SEQUENCES, PATHS, BALLOT NUMBERS

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1. INTRODUCTION

Consider the sequences of positive integers $(a_1, a_2, \cdots, a_n)$ that satisfy the following conditions:

$$1 = a_1 \le a_2 \le \cdots \le a_n \tag{1.1}$$

and

$$a_i \le i \qquad (1 \le i \le n) . \tag{1.2}$$

The number of such sequences with $a_n = k$, where k is fixed, $1 \le k \le n$, will be denoted by $f(n,k)$. Thus the total number of sequences satisfying (1.1) and (1.2) is equal to

$$\sum_{k=1}^{n} f(n,k) \quad . \tag{1.3}$$

The numbers $f(n,k)$ were called two-element lattice permutations by MacMahon [6, p. 167]. Two-element lattice permutations have n elements of one kind, k of a second kind with $k \le n$, and are such that if a_r is the number of the first kind in the first r and b_r is the corresponding number of the second kind, then $a_r \ge b_r$ for every r. Another way of putting it is that the elements of the first kind are thought of as votes for candidate A, those of the second kind as votes for candidate B; the lattice permutation is then an election return with final vote (n,k) which is such that all partial returns correctly predict the winner. As still another interpretation, let each element of the first kind be represented as a unit horizontal line and each of the second kind as a unit vertical line, then the permutation represents a path from (1,1) to (n,k) which does not cross the line $y = k$. The illustration at the top of the following page shows an admissible path from (1,1) to (7,6).

In the present paper we discuss some of the basic properties of $f(n,k)$ and related functions. We also discuss briefly some extensions, in particular the q-analog [3]

$$f(n,k,q) = \sum q^{a_1+a_2+\cdots+a_n} , \tag{1.4}$$

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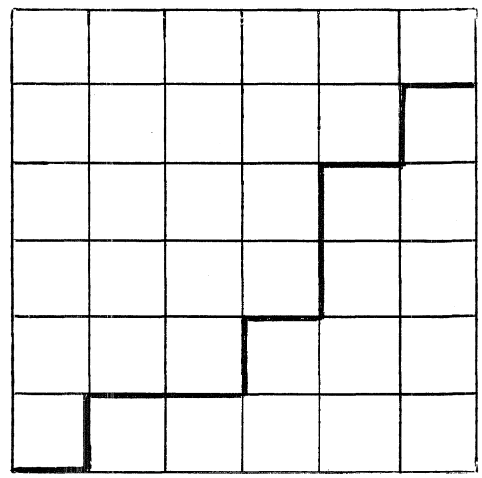

where the summation is over all $a_1, a_2, \cdots, a_n$ such that

$$1 = a_1 \le a_2 \le \cdots \le a_n = k$$

and

$$a_i \le i \qquad (1 \le i \le n) ,$$

furnishes a useful generalization of $f(n,k)$. Many of the properties of $f(n,k)$ carry over to the general case.

The list of references at the end of the paper is by no means complete. A comprehensive bibliography of Catalan numbers is included in [1].

2. It follows at once from the definition that

$$f(n,k) = \sum_{j=1}^{k} f(n-1,j) \qquad (1 \le k \le n;\ n > 1) , \tag{2.1}$$

where it is understood that $f(n-1, n) = 0$. From (2.1) we get

$$f(n,k) = f(n, k-1) + f(n-1, k) , \tag{2.2}$$

where again $1 \le k \le n,\ n > 1$.

Making use of either (2.1) or (2.2) we can easily compute the table shown at the top of the following page.

It is evident from (2.1) that the total number of sequences satisfying (1.1) and (1.2) is equal to

$$f(n+1, n+1) = f(n+1, n) \quad . \tag{2.3}$$

f(n,k) :

n \ k	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	2	2				
4	1	3	5	5			
5	1	4	9	14	14		
6	1	5	14	28	42	42	
7	1	6	20	48	90	132	132

We now define

(2.4) $$b(n,k) = \sum_{j=1}^{k} 2^{k-j} f(n,j) .$$

Using (2.2) we get

$$b(n,k) = \sum_{j=1}^{k} 2^{k-j} \{f(n, j - 1) + f(n - 1, j)\}$$

$$= \sum_{j=1}^{k-1} 2^{k-j-1} f(n,j) + \sum_{j=1}^{k} 2^{k-j} (f(n - 1, j) ,$$

so that

(2.5) $$b(n,k) = b(n, k - 1) + b(n - 1, k) \qquad (1 \le k < n) .$$

However, for k = n, we get

(2.6) $$b(n,n) = b(n, n - 1) + 2b(n - 1, n - 1) .$$

It follows from (2.5) that

(2.7) $$b(n,k) = \sum_{j=1}^{k} b(n - 1, j) \qquad (1 \le k \le n) ;$$

however

(2.8) $$b(n,n) = \sum_{j=1}^{n-1} b(n - 1, j) + 2b(n - 1, n - 1) .$$

The table shown at the top of the following page is easily computed using either (2.4) or (2.5) and (2.6).

