Dihedral-symmetry middle-levels problem via a Catalan system of numeration

Italo J. Dejter
University of Puerto Rico
Rio Piedras, PR 00936-8377
italo.dejter@gmail.com

Abstract

Let $0 < k \in \mathbb{Z}$. A system of numeration according to which the nonnegative integers are presented as restricted growth strings has the k-th Catalan number $C_k = \binom{2k+1}{k}/(2k+1)$ presented as the restricted growth string 10^k. This system yields a linear ordering of the vertex set of a quotient of the graph M_k induced by the k- and $(k+1)$-levels of the n-cube graph under a natural dihedral group action. Mütze proved the existence of Hamilton cycles of M_k but we still ask for the existence of Hamilton cycles of M_k with dihedral symmetry so the cited system encodes them.

1 Introduction

Let $0 < k \in \mathbb{Z}$ and let $n = 2k+1$. Assume that the dihedral group D_{2n} of order $2n$ acts on a graph G, and in that case let H be a D_{2n}-invariant subgraph of G ([11], pg. 20). If this subgraph H is a Hamilton cycle of G, then we say that H is a *Hamilton cycle with dihedral symmetry of* G.

We present each nonnegative integer as a *restricted growth string* (or *RGS*) α, so called by Arndt [2] and Ruskey [17]; see Section 2 below. These RGSs are related to the Catalan numbers $C_k = \frac{1}{n}\binom{n}{k}$ (A000108 in [21]) in that C_k becomes the RGS $\alpha = 10^k$, as observed by Arndt in [2] pg. 325. A system of numeration (A239903 in [21]) composed by the RGSs α is introduced in Section 2 and applied to a dihedral-symmetry version of Hável's [12] or Buck-Wiedemann's [4] conjecture on the existence of Hamilton cycles in the middle-levels graphs M_k ($0 < k \in \mathbb{Z}$). The graphs M_k are treated from Section 4 on, after associating the RGSs α to corresponding n-tuples $F(\alpha)$ that represent the vertices of M_k by means of an "ascending castling" operation (Theorem 2 in Section 3). In turn, the n-tuples $F(\alpha)$ become, in Section 9, the vertices of a quotient graph R_k of M_k under an action of D_{2n} (Section 7). The general version of Hável's conjecture was proved by Mütze [16].

We point out that Lemma 3 in [18] means that the existence of Hamilton paths in R_k between certain distinguished vertices insures the existence of Hamilton cycles with dihedral symmetry in M_k, which is the dihedral-symmetry version of Hável's conjecture. We also point out that these paths can be presented in terms of the Kierstead-Trotter lexical matchings of M_k [13], which highlights the relevance of the said system, particularly in pursuing the existence of such paths, their enumeration (i.e., if existence is established, perhaps there are less than $2^{2^{\Omega(n)}}$ paths, see next paragraph) and their sorting (according to both the α ordering and the lexical matchings, see Section 11).

These tasks remain in spite of the establishment of Hável's conjecture by Mütze, who reviewed its history in [16], to which bibliography we refer, and proved that there are $2^{2^{\Omega(n)}}$ such cycles.

In our treatment below lexical 1-factorizations in the M_k s are defined via the Kierstead-Trotter lexical matchings [13] from Section 8 on. Ammerlaan and Vassilev [1] showed that any Hamilton cycle in an M_k has the same number of edges along every coordinate direction of the n-cube H_n. This is the case of the Shields-Savage construction of a Hamilton cycle η_k with dihedral symmetry of M_k for it uses the cyclic nature mod n of M_k in taking a Hamilton path ξ_k as required and with: **(i)** its vertices in 1-1 correspondence with the first C_k strings α and **(ii)** dihedral unfolding of M_k formed by concatenating ξ_k with its translates mod n in order to compose η_k, (Section 11). In fact, such ξ_k must visit just once each vertex class of M_k under an adequate equivalence relation (Section 5). The Shields-Savage construction (further employed in [19, 20] with no visible association to any system of numeration, naturally related, or not, to the Catalan numbers) motivates our inquiry about existence, enumeration and sorting of such Hamilton cycles η_k. Each vertex of ξ_k corresponds uniquely to a *k-word* α in the system of numeration A239903, concept made precise in Section 2.

Thus, we ask: Are there as many Hamilton cycles with dihedral symmetry of the M_k s as Mütze showed by his constructions in the general case in [16]? Moreover, can Mütze's construction be particularized to Hamilton cycles with dihedral symmetry of the M_k s?

