defined as that solution, in the class, with the least norm. It is then sufficient to program a search for characteristic solutions only.

wavy The number of solutions is denoted by T_n and the number of characteristic solutions by C_n

To determine T_n we choose the dihedral group of the square as the group of transformations. This group consists of the 8 transformations $\{I, R, R^2, R^3, RS, R^2S, R^3S\}$, where I denotes the identity transformation, R a rotation of the board in its own plane through $\pi/2$ and S a reflection of the board. This group enables solutions to be classified as equivalent under simple rotations and reflections of the board (Alexandroff [2, p50], Kraitchik [4, ch 10], Ledermann [5, pp48-55]).

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The equivalence class containing a solution ω consists of:

i) 8 members if $\omega \neq R^2.\omega$ $\{\omega$ is unsymmetric$\}$

ii) 4 members if $\omega = R^2.\omega$ but $\omega \neq R.\omega$ $\{\omega$ is symmetric$\}$

iii) 2 members if $\omega = R.\omega$ $\{\omega$ is doubly-symmetric$\}$

The number of unsymmetric characteristic solutions is denoted by U_n, the number of symmetric characteristic solutions by S_n and the number of doubly-symmetric characteristic solutions by D_n.

Thus:

$$C_n = U_n + S_n + D_n$$

$$T_n = 8U_n + 4S_n + 2D_n$$

n	P_n	D_n	S_n	U_n	C_n	T_n
4	1	1	0	0	1	2
5	1	1	0	1	2	10
6	0	0	1	0	1	4
7	0	0	2	4	6	40
8	0	0	1	11	12	92
9	0	0	4	42	46	352
10	0	0	3	89	92	724
11	0	0	12	329	341	2680
12	1	4	18	1765	1787	14200
13	1	4	32	9197	9233	73712
14	0	0	105	45647	45752	395596
15	0	0	310			
16	4	32	734			
17	8	64	2006			
18	0	0	4526			
19	0	0				
20	15	240				
21	22	352				
22	0	0				
23	0	0				
24	52	1664				

25	51	1632					

26	0	0
27	0	0
28	257	16448 ~~21888~~
29	342	21888
30	0	0
31	0	0
32	1589	203392
[illegible]	2609	333952
34	0	0
35	0	0
36	11417	2922752
37	16896	4325376
38	0	0
39	0	0
40	75375	38592000
41	99114	50746368