Examination of the table suggests the following formula.

(2.9) $$b(n,k) = \binom{n + k - 1}{k - 1} \qquad (1 \le k \le n) .$$

It is clear from (2.4) that

b(n, k) :

n \ k	1	2	3	4	5	6
1	1					
2	1	3				
3	1	4	10			
4	1	5	15	35		
5	1	6	21	56	126	
6	1	7	28	84	210	462

(2.10) $$b(n,1) = 1 \qquad (n = 1, 2, 3, \cdots),$$

in agreement with (2.9). Assume that (2.9) holds for $n = 1, 2, \cdots, m$ and $1 \le k \le n$. Then by (2.7), for $k \le m$,

$$b(m+1, k) = \sum_{j=1}^{k} b(m,j) = \sum_{j=1}^{k} \binom{m+j-1}{j-1} = \sum_{j=1}^{k} \binom{m+j-1}{m}$$
$$= \binom{m+k}{m+1} = \binom{m+k}{k-1}.$$

On the other hand, by (2.8),

$$b(m+1, m+1) = \sum_{j=1}^{m} b(m,j) + 2b(m,m)$$
$$= \sum_{j=1}^{m} \binom{m+j-1}{j-1} + 2\binom{2m-1}{m-1}$$
$$= \binom{2m}{m-1} + \binom{2m}{m} = \binom{2m+1}{m}.$$

This evidently completes the proof of (2.9).

Returning to (2.4), it is evident that

(2.11) $$f(n,k) = b(n,k) - 2b(n, k-1) \qquad (1 \le k \le n).$$

Therefore, by (2.9),

(2.12) $$f(n,k) = \binom{n+k-1}{k-1} - 2\binom{n+k-2}{k-2} = \frac{n-k+1}{n}\binom{n+k-2}{n-1} \qquad (1 \le k \le n).$$

In particular, for $k = n$,

(2.13) $$f(n,n) = \frac{1}{n}\binom{2n-2}{n-1}.$$

Thus, by (2.3), the number of sequences that satisfy (1.1) and (1.2) is equal to the Catalan number

(2.14) $$c(n) = \frac{1}{n+1}\binom{2n}{n} .$$

Making use of (2.12), it is easy to verify that

(2.15) $$\begin{vmatrix} f(n, k) & f(n, k+1) \\ f(n+1, k) & f(n+1, k+1) \end{vmatrix} > 0$$

and

(2.16) $$f^2(n,k) > f(n, k-1)f(n, k+1) .$$

3. Put

(3.1) $$C(x) = \sum_{n=0}^{\infty} c(n)\, x^n .$$

Since

$$(1 - 4x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} (4x)^n$$

$$= 1 - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1} x^n$$

$$= 1 - 2x\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} x^n ,$$

it follows from (2.14) and (3.1) that

(3.2) $$C(x) = \frac{1}{2x}(1 - \sqrt{1 - 4x}) .$$

Thus

(3.3) $$xC^2(x) = C(x) - 1 ,$$

which is equivalent to

(3.4) $$c(n+1) = \sum_{k=0}^{n} c(k)c(n-k) .$$

In a letter to the writer, Dr. Jürg Rätz had inquired about the possibility of proving (3.4) without the use of generating functions. This can be done in the following way. Put

$$\frac{(x+\frac{1}{2})_n}{(x)_{n+1}} = \sum_{k=0}^{n} \frac{A_k}{x + k} ,$$

where

$$(a)_n = a(a+1)\cdots(a+n-1)\,.$$

Then

$$A_k\left[\frac{(x)_{n+1}}{x+k}\right]_{x=-k} = \binom{\frac{1}{2}-k}{}$$

and a little manipulation leads to

$$2^{2n}A_k = \binom{2k}{k}\binom{2n-2k}{n-k}\,.$$

Thus we have proved the identity

$$2^{2n}\frac{(x+\frac{1}{2})_n}{(x)_{n+1}} = \sum_{k=0}^{n}\frac{1}{x+k}\binom{2k}{k}\binom{2n-2k}{n-k}\,. \tag{3.5}$$