2 Catalan system of numeration

According to the proposed system of numeration, the increasing sequence of non-negative integers is represented via the RGS s, starting with:

$$0, 1, 10, 11, 12, 100, 101, 110, 111, 112, 120, 121, 122, 123, 1000, \dots \tag{1}$$

where the subsequence 1, 10, 100, 1000, ..., $10\cdots0$, ..., (the last term shown with t zeros represented symbolically by 10^t, $t \geq 0$) corresponds to the numbers $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, ..., $C_{t+1} = \frac{1}{2t+3}\binom{2t+3}{t+1}$, etc. Because of this, we refer to the system as the *Catalan* system of numeration, defined formally as follows. The RGS s in (1), representing the consecutive integers from 0 to 14, and so on, are written as expressions $a_{k-1}a_{k-2}\cdots a_2a_1$ by prefixing enough zeros if necessary to the entries in (1) and so on, for any adequate k. Such expressions $a_{k-1}a_{k-2}\cdots a_2a_1$, called *k-words* from now on, are defined for $1 < k \in \mathbb{Z}$ by means of the following two rules:

1. The leftmost position in a k-word $a_{k-1}a_{k-2}\cdots a_2a_1$, namely position $k-1$, contains a digit $a_{k-1} \in \{0, 1\}$.

2. Given a position $i > 1$ with $i < k$ in a k-word $a_{k-1}a_{k-2}\cdots a_2a_1$, then to the immediate right of the corresponding digit a_i, the digit a_{i-1} (meaning at position $i-1$) satisfies $0 \le a_{i-1} \le a_i + 1$.

The reader may compare this with the essentially similar *Catalan* RGSs in Section 15.2 of [2], and with the mixed radix systems [5], including the factorial number, or factoradic, system [9], [10], [14] pg. 192, [15] pg. 12, or A007623 in [21]. We refer as well to Stanley's interpretation of Catalan numbers [22], Exercise (u), as mentioned in A239903 of [21].

Every k-word $a_{k-1}a_{k-2}\cdots a_2a_1$ yields a $(k+1)$-word $a_ka_{k-1}a_{k-2}\cdots a_2a_1 = 0a_{k-1}a_{k-2}\cdots a_2a_1$. A k-word $\neq 0$ stripped of the null digits to the left of the leftmost position containing digit 1 is called a *Catalan string*. We also consider (improper) *Catalan string* 0 corresponding to the null k-words, $0 < k \in \mathbb{Z}$.

The k-words are ordered as follows: Given any two k-words, say

$$\alpha = a_{k-1}\cdots a_2a_1 \text{ and } \beta = b_{k-1}\cdots b_2b_1,$$

where $\alpha \neq \beta$, we say that α precedes β, written $\alpha < \beta$, whenever either

(i) $a_{k-1} < b_{k-1}$ or

(ii) $a_j = b_j$, for $k-1 \le j \le i+1$, and $a_i < b_i$, for some $1 \le i < k-1$.

The order defined on the k-words this way is said to be their *stair-wise* order.

Observation 1. *The sequence of nonzero Catalan strings has the terms corresponding to the Catalan numbers* $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $\ldots$, $C_{t+1} = \frac{1}{2t+3}\binom{2t+3}{t+1}$, $\ldots$, *written respectively as* $1, 10, 100, 1000, \ldots, 10^t, \ldots$, *where* $0 \le t \in \mathbb{Z}$. *Moreover, there exists exactly* C_{k+1} k*-words* $< 10^k$, *for each* $k > 0$.

To determine the Catalan string corresponding to a given decimal integer x_0, or vice versa, one employs *Catalan's triangle* Δ, namely a triangular arrangement composed by positive integers starting with the following rows Δ_j, for $j = 0, \ldots, 8$:

1								
1	1							
1	2	2						
1	3	5	5					
1	4	9	14	14				
1	5	14	28	42	42			
1	6	20	48	90	132	132		
1	7	27	75	165	297	429	429	
1	8	35	110	275	572	1001	1430	1430

and with a linear reading of the successive rows conforming the sequence A009766 in [21]. Specifically, the numbers τ_i^j in row Δ_j of Δ ($0 \le j \in \mathbb{Z}$) satisfy the following items:

1. $\tau_0^j = 1$, for every $j \ge 0$;
2. $\tau_1^j = j$ and $\tau_j^j = \tau_{j-1}^j$, for every $j \ge 1$;
3. $\tau_i^j = \tau_i^{j-1} + \tau_{i-1}^j$, for every $j \ge 2$ and $i = 1, \ldots, j-2$;
4. $\sum_{i=0}^j \tau_i^j = \tau_j^{j+1} = \tau_{j+1}^{j+1} = C_j$, for every $j \ge 1$.

