# Catalan's Books

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**Abstract**

Briefly here is presented a possible alternative interpretation of the Catalan's triangle in terms of mathematical alphabets and books.

## 1 The books: Words and Letters (Definition)

Let be named a "*Catalan's Book*" any book built from all the possible words of fixed length written on a given alphabet, such that the letters are read in lexicographic non-decreasing order from left to right, and there are no less words than repeated letters in the book.

## 2 "No less words than repeated letters": (The underlying combinatorics)

**The general properties of such kind of books without making a comparison of the mentioned counters.**

It might be observed by first time with numbers[1] written in decimal[2] that there exists "a function" that correlates the fixed length in each word with the quantity of words composing the book. Then this function perhaps might be generalized to any alphabet. Such function is no other than:

$$\binom{alphabetSize - 1 + wordSize}{alphabetSize - 1} \tag{1}$$

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[1]These would be the words.

[2]This is, the alphabet consisting in the base 10 digits: $\{0, 1, 2, ..., 9\}$

It must be emphasized here that this is a property shared by all the books made from all the fixed length words in a given alphabet such that their letters are read from left to right in lexicographic non-decreasing order. Relative to these class of books, the just defined *Catalan's Books* are a subclass.

The 2nd key property of interest is: Given a book belonging the superclass for the *Catalan's Books*, which function counts the number of times that any letter of its alphabet is repeated?. At purpose!, are these letters uniformly repeated in the book?.

Well, fortunately the answer for the second question is affirmative: For such book each letter of its alphabet is repeated the same number of times, so the first question (as expected), have a simple and general answer:

$$\binom{alphabetSize - 1 + wordSize}{alphabetSize} \tag{2}$$

**Satisfying the restriction defining the Catalan's books.**

Based upon the present exposition, mathematically talking a *Catalan's Book* is such that:

$$WordsByBook - UniformFrequency \geq 0$$

This is:

$$\binom{alphabetSize - 1 + wordSize}{alphabetSize - 1} - \binom{alphabetSize - 1 + wordSize}{alphabetSize} \geq 0$$

By making the replacements: $s$ instead of $alphabetSize$ and $w$ instead of $wordSize$, we have:

$$\binom{(s-1)+w}{(s-1)} - \binom{(s-1)+w}{s} \geq 0$$

$$\frac{1}{(s-1)!}\prod_{j=1}^{(s-1)}[(s-1)+w-j+1] - \frac{1}{s!}\prod_{j=1}^{s}[(s-1)+w-j+1] \geq 0$$

$$\frac{1}{(s-1)!}\prod_{j=1}^{(s-1)}[(s-1)+w-j+1]\left[1 - \frac{1}{s}(\cancel{s}-1+w-\cancel{s}+1)\right] \geq 0$$

$$\mathbf{w \leq s}$$

**Consequence by definition**

$\mathbf{w} \leq \mathbf{s}$, This is: In any *Catalan's Book* the length of the words does not exceed the length of the alphabet.

# 3 Connection with the Catalan's triangle

The preceding consequence is consistent with the construction of a triangle based upon the studied non-negative differences between the number of words composing a *Catalan's Book* and the common frequency of the letters for the corresponding alphabet.

Such triangle indeed results identical to the Catalan's triangle if it is accepted to talk artificially of "words with length zero", which leads to the following fallacy and contradiction: Conceptually talking, A book with zero sized words wouldn't be empty since $\binom{s-1+w}{s-1}$ is 1 when $w = 0$, so there should be a word, however regardless which alphabet is associated, none letter is repeated at least once, since $\binom{s-1+w}{s}$ is zero when $w$ is zero. Also notice this: From the previous calculation it is straightforward to verify that is $\mathbf{w} = \mathbf{s}$ then the difference of the counters is zero, therefore such differences are not included there in the *Catalan's triangle* since it doesn't end in zeros at the right side. Finally the word length must be ranged from zero until the alphabet size minus 1, in order to generate the *Catalan's triangle*. Curiously, there is no problem if the size or length for the alphabet is 1 since[3] by either convention or definition: $\binom{w}{0} = 1$ and $\binom{w}{1} = w$.

[3]This behavior is already implemented for example in number theory oriented software like *PARI-GP*.

# Appendix. PARI-GP script:

Illustrating[4] the alternative interpretation of the *Catalan's Triangle*

```
/* R. J. Cano, Jul 22 2014

  Catalan's books: An enumerative combinatorics exercise.

  Some relevant counters associated to books made with
  all the possible words written on a fixed number of
  letters in a given alphabet, such that these letters
  are read in lexicographic non-decreasing order from
  left to right.
 */

/* Saves a book as a plain text file */

book=(wordSize,alphabetSize,filename)->\
 forvec(y=vector(wordSize,k,[0,alphabetSize-1]),write(filename,y),1);

/* Number of times that any letter is repeated in a book,
   if all the words are of the same size */

howManyTimesAnyLetterInTheBook=(wordSize,alphabetSize)->\
binomial(alphabetSize-1+wordSize,alphabetSize);

/* Number of words in a book,
   if all the words are of the same size */

howManyWordsInTheBook=(wordSize,alphabetSize)->\
binomial(alphabetSize-1+wordSize,alphabetSize-1);

/*
   Connection with the Catalan's triangle
   (Sequence A009766 at OEIS)

   The numbers at the rows in the Catalan's triangle count
   the differences when there are NO less words in a book
```

[4] A copy of this material is available at: `http://oeis.org/w/images/0/00/CatalanBooks.txt`.; If you don't have PARI-GP installed, you can try `http://www.compileonline.com/execute_pari_online.php`.;

```
   than the number of times that any letter is repeated inside
   such book. The size of the alphabet would be row+1 relative
   to the triangle, and the size of the word would be column-1
   assuming that the columns are labeled from left to right
   starting with 1. If such labels start with zero, the size
   of the words is simply column.

   Note: For the zeroth row of the triangle and/or the first
   column in the representation generated below, this
   interpretation doesn't apply, however the mathematical
   behavior here described is hold due the underlying
   conventions and definitions for binomial coefficients.
 */

/* Return -1 if the answer is "No".
   Otherwise returns the difference */

NoLessWordsThanTimesRepeatedAnyLetter=(wordSize,alphabetSize)->\
 {my(Answer=howManyWordsInTheBook(wordSize,alphabetSize)-\
 howManyTimesAnyLetterInTheBook(wordSize,alphabetSize));\
 if(Answer>=0,Answer,-1)}

/* The Catalan's triangle is printed as follows:
   (By default only the first 10 rows are shown) */

CatalanTriangle=(rows=10)-> for(alphabetSize=1,rows,\
 for(wordSize=0,alphabetSize-1,\
 print1(" "\
 NoLessWordsThanTimesRepeatedAnyLetter(wordSize,alphabetSize)));\
 print1("\n")\
);

{print("\nFirst 10 rows of the triangle:\n");CatalanTriangle();}

/* Acknowledgments to:
 *  The On-Line Encyclopedia Of Integer Sequences (OEIS)
 *  and The PARI-GP development team, for their valuable
 *  time, moral support and the formidable work they do.
*/
```