It is easily verified that

$$\frac{2^{2n}(x+\frac{1}{2})_n}{(x)_{n+1}} = \frac{(2x)_{2n}}{(x)_n(x+1)_n}\,,$$

so that (3.5) becomes

$$\sum_{k=0}^{n}\frac{1}{x+k}\binom{2k}{k}\binom{2n-2k}{n-k} = \frac{(2x)_{2n}}{(x)_n(x+1)_n}\,.$$

In particular, for $x = 1$, we have

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k} = \frac{(2n+1)!}{n!\,(n+1)!} = \binom{2n+1}{n}\,.$$

It follows that

$$\binom{2n+2}{n+1} = 2\binom{2n+1}{n} = \sum_{k=0}^{n}\frac{1}{k+1}\binom{2k}{k}\binom{2n-2k}{n-k} + \sum_{k=0}^{n}\frac{1}{n-k+1}\binom{2k}{k}\binom{2n-2k}{n-k}$$

$$= \sum_{k=0}^{n}\frac{n+2}{(k+1)(n-k+1)}\binom{2k}{k}\binom{2n-2k}{n-k} = (n+2)\sum_{k=0}^{n}c_k\,c_{n-k}\,.$$

This evidently proves (3.4).

4. We now define

$$F(x,y) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} f(n,k)\, x^n y^k . \tag{4.1}$$

Then, by (2.1),

$$\begin{aligned} F(x,y) &= xy + x \sum_{n=1}^{\infty} \sum_{k=1}^{n+1} f(n+1,k)\, x^n y^k \\ &= xy + x \sum_{n=1}^{\infty} \sum_{k=1}^{n+1} \sum_{j=1}^{k} f(n,j)\, x^n y^k \\ &= xy + x \sum_{n=1}^{\infty} \sum_{j=1}^{n} f(n,j)\, x^n y^j \sum_{k=j}^{n+1} y^{k-j} \\ &= xy + \frac{x}{1-y} \sum_{n=1}^{\infty} \sum_{j=1}^{n} f(n,j)\, x^n y^j (1 - y^{n-j+2}) \\ &= xy + \frac{x}{1-y} F(x,y) - \frac{xy^2}{1-y} \sum_{n=1}^{\infty} \sum_{j=1}^{n} f(n,j)\, x^n y^n . \end{aligned}$$

Since, by (2.13) and (2.14),

$$\sum_{j=1}^{n} f(n,j) = f(n+1,\ n+1) = c(n) ,$$

we get

$$(1 - x - y)F(x,y) = xy(1-y) - xy^2 \sum_{n=1}^{\infty} c(n)(xy)^n$$

and therefore

$$(1 - x - y)F(x,y) = xy - xy^2 C(xy) . \tag{4.2}$$

Now put

$$F_n(x) = \sum_{k=1}^{\infty} f(n+k,\ k)x^{n+k} \qquad (n \geq 0) . \tag{4.3}$$

Then

$$F(x,y) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} f(n+k,\ k)x^{n+k} y^k = \sum_{n=0}^{\infty} y^{-n} \sum_{k=1}^{\infty} f(n+k,k)(xy)^{n+k} ,$$

so that

$$F(x,y) = \sum_{n=o}^{\infty} y^{-n} F_n(xy) . \tag{4.4}$$

It follows from (4.2) and (4.4), with x replaced by xy^{-1}, that

$$(1 - xy^{-1} - y) \sum_{n=0}^{\infty} y^{-n} F_n(x) = x - xy\, C(x) ,$$

or preferably

$$(1 - y + xy^2) \sum_{n=0}^{\infty} y^n F_n(x) = x\, C(x) - xy . \tag{4.5}$$

Comparison of coefficients yields

$$\begin{aligned} F_0(x) &= x\, C(x) - F_1(x) + F_0(x) = x , \\ F_n(x) &- F_{n-1}(x) + xF_{n-2}(x) = 0 \qquad (n \geq 2) . \end{aligned} \tag{4.6}$$

Thus

$$F_1(x) = F_0(x) - x = x\, C(x) - x = x^2 C^2(x) ,$$

by (3.3). Next

$$F_2(x) = F_1(x) - xF_0(x) = x^2 C(x)\big(C(x) - 1\big) = x^3 C^3(x) .$$

Generally we have

$$F_n(x) = x^{n+1} C^{n+1}(x) , \tag{4.7}$$

as is easily proved by induction, using (4.6).