A unified formula for the numbers τ_j^k, $(j = 0, 1 \ldots, k)$, is given by:

$$\tau_j^k = \frac{(k+j)!(k-j+1)}{j!(k+1)!}$$

Now, the determination of the Catalan string corresponding to a decimal integer x_0 proceeds as follows. Let $y_0 = \tau_k^{k+1}$ be the largest member of the second diagonal of Δ with $y_0 \leq x_0$. Let $x_1 = x_0 - y_0$. If $x_1 > 0$, then let $Y_1 = \{\tau_{k-1}^j\}_{j=k}^{k+b_1}$ be the largest set of successive terms in the $(k-1)$-column of Δ with $y_1 = \sum(Y_1) \leq x_1$. Either $Y_1 = \emptyset$, in which case we take $b_1 = -1$, or not, in which case $b_1 = |Y_1| - 1$. Let $x_2 = x_1 - y_1$. If $x_2 > 0$, then let $Y_2 = \{\tau_{k-2}^j\}_{j=k}^{k+b_2}$ be the largest set of successive terms in the $(k-2)$-column of Δ with $y_2 = \sum(Y_2) \leq x_2$. Either $Y_2 = \emptyset$, in which case we take $b_2 = -1$, or not, in which case $b_2 = |Y_3| - 1$. Proceeding this way, we arrive at a null x_k. Then the Catalan string corresponding to x_0 is $a_{k-1}a_{k-2}\cdots a_1$, where $a_{k-1} = 1$, $a_{k-2} = 1 + b_1$, ..., and $a_1 = 1 + b_k$.

For example, if $x_0 = 38$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 38 - 14 = 24$, $y_1 = \tau_2^3 + \tau_2^4 = 5 + 9 = 14$, $x_2 = x_1 - y_1 = 24 - 14 = 10$, $y_2 = \tau_1^2 + \tau_1^3 + \tau_1^4 = 2 + 3 + 4 = 9$, $x_3 = x_2 - y_2 = 10 - 9 = 1$, $y_3 = \tau_0^1 = 1$ and $x_4 = x_3 - y_3 = 1 - 1 = 0$, so that $b_1 = 1$, $b_2 = 2$, and $b_3 = 0$, taking to $a_4 = 1$, $a_3 = 1 + b_1 = 2$, $a_2 = 1 + b_2 = 3$ and $a_1 = 1 + b_3 = 1$, determining the 5-word of 38 to be $a_4a_3a_2a_1 = 1231$. If $x_0 = 20$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 20 - 14 = 6$, $y_1 = \tau_2^3 = 5$, $x_2 = x_1 - y_1 = 1$, $y_2 = 0$ is an empty sum (since its possible summand $\tau_1^2 > 1 = x_2$), $x_3 = x_2 - y_2 = 1$, $y_3 = \tau_0^1 = 1$ and $x_4 = x_3 - x_3 = 1 - 1 = 0$, determining the 5-word of 20 to be $a_4a_3a_2a_1 = 1101$. Moreover, if $x_0 = 19$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 19 - 14 = 5$, $y_1 = \tau_2^3 = 5$, $x_2 = x_1 - y_1 = 5 - 5 = 0$, determining the 5-word $a_4a_3a_2a_1 = 1100$.

Given a Catalan string or k-word $a_{k-1}\cdots a_1$, the considerations above can easily be played backwards to recover the corresponding decimal integer x_0.

3 Descending castling

Theorem 2. *To each k-word $\alpha = a_{k-1}\cdots a_1$ corresponds an n-tuple $F(\alpha)$ whose entries are the integers $0, 1, \ldots, k$ (once each) together with k asterisks $*$ and such that:* ***(a)*** *the leftmost entry of $F(\alpha)$ is k;* ***(b)*** *each integer entry to the immediate right of an integer entry b is an integer less than b; in particular, $F(0^{k-1}) = k(k-1)(k-2)\cdots 210 * \cdots *$;* ***(c)*** *first, to each k-word $\alpha \neq 00\cdots 0$ corresponds a k-word $\beta = b_{k-1}\cdots b_1$ smaller than α in the stair-wise order of k-words and differing from α in exactly one entry, that is $b_i \neq a_i$ for just one i with $k-1 \geq i \geq 1$, being β maximal under these restrictions; second, $F(\alpha) = f_0f_1\cdots f_{2k}$ is obtained from $F(\beta) = g_0g_1\cdots g_{2k}$ by the* descending castling *operation consisting of:*

1. *setting $g_0 = f_0$, $g_1 = f_1$, ..., $g_{i-1} = f_{i-1}$; $g_{2k} = f_{2k}$, $g_{2k-1} = f_{2k-1}$, ..., $g_{2k-i+1} = f_{2k-i+1}$;*

2. *denoting the i-substrings formed by the entries in item 1 by W^i on the left and Z^i on the right, and writing $F(\beta) \setminus (W^i \cup Z^i) = X|Y$, ($X$ concatenated with Y), with the substring Y starting at entry $\ell - 1$, where $\ell > 0$ is the leftmost entry of X;*