Clearly (4.7) implies

$$F_n(x) = xC(x) F_{n-1}(x) \qquad (n \geq 1) . \tag{4.8}$$

Since

$$xC(x) = \sum_{n=1}^{\infty} f(n,n) x^n ,$$

it follows from (4.8) that

$$f(n + k, k) = \sum_{i=1}^{k} f(j,j) f(n + k - j, k - j + 1) \qquad (n \geq 1) . \tag{4.9}$$

When $n = 1$, (4.9) reduces to (3.4).

If we define

$$f_n(x) = \sum_{k=1}^{n} f(n,k) x^k , \tag{4.10}$$

we have

$$(1 - x)f_n(x) = \sum_{k=1}^{n} \{f(n,k) - f(n, k - 1)\} x^k - f(n,n)x^{n+1}$$

$$= \sum_{k=1}^{n-1} f(n - 1,k)x^k - f(n,n)x^{n+1} ,$$

so that

(4.11) $$(1 - x)f_n(x) = f_{n-1}(x) - f(n,n)x^{n+1} .$$

By iteration of (4.11),

$$(1 - x)^2 f_n(x) = f_{n-2}(x) - f(n - 1, n - 1)x^n - f(n,n)x^n(1 - x),$$

$$(1 - x)^3 f_n(x) = f_{n-3}(x) - f(n - 2, n - 2)x^{n-1} - f(n - 1, n - 1)x^n(1 - x) - f(n,n)x^{n+1}(1 - x)^2$$

and generally

(4.12) $$(1 - x)^k f_n(x) = f_{n-k}(x) - \sum_{j=o}^{k-1} f(n - j, n - j)x^{n-j+1}(1 - x)^{k-j-1} .$$

In particular, for $k = n - 1$, Eq. (4.12) becomes

$$(1 - x)^{n-1} f_n(x) = x - \sum_{j=o}^{n-2} f(n - j, n - j)x^{n-j+1}(1 - x)^{n-j-2} ,$$

so that

(4.13) $$(1 - x)^n f_n(x) = x - \sum_{j=1}^{n} f(j,j)x^{j+1}(1 - x)^{j-1} .$$

For example, for $n = 3$,

$$(1 - x)^3(x + 2x^2 + 2x^3) = x - x^2 - x^3 - x^4 + 4x^5 - 2x^6$$

$$= x - x^2 - x^3(1 - x) - 2x^4(1 - x)^2 .$$

If we put

(4.14) $$G_n(x) = x - \sum_{j=1}^{n} f(j,j)x^{j+1}(1 - x)^{j-1} ,$$

(4.13) becomes

(4.15) $$(1 - x)^n f_n(x) = G_n(x) .$$

We shall show that (4.15) characterizes the $f(n,k)$ in the following sense. Let

$$(a_1,\ a_2,\ a_3,\ \cdots)$$

be a sequence of numbers such that

$$A_n(x) = x - \sum_{j=1}^{n} a_j x^{j+1}(1 - x)^{j-1} \tag{4.16}$$

is divisible by $(1 - x)^n$ for $n = 1, 2, 3, \cdots$. Define $g_n(x)$ by means of

$$A_n(x) = (1 - x)^n g_n(x) , \tag{4.17}$$

so that $g_n(x)$ is a polynomial of degree n. We shall show that

$$g_n(x) = f_n(x) \qquad (n = 1, 2, 3, \cdots) . \tag{4.18}$$

For $n = 1$, it follows from (4.16) and (4.17) that $a_1 = 1$, $g_1(x) = x$. For $n = 2$ we have

$$x - x^2 - a_2 x^3(1 - x) = (1 - x)^2 g_2(x) ,$$

so that $a_2 = 1$, $g_2(x) = 1 + x$. For $n = 3$ we have

$$x - x^2 - x^3(1 - x) - a_3 x^4(1 - x)^2 = (1 - x)^3 g_3(x) ,$$

which gives

$$a_3 = 2, \qquad g_3(x) = 1 + 2x + 2x^2 .$$

It follows from (4.16) that

$$A_{n-1}(x) - A_n(x) = a_n x^{n+1} (1 - x)^{n-1} ,$$

so that, by (4.17),

$$(1 - x)^{n-1} g_{n-1}(x) - (1 - x)^n g_n(x) = a_n x^{n+1}(1 - x)^{n-1} .$$

Thus

$$g_{n-1}(x) - (1 - x)g_n(x) = a_n x^{n+1} . \tag{4.19}$$

On the other hand, by (4.11),

$$f_{n-1}(x) - (1 - x)f_n(x) = f(n,n)x^{n+1} . \tag{4.20}$$

Now assume that (4.18) holds for $n = 1, 2, \cdots, m - 1$. Then by (4.19) and (4.20) we have

$$(1 - x)\left[f_m(x) - g_m(x)\right] = \left[a_m - f(m,m)\right]x^{m+1} .$$

This implies

$$a_m = f(m,m), \qquad f_m(x) = g_m(x) ,$$

thus completing the proof of (4.18).

In the next place, we have, by (4.13),

$$\begin{aligned} f_n(x) &= x(1 - x)^{-n} - \sum_{j=1}^{n} f(j,j)x^{j+1}(1 - x)^{-n+j-1} \\ &= x\sum_{k=o}^{\infty}\binom{n + k - 1}{k}x^k - \sum_{j=1}^{n} f(j,j)x^{j+1}\sum_{k=o}^{\infty}\binom{n - j + k}{k}x^k \\ &= x\sum_{k=o}^{\infty}\binom{n + k - 1}{k}x^k - \sum_{k=o}^{\infty} x^{k+1}\sum_{\substack{j=1\\ j\le k}}^{n}\binom{n - 2j + k}{k - j}f(j,j) . \end{aligned}$$

Equating coefficients, we get

$$f(n, k + 1) = \binom{n + k - 1}{k} - \sum_{j=1}^{k}\binom{n - 2j + k}{k - j}f(j,j) \qquad (0 \le k \le n) . \tag{4.21}$$

and

$$\sum_{j=1}^{n}\binom{n - 2j + k}{k - j}f(j,j) = \binom{n + k - 1}{k} \qquad (k \ge n) . \tag{4.22}$$

In particular, for $k = n$, (4.22) becomes

$$\sum_{j=1}^{n}\binom{2n - 2j}{n - j}f(j,j) = \binom{2n - 1}{n - 1} = \frac{1}{2}\binom{2n}{n} \qquad (n \ge 1) . \tag{4.23}$$

Then

$$\frac{1}{2}\sum_{n=1}^{\infty}\binom{2n}{n}x^n = \sum_{j=1}^{\infty} f(j,j)x^j\sum_{n=0}^{\infty}\binom{2n}{n}x^n .$$

Since

$$(1 - 4x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n ,$$

it follows that

$$\sum_{j=1}^{\infty} f(j,j)\, x^j = \tfrac{1}{2}(1 - 4x)^{\frac{1}{2}} \left\{ (1 - 4x)^{-\frac{1}{2}} - 1 \right\}$$

$$= \tfrac{1}{2} \left\{ 1 - (1 - 4x)^{\frac{1}{2}} \right\} .$$

Thus we have obtained another proof of (3.2).

We can also prove (3.2) — or the equivalent formula (3.4) — directly from the definition of $f(n,k)$. Consider the sequence $(a_1, a_2, \cdots, a_{n+1})$ with

$$1 = a_1 \le a_2 \le \cdots \le a_{n+1} = n; \qquad a_i \le i \qquad (i = 1, 2, \cdots, n) .$$

Let k be the largest integer such that $a_k = k$. Clearly $k \le n$ and $a_{k+1} = k$. Now break the given sequence into two subsequences

$$(a_1, \cdots, a_k), \qquad (a_{k+1}, \cdots, a_{n+1}) .$$

Put

$$b_j = a_{k+j} - k + 1 \qquad (j = 1, 2, \cdots, n - k + 1) .$$

Then

$$\begin{cases} 1 = b_1 \le b_2 \le \cdots \le b_{n-k+1} = n - k + 1 ; \\ b_j \le j \qquad (j = 1, 2, \cdots, n - k + 1) . \end{cases}$$

It follows that

$$f(n + 1, n) = \sum_{k=1}^{n} f(k,k) f(n - k + 1, n - k + 1) . \tag{4.24}$$

Since

$$c(n) = f(n + 1, n) = f(n + 1, n + 1) ,$$

(4.24) reduces to

$$c(n) = \sum_{k=1}^{n} c(k) c(n - k) .$$

More generally consider the sequence $(a_1, a_2, \cdots, a_{n+1})$ with

$$\begin{cases} 1 = a_1 \le a_2 \le \cdots \le a_{n+1} = m \le n ; \\ a_i \le i \qquad (i = 1, 2, \cdots, n) . \end{cases}$$