3. *by noticing that $F(\beta) = W^i|X|Y|Z^i$, finally setting $F(\alpha) = W^i|Y|X|Z^i$.*

In order to proceed with the proof of Theorem 2, let us note that there is a rooted tree $\mathcal{T}_k$ whose nodes are the k-words. In fact, $\mathcal{T}_k$ is associated to the stair-wise order of the k-words. The root of $\mathcal{T}_k$ is 0^{k-1} and any other k-word $\alpha \neq 0^{k-1}$ is a child in $\mathcal{T}_k$ of a k-word β that differs from α in just one entry a_i $(0 < i < k)$. By representing $\mathcal{T}_k$ with the children of every k-word α enclosed between parentheses after α and separating siblings with commas, we can write, say for $k = 4$: $\mathcal{T}_4 = 000(001, 010(011(012)), 100(101, 110(111(121)), 120(121(122(123)))))$.

Proof. In sub-item 2 of item (c), we have: **(i)** W^i is a proper subsequence of the maximal starting descending integer sequence $\bar{W}$ such that $\bar{W} \setminus W^i$ starts with ℓ, the head of X; **(ii)** Z^i is solely composed by asterisks. The k-tuple α contains precise instructions for the recursive operation of descending castling to work as prescribed in the statement, away from those W^i and Z^i equally long in $F(\beta)$ and $F(\alpha)$: By respecting the rule that in every sequence of applications of sub-items 1-3 along descending paths in $\mathcal{T}_k$, unit augmentation of a_i for larger values of i, $(0 < i < k)$, must occur first, and only then in descending order of i, thus thinning the inner sub-string $X|Y$ after each application, the resulting process effectively preserves the stated properties, for by changing the order of the appearing sub-strings X and Y, that have their first elements being respectively ℓ and $\ell - 1$ in successive decreasing order, the descending nature of this operation is effectively guaranteed. □

Let us illustrate the descending castling operation. For $k = 2, 3, 4$, the k-words α are presented in their stair-wise order in Table I, both on the left and the right columns, and their corresponding images under F, or F-images, on the penultimate column. In each of the three listings, each with C_k rows ($C_2 = 2$, $C_3 = 5$ and $C_4 = 14$), the columns are filled, from the second row on, as follows: **(i)** α, appearing in downward stair-wise order; **(ii)** β as in item (c) of Theorem 2, which allows to determine the subindex i in item (iv) below, where α and β differ; **(iii)** $F(\beta)$, from the penultimate column in the previous row; **(iv)** the only subindex i $(k - 1 \geq i \geq 1)$ for which the i-th entries in α and β differ, with β as large as possible such that $\beta < \alpha$; **(v)** the decomposition $W^i|Y|X|Z^i$ of $F(\beta)$; **(vi)** the result of the descending castling operation; **(vii)** the corresponding re-concatenation in the column $F(\alpha)$; and **(viii)** again α, as $F^{-1}(F(\alpha))$.

Clearly, for each k-word α different from the zero k-word there exists a well determined β obtained as indicated in item (c) of Theorem 2 and observable by comparing the first two columns in Table I, which in addition highlights the sub-index i of the fourth column. On the other hand, not all the n-tuples satisfying items (a)-(b) of Theorem 2 appear in the finite recursion implied by item (c). For example, $\beta' = 431 * 2 * *0*$ is not image of a permissible α for $k = 4$, (see Table I, for $k = 4$). In fact, trying to apply sub-items 1.-3. for $i = 1$ to β' would result in $\alpha' = 42 * *031 * *$ which does not respect item (b). For $i = 2$, we would have β' decomposing as $43|1 * 2 * *|0*$, and since the first sub-word of the middle part starts with 1, we would like to take 0 as the second sub-word, but 0 is outside the middle part. Again this is not a case in our treatment, as indicated in item (ii) of the proof of Theorem 2.

One may perform sub-items 1-3 departing from a k-word β independently of a specific α by taking a subindex i in the text of item (D) and after replacing numbers and asterisks respectively by 0 s and 1 s, obtaining a k-work α' provided already by the textual item (D). Say we depart for $k = 3$ from $F(\beta) = F(01)$ =310**2* and take $i = 1$. Sub-items 1-3 leads here to $F(\alpha') = 30**21*$, which yields 0011001, a translation mod 7 of 31*20**= $F(11)$, obtained already in a different way in Table I.

TABLE I

α	β	$F(\beta)$	i	$W^i \mid X \mid Y \mid Z^i$	$W^i \mid Y \mid X \mid Z^i$	$F(\alpha)$	α
0	–	–	–	–	–	$210**$	0
1	0	$210**$	1	$2\mid 1\mid 0*\mid *$	$2\mid 0*\mid 1\mid *$	$20*1*$	1
00	–	–	–	–	–	$3210***$	00
01	00	$3210***$	1	$3\mid 2\mid 10**\mid *$	$3\mid 10**\mid 2\mid *$	$310**2*$	01
10	00	$3210***$	2	$32\mid 1\mid 0*\mid **$	$32\mid 0*\mid 1\mid **$	$320*1**$	10
11	10	$320*1**$	1	$3\mid 20*\mid 1*\mid *$	$3\mid 1*\mid 20*\mid *$	$31*20**$	11
12	11	$31*20**$	1	$3\mid 1*2\mid 0*\mid *$	$3\mid 0*\mid 1*2\mid *$	$30*1*2*$	12
000	–	–	–	–	–	$43210****$	000
001	000	$43210****$	1	$4\|3\|210***\|*$	$4\|210***\|3\|*$	$4210***3*$	001
010	000	$43210****$	2	$43\|2\|10**\|**$	$43\|10**\|2\|**$	$4310**2**$	010
011	010	$4310**2**$	1	$4\|310**\|2*\|*$	$4\|2*\|310**\|*$	$42*310***$	011
012	011	$42*310***$	1	$4\|2*3\|10**\|*$	$4\|10**\|2*3\|*$	$410**2*3*$	012
100	000	$43210****$	3	$432\|1\|0*\|***$	$432\|0*\|1\|***$	$4320*1***$	100
101	100	$4320*1***$	1	$4\|3\|20*1**\|*$	$4\|20*1**\|3\|*$	$420*1**3*$	101
110	100	$4320*1***$	2	$43\|20*\|1*\|**$	$43\|1*\|20*\|**$	$431*20***$	110
111	110	$431*20***$	1	$4\|31*\|20**\|*$	$4\|20**\|31*\|*$	$420**31**$	111
112	111	$420**31**$	1	$4\|20**3\|1*\|*$	$4\|1*\|20**3\|*$	$41*20**3*$	112
120	110	$431*20***$	2	$43\|1*2\|0*\|**$	$43\|0*\|1*2\|**$	$430*1*2**$	120
121	120	$430*1*2**$	1	$4\|30*1*\|2*\|*$	$4\|2*\|30*1*\|*$	$42*30*1**$	121
122	121	$42*30*1**$	1	$4\|2*30*\|1*\|*$	$4\|1*\|2*30*\|*$	$41*2*30**$	122
123	122	$41*2*30**$	1	$4\|1*2*3\|0*\|*$	$4\|0*\|1*2*3\|*$	$40*1*2*3*$	123

To each $F(\alpha)$ corresponds a binary n-tuple $\phi(\alpha)$ of weight k obtained by replacing each integer entry in $\{0,1,\dots,k\}$ by 0 and each asterisk $*$ by 1. By attaching the entries of $F(\alpha)$ as subscripts to the corresponding entries of $\phi(\alpha)$, a subscripted binary n-tuple $\bar{\phi}(\alpha)$ is obtained. Let $\aleph(\phi(\alpha))$ be given by the *reverse complement* of $\phi(\alpha)$, that is

$$\text{if } \phi(\alpha) = a_0a_1\cdots a_{2k}, \text{ then } \aleph(\phi(\alpha)) = \bar{a}_{2k}\cdots\bar{a}_1\bar{a}_0, \tag{2}$$

where $\bar{0} = 1$ and $\bar{1} = 0$. A subscripted version $\bar{\aleph}$ of $\aleph$ is immediately obtained for $\bar{\phi}(\alpha)$. Observe that every image under $\aleph$ is an n-tuple of weight $k+1$ and has the 1 s with integer subscripts and the 0 s with asterisk subscripts. The integer subscripts reappear from Section 8 to Section 10 as *lexical colors* [13]. Table II illustrates the notions just presented, for $k = 2, 3$.

An interpretation of this related to the middle-levels graphs is started at the end of Section 5 in relation to the subscripts $0, 1, \dots, k$ and concluded as Corollary 6 at the end of Section 9.