As above let k be the greatest integer such that $a_k = k$. Then $k \le m \le n$ and $a_{k+1} = k$. Break the given sequence into two pieces:

$$(a_1, \cdots, a_k), \qquad (a_{k+1}, \cdots, a_{n+1})$$

and put

$$b_j = a_{k+j} - k + 1 \qquad (j = 1, 2, \cdots, n - k + 1).$$

Then

$$\begin{cases} 1 = b_1 \le b_2 \le \cdots \le b_{n-k+1} = m - k + 1; \\ b_j \le j \qquad (j = 1, 2, \cdots, n - k + 1). \end{cases}$$

It follows that

$$f(n + 1, m) = \sum_{k=1}^{m} f(k, k) f(n - k + 1, m - k + 1). \tag{4.25}$$

Replacing $n + 1$ by $r + m$, (4.25) becomes

$$f(r + m, m) = \sum_{k=1}^{m} f(k, k) f(r + m - k, m - k + 1). \tag{4.26}$$

This furnishes another proof of (4.9).

5. We now consider the following generalization of $f(n,k)$:

$$f(n, k, q) = \sum q^{a_1 + a_2 + \cdots + a_n}, \tag{5.1}$$

where the summation is over all $a_1, a_2, \cdots, a_n$ that satisfy

$$1 = a_1 \le a_2 \le \cdots \le a_n = k \tag{5.2}$$

and

$$a_i \le i \qquad (i = 1, 2, \cdots, n). \tag{5.3}$$

It is clear that $f(n, k, 1) = f(n, k)$.

It follows at once from the definition that

$$f(n, k, q) = q^k \sum_{j=1}^{k} f(n - 1, j, q), \tag{5.4}$$

where $f(n - 1, n, q) = 0$. From (5.4) we get

(5.5) $$f(n,k,q) = qf(n, k-1, q) + q^k f(n-1, k, q) .$$

Making use of either (5.4) or (5.5) we can compute the following table.

n \ k	1	2	3	4	5
1	q				
2	q^2	q^3			
3	q^3	$q^4 + q^5$	$q^5 + q^6$		
4	q^4	$q^5 + q^6 + q^7$	$q^6 + q^7 + 2q^8 + q^9$	$q^7 + q^8 + 2q^9 + q^{10}$	
5	q^5	$q^6 + q^7 + q^8 + q^9$	$q^7 + q^8 + 2q^9 + 2q^{10} + 2q^{11} + q^{12}$	$q^8 + q^9 + 2q^{10} + 3q^{11} + 3q^{12} + 3q^{13} + q^{14}$	$q^9 + q^{10} + 2q^{11} + 3q^{12} + 3q^{13} + 3q^{14} + q^{15}$

It is evident that

(5.6) $$f(n,1,q) = q^n \qquad (n = 1, 2, 3, \cdots)$$

and

(5.7) $$f(n,n,q) = qf(n, n-1, q) .$$

Also, by (5.4), the sum

(5.8) $$f(n,q) = \sum \lambda^{a_1+a_2+\cdots+a_n} ,$$

where now

(5.9) $$1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n; \qquad a_i \leq i ,$$

satisfies

(5.10) $$f(n,q) = q^{-n} f(n+1, n, q) = q^{-n-1} f(n+1, n+1, q) .$$

It is also easily verified that

(5.11) $$f(n,2,q) = q^{n+1} \frac{1 - q^{n-1}}{1 - q} .$$

6. It is convenient to consider the polynomial $a_n(x,q)$ defined by means of

(6.1) $$(a)_{n+1}\, a_n(x,q) = 1 - x \sum_{k=0}^{n} a(k,k,q)(qx)^k (x)_k ,$$

where

(6.2) $$(x)_n = (1 - x)(1 - qx) \cdots (1 - q^{n-1}x), \qquad (x)_0 = 1 .$$

The definition (6.1) may be compared with (4.13).

For $n = 0$, (6.1) becomes

$$(1 - x)a_0(x,q) = 1 - xa(0,0,q) ,$$

so that

$$a(0,0,q) = 1, \qquad a_0(x,q) = 1 .$$

For $n = 1$ we have

$$(1 - x)(1 - qx)a_1(x,q) = 1 - xa(0,0,q) - qx^2(1 - x)a(1,1,q) ,$$

which implies

$$a(1,1,q) = q, \qquad a_1(x,q) = 1 + qx .$$

For $n = 2$ we have

$$(1 - x)(1 - qx)(1 - q^2x)a_2(x,q) = 1 - x - q^2x^2(1 - x) - q^2x^3(1 - x)(1 - qx)a(2,2,q).$$

This yields

$$a(2,2,q) = 1 + q, \qquad a_2(x,q) = 1 + (q + q^2)x + (q^3 + q^4)x^2 .$$

We now show that (6.1) uniquely determines $a(n,n,q)$ and $a_n(x,q)$. Clearly (6.1) implies

$$(x)_{n+1}\, a_n(x,q) = (x)_n\, a_{n-1}(x,q) - xa(n,n,q)(qx)^n(x)_n ,$$

or

$$(1 - q^nx)a_n(x,q) = a_{n-1}(x,q) - q^n\, a(n,n,q)x^{n+1} . \tag{6.3}$$

For $x = q^{-n}$ this becomes

$$a(n,n,q) = q^n\, a_{n-1}(q^{-n}, q) . \tag{6.4}$$

Substitution from (6.4) in (6.2) shows that $a_n(x,q)$ is uniquely determined and is of degree n in x. We may accordingly put

$$a_n(x,q) = \sum_{k=0}^{n} a(n,k,q)x^n , \tag{6.5}$$

thus incidentally justifying the notation $a(n,n,q)$ in (6.1).

It now follows from (6.2) that

$$\begin{cases} a(n,k,q) = q^na(n,\ k - 1,\ q) + a(n - 1,\ k,\ q) \\ a(n,n,q) = q^na(n,\ n - 1,\ q) \end{cases} . \tag{6.6}$$

Iteration of the first of (6.6) leads to

$$a(n,k,q) = \sum_{j=0}^{k} q^{jn}a(n - 1,\ k - j,\ q) . \tag{6.7}$$

If we put

$$a(n,k,q) = q^{\frac{1}{2}k(k+1)} b(n,k,q) , \tag{6.8}$$

(6.6) becomes

$$\begin{cases} b(n,k,q) = b(n-1, k, q) + q^{n-k} b(n, k-1, q) \\ b(n,n,q) = b(n, n-1, q) . \end{cases} \tag{6.9}$$

The following table is easily computed.

n \ k	0	1	2	3	4
0	1				
1	1	1			
2	1	$1 + q$	$1 + q$		
3	1	$1 + q + q^2$	$1 + 2q + q^2 + q^3$	$1 + 2q + q^2 + q^3$	
4	1	$1 + q + q^2 + q^3$	$1 + 2q + 2q^2 + 2q^3 + q^4 + q^5$	$1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$	$1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6$

Comparison with the table for $f(n,k,q)$ suggests that

$$f(n+1, k+1, q) = q^{(k+1)(n+1)-\frac{1}{2}k(k+1)} b(n,k,q^{-1}) . \tag{6.10}$$

To prove (6.10), substitute from (6.9) in (5.5). We get

$$q^{(k+1)(n+1)-\frac{1}{2}k(k+1)} b(n,k,q^{-1}) = q^{k(n+1)-\frac{1}{2}k(k-1)+1} b(n, k-1, q^{-1}) + q^{(k+1)n-\frac{1}{2}k(k+1)+k+1} b(n-1, k, q^{-1}) ,$$

that is

$$b(n,k,q^{-1}) = q^{k-n} b(n, k-1, q^{-1}) + b(n-1, k, q^{-1}) .$$

Replacing q by q^{-1}, this becomes

$$b(n,k,q) = q^{n-k} b(n, k-1, q) + b(n-1, k, q) ,$$

which is identical with (6.9). This evidently proves (6.10).

7. It follows from (6.9) that $b(n,k,q)$ is a polynomial of degree k in q^n. Put

$$b(n,k,q) = \frac{1}{(q)_k} \sum_{s=0}^{k} c(k,s) q^{ns} , \tag{7.1}$$

where $c(k,s) = c(k,s,q)$ is independent of n. Using (6.9) we get

$$c(k,s) = -q^{s-k}\frac{1-q^k}{1-q^s}c(k-1,\ s-1)\ .$$

This yields

(7.2) $$c(k,s) = (-1)^s q^{s(s-k)}\begin{bmatrix}k\\s\end{bmatrix}c(k-s)\ ,$$

where $c(k-s) = c(k-s,0)$ and

$$\begin{bmatrix}k\\s\end{bmatrix} = \frac{(1-q^k)(1-q^{k-1})\cdots(1-q^{k-s+1})}{(1-q)(1-q^2)\cdots(1-q^s)}\ .$$