TABLE II

α	$\phi(\alpha)$	$\bar{\phi}(\alpha)$	$\bar{\aleph}(\phi(\alpha)) = \aleph(\bar{\phi}(\alpha))$	$\aleph(\phi(\alpha))$
0	00011	$0_2 0_1 0_0 1_* 1_*$	$0_* 0_* 1_0 1_1 1_2$	00111
1	00101	$0_2 0_0 1_* 0_1 1_*$	$0_* 1_1 0_* 1_0 1_2$	01011
00	0000111	$0_3 0_2 0_1 0_0 1_* 1_* 1_*$	$0_* 0_* 0_* 1_0 1_1 1_2 1_3$	0001111
01	0001101	$0_3 0_1 0_0 1_* 1_* 0_2 1_*$	$0_* 1_2 0_* 0_* 1_0 1_1 1_3$	0100111
10	0001011	$0_3 0_2 0_0 1_* 0_1 1_* 1_*$	$0_* 0_* 1_1 0_* 1_0 1_2 1_3$	0010111
11	0010011	$0_3 0_1 1_* 0_2 0_0 1_* 1_*$	$0_* 0_* 1_0 1_2 0_* 1_1 1_3$	0011011
12	0010101	$0_3 0_0 1_* 0_1 1_* 0_2 1_*$	$0_* 1_2 0_* 1_1 0_* 1_0 1_3$	0101011

4 The middle-levels graphs

Let $1 < n \in \mathbb{Z}$. The *n-cube graph* H_n is the Hasse diagram of the Boolean lattice on the coordinate set $[n] = \{0, \ldots, n-1\}$. Vertices of H_n are cited in three different ways:

(a) as the subsets $A = \{a_0, a_1, \ldots, a_{r-1}\} = a_0 a_1 \cdots a_{r-1}$ of $[n]$ they stand for, where $0 < r \leq n$;

(b) as the characteristic n-vectors $B_A = (b_0, b_1, \ldots, b_{n-1})$ over the field $\mathbb{F}_2$ that the subsets A represent, meaning they are given by $b_i = 1$ if and only if $i \in A$ ($i \in [n]$), and represented for short by $B_A = b_0 b_1 \cdots b_{n-1}$;

(c) as the polynomials $\epsilon_A(x) = b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$ associated with the vectors B_A.

A subset A as in (a) is said to be the *support* of the vector B_A in (b). For each $j \in [n]$, the *j-level* L_j is the vertex subset in H_n formed by those $A \subseteq [n]$ with $|A| = j$. For $1 \leq k \in \mathbb{Z}$, the *middle-levels graph* M_k is defined as the subgraph of H_n induced by $L_k \cup L_{k+1}$.

5 Quotient graphs under cyclic action

By viewing the vertices of M_k as polynomials, as in item (c) of Section 4, an equivalence relation π is seen to exists in the vertex set $V(M_k)$ of M_k by means of the logical expression:

$$\epsilon_A(x) \pi \epsilon_{A'}(x) \Longleftrightarrow \exists\, i \in \mathbb{Z} \text{ such that } \epsilon_{A'}(x) \equiv x^i \epsilon_A(x) \pmod{1 + x^n}.$$

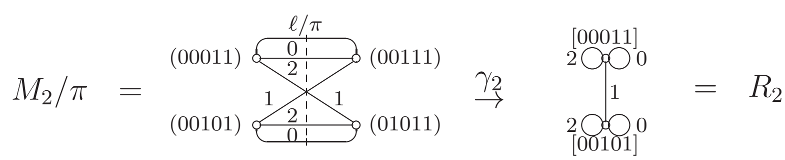

Figure 1: Reflection symmetry of M_2/π about a line ℓ/π and resulting graph map γ_2

This implies that there exists a quotient graph M_k/π under a regular (i.e. transitive and free) action

$$\Upsilon' : \mathbb{Z}_n \times M_k \to M_k \tag{3}$$

given by $\Upsilon'(i,v)=v(x)x^i \bmod 1+x^n$ in polynomial terms as in item (c) of Section 4, where $v\in V(M_k)$ and $i\in\mathbb{Z}_n$. Here, M_k/π is the graph whose vertices are the equivalence classes of vertices of M_k under π and whose edges are the equivalence classes that π induces on the edge set $E(M_k)$ of M_k.

For example, M_2/π is the domain of the graph map γ_2 suggested in Figure 1 and associated with reflective symmetry of both M_2/π and M_2 about respective dashed vertical lines ℓ/π and ℓ acting as symmetry axes (generalized below in Section 6) with $V(M_2/\pi)=L_2/\pi\cup L_3/\pi$, where $L_2/\pi=\{(00011),(00101)\}$ and $L_3/\pi=\{(00111),(01011)\}$, each π-class expressed between parentheses about one of its representatives written as in (b) of Section 4 and composed by the following elements (of L_2/π and L_3/π):

$$L_2/\pi=\{(00011)=\{00011,10001,11000,01100,00110\},\quad (00101)=\{00101,10010,01001,10100,01010\}\};$$
$$L_3/\pi=\{(00111)=\{00111,10011,11001,11100,01110\},\quad (01011)=\{01011,10101,11010,01101,10110\}\}.$$

Let $k>1$ be a fixed integer. We associate with each binary weight-k n-tuple $F(A)$ the class $(F(A))$ generated by $F(A)$ in L_k/π and the class $(\aleph(F(A)))$ generated by $\aleph(F(A))$ in L_{k+1}/π. These associations start the interpretation of Section 3 above to be concluded as Corollary 6 in Section 9.