Thus (7.1) becomes

(7.3) $$b(n,k,q) = \frac{1}{(q)_k}\sum_{s=o}^{k}(-1)^s q^{s(s-k)}\begin{bmatrix}k\\s\end{bmatrix}c(k-s)q^{ns}\ .$$

By (6.9) and (7.3) we have

$$b(j,\ k+1,\ q) - b(j-1,\ k+1,\ q) = \frac{q^{-k-1}}{(q)_k}\sum_{s=o}^{k}(-1)^s q^{s(s-k)}\begin{bmatrix}k\\s\end{bmatrix}c(k-s)q^{j(s+1)} \qquad (j > k).$$

Summing over j gives

$$\begin{aligned} b(n,\ k+1,\ q) &= \frac{q^{-k-1}}{(q)_k}\sum_{s=o}^{k}(-1)^s q^{s(s-k)}\begin{bmatrix}k\\s\end{bmatrix}c(k-s)\ \frac{q^{(k+1)(s+1)} - q^{(n+1)(s+1)}}{1-q^{s+1}} \\ &= \frac{1}{(q)_{k+1}}\sum_{s=o}^{k+1}(-1)^s q^{s(s-k-1)}\begin{bmatrix}k+1\\s\end{bmatrix}c(k-s+1)q^{ns} \\ &= \frac{1}{(q)_{k+1}}\sum_{s=1}^{k+1}(-1)^s q^{s(s-1)}\begin{bmatrix}k+1\\s\end{bmatrix}c(k-s+1)\ . \end{aligned}$$

Comparison with (7.3) yields

$$c_{k+1} = -\sum_{s=1}^{k+1}(-1)^s q^{s(s-1)}\begin{bmatrix}k+1\\s\end{bmatrix}c(k-s+1)\ ,$$

that is,

(7.4) $$\sum_{s=o}^{k}(-1)^s q^{s(s-1)}\begin{bmatrix}k\\s\end{bmatrix}c(k-s) = 0 \qquad (k > 0)\ .$$

The $c(k)$ are uniquely determined by (7.4) and $c(0) = 1$. In particular

$$c(1) = 1, \quad c(2) = 1 + q - q^2, \quad c(3) = 1 + 2q + 2q^2 + q^3 - q^6 .$$

To get a generating function for $c(k)$, put

$$f(t) = \sum_{k=0}^{\infty} (-1)^k q^{k(k-1)} t^k/(q)_k . \tag{7.5}$$

Then, by (7.4),

$$\sum_{n=o}^{\infty} c(n)t^n/(q)_n = \frac{1}{f(t)} . \tag{7.6}$$

In the next place put

$$\frac{f(xt)}{f(t)} = \sum_{n=0}^{\infty} \psi_n(x) \frac{t^n}{(q)_n} , \tag{7.7}$$

where

$$\psi_n(n) = \sum_{s=o}^{n} (-1)^s q^{s(s-1)} \begin{bmatrix} n \\ s \end{bmatrix} c(n - s)x^s . \tag{7.8}$$

In particular, by (7.3),

$$b(n + k - 1, k, q) = \frac{1}{(q)_k} \psi_k(q^n) , \qquad b(k,k,q) = \frac{1}{(q)_k} \psi_k(q) \tag{7.9}$$

Therefore, by (7.7),

$$\sum_{k=0}^{\infty} b(n + k - 1, k, q) t^k = \frac{f(q^n t)}{f(t)} \qquad (n > 0) . \tag{7.10}$$

While $f(t)$ resembles the familiar series

$$\sum_{n=o}^{\infty} t^n/q^n = \prod_{n=o}^{\infty} (1 - q^n t)^{-1}$$

and

$$\sum_{n=o}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} t^n/(q)_n = \prod_{n=o}^{\infty} (1 - q^n t) ,$$

its properties seem to be less simple.

It follows from (7.5) that

$$f(t) - f(qt) + tf(q^2t) = 0 . \tag{7.11}$$

Repeated application of (7.11) leads to

$$q^{\frac{1}{2}n(n-1)}t^n f(q^{n+1}t) = -A_n(t)f(t) + B_n(t)f(qt) \qquad (n > 0) , \tag{7.12}$$

where $A_{n+1}(t) = B_n(qt)$ and

$$B_n(t) = \sum_{2s \le n} (-1)^s q^{s(s-1)} \begin{bmatrix} n - s \\ s \end{bmatrix} t^s . \tag{7.13}$$

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