6 Reflective-symmetry graph involutions

A *graph involution* of a graph G is a graph map $\aleph:G\to G$ such that $\aleph^2$ is the identity graph map. Clearly, a graph involution is a graph isomorphism. In a way similar to the example for $k=2$ in Section 5, but now for any $k\geq 2$, and in order to make explicit a graph involution $\aleph$ of M_k/π given by reflective symmetry, as suggested in Figure 1 about the symmetry axis ℓ/π, we want now to list and represent vertically the vertex parts L_k/π and L_{k+1}/π of M_k/π (resp. L_k and L_{k+1} of M_k) by placing their vertices into pairs, each pair displayed on an horizontal line with its two composing vertices equidistant from a dashed vertical line ℓ/π (resp. ℓ), like the ℓ/π in the representation of M_2/π in Figure 1. To specify the desired vertex setting, the definition of $\aleph$ in display (2) can be immediately extended to a bijection $\aleph:L_k\to L_{k+1}$, where the image of an element of L_k through $\aleph$ is again said to be its *reverse complement*. Let us take each resulting horizontal vertex pair to be of the form $(B_A,\aleph(B_A))$ and ordered from left to right. Let $\rho_i:L_i\to L_i/\pi$ be the canonical projection given by assigning $b_0b_1\cdots b_{n-1}\in L_i$ to $(b_0b_1\cdots b_{n-1})\in L_i/\pi$, for $i=k,k+1$, and let $\aleph_\pi:L_k/\pi\to L_{k+1}/\pi$ be given by $\aleph_\pi((b_0b_1\cdots b_{n-1}))=(\bar{b}_{n-1}\cdots\bar{b}_1\bar{b}_0)$. Then $\aleph_\pi$ is a bijection and we have the commutative identity $\rho_{k+1}\aleph=\aleph_\pi\rho_k$. In what follows, we say that a non-horizontal edge of M_k/π is a *skew edge*.

Theorem 3. *To each skew edge $e=(B_A)(B_{A'})$ of M_k/π corresponds a different skew edge $\aleph_\pi((B_A))\aleph_\pi^{-1}((B_{A'}))$ obtained from e by reflection on the line ℓ/π, which is equidistant from $(B_A)=\aleph_\pi^{-1}((B_{A'}))\in L_k/\pi$ and $\aleph_\pi((B_A))=(B_{A'})\in L_{k+1}/\pi$. Thus:* ***(i)*** *the skew edges of M_k/π appear in pairs, having their endpoints in each pair forming two pairs of horizontal vertices equidistant from ℓ/π;* ***(ii)*** *the horizontal edges of M_k/π have multiplicity ≤ 2.*

Proof. With the representation adopted for the vertices of M_k, the skew edges $B_AB_{A'}$ and $\aleph^{-1}(B_{A'})\aleph(B_A)$ of M_k are now seen to be reflection of each other about ℓ and also having their pairs $(B_A, \aleph(B_A))$ and $(\aleph^{-1}(B_{A'}), B_{A'})$ of endpoints lying each on a corresponding horizontal line. Now, ρ_k and ρ_{k+1} extend together to a covering graph map $\rho : M_k \to M_k/\pi$, since the edges accompany the projections correspondingly, as for example for $k = 2$, where:

$$\begin{array}{r}\aleph((00011))=\quad\aleph(\{00011,10001,11000,01100,00110\})=\{00111,01110,11100,11001,10011\}=(00111),\\ \aleph^{-1}((01011))=\aleph^{-1}(\{01011,10110,10110,11010,10101\})=\{00101,10010,01001,10100,01010\}=(00101),\end{array}$$

showing the order of the elements in the images or preimages under $\aleph$ of the classes mod π as displayed in Figure 1, that is: presented backwards (from right to left), cyclically between braces, and continuing on the right once one reaches a leftmost brace. This backwards behavior holds for any $k > 2$, that is:

$$\begin{array}{r}\aleph((b_0\cdots b_{2k}))=\quad\aleph(\{b_0\cdots b_{2k},\, b_{2k}\ldots b_{2k-1},\, ...,\, b_1\cdots b_0\})=\{\bar b_{2k}\cdots\bar b_0,\, \bar b_{2k-1}\cdots\bar b_{2k},\, ...,\, \bar b_1\cdots\bar b_0\}=(\bar b_{2k}\cdots\bar b_0),\\ \aleph^{-1}((\bar b'_{2k}\cdots\bar b'_0))=\aleph^{-1}(\{\bar b'_{2k}\cdots\bar b'_0,\, \bar b'_{2k-1}\cdots\bar b'_{2k},\, ...,\, \bar b'_1\cdots\bar b'_0\})=\{b'_0\cdots b'_{2k},\, b'_{2k}\cdots b'_{2k-1},\, ...,\, b'_1\cdots b'_0\}=(b'_0\cdots b'_{2k}),\end{array}$$

for any vertices $(b_0 \cdots b_{2k}) \in L_k/\pi$ and $(b'_0 \cdots b'_{2k}) \in L_{k+1}/\pi$. This establishes item (i). Now, every edge of M_k from a vertex in $\aleph^{-1}(v)$ to a vertex in $\aleph^{-1}(\rho(v))$, for some $v \in L_k$, projects onto an horizontal edge of M_k, while all other edges of M_k project onto corresponding skew edges of M_k. It is easy to see that an horizontal edge of M_k/π has its endpoint in L_k/π represented by a vertex $\bar b_k \cdots \bar b_1 0 b_1 \cdots b_k \in L_k$ so there are 2^k such vertices in L_k and less than 2^k corresponding vertices of L_k/π; for example, $(0^{k+1}1^k)$ and $(0(01)^k)$, mentioned in Shields-Savage Lemma 3 [18], are endpoints of two horizontal edges, each in M_k/π. To prove that this implies item (ii), we have to see that there cannot be more than two representatives $\bar b_k \cdots \bar b_1 b_0 b_1 \cdots b_k$ and $\bar c_k \cdots \bar c_1 c_0 c_1 \cdots c_k$ of a vertex $v \in L_k/\pi$, with $b_0 = c_0 = 0$. Let $v = (d_0 \cdots b_0 d_{i+1} \cdots d_{j-1} c_0 \cdots d_{2k})$, with $b_0 = d_i$, $c_0 = d_j$ and $0 < j - i \leq k$. A substring $\sigma = d_{i+1} \cdots d_{j-1}$ with $0 < j-i \leq k$ is said to be $(j-i)$-*feasible* if v fulfills (ii) with multiplicity at least 2. Any $(j - i)$-feasible substring σ forces in L_k/π only endpoints ω incident to two different (parallel) horizontal edges in M_k/π because periodic continuation mod n of $d_0 \cdots d_{2k}$ both to the right of $d_j = c_0$ with minimal cyclic substring $\bar d_{j-1} \cdots \bar d_{i+1} 1 d_{i+1} \cdots d_{j-1} 0 = P_r$ and to the left of $d_i = b_0$ with minimal cyclic substring $0 d_{i+1} \cdots d_{j-1} 1 \bar d_{j-1} \cdots \bar d_{i+1} = P_\ell$ yields a two-way infinite string that winds up onto $(d_0 \cdots d_{2k})$, corresponding to ω. For example, the initial feasible substrings σ, with 'o' indicating the positions $b_0 = 0$ and $c_0 = 0$, are

(∅,(oo1)), (0,(o0o11)), (1,(o1o)), (0^2,(o00o111)), (01,(o01o011)), ((1^2, o11o0)), (0^3,(o000o1111)),
(010,(o010o101101)), (01^2,(o011o)), (101,(o101o)), (1^3,(o111o00)),

where n has successive values $n = 3, 5, 3, 7, 7, 5, 9, 11, 5, 5, 7$. (However, the substrings $0^2 1$ and 10^2 are non-feasible). If σ is a feasible substring and $\bar\sigma$ is its reverse complement via $\aleph$, then the possible symmetrical substrings about oσo $= 0\sigma 0$ in (the notation of) a vertex ω of L_k/π are in order of ascending length:

$$\begin{array}{c}0\sigma0,\\ \bar\sigma0\sigma0\bar\sigma,\\ 1\bar\sigma0\sigma0\bar\sigma1,\\ \sigma1\bar\sigma0\sigma0\bar\sigma1\sigma,\\ 0\sigma1\bar\sigma0\sigma0\bar\sigma1\sigma0,\\ \bar\sigma0\sigma1\bar\sigma0\sigma0\bar\sigma1\sigma0\bar\sigma,\\ 1\bar\sigma0\sigma1\bar\sigma0\sigma0\bar\sigma1\sigma0\bar\sigma1,\end{array}$$

etc., where we use again '0' instead of 'o' for the entries immediately preceding and following the shown central copy of σ. Due to this, the finite lateral periods of the resulting P_r and P_ℓ do not allow a third horizontal edge (at v in M_k) up to returning to b_0 or c_0 since no entry $e_0 = 0$

of $(d_0 \cdots d_{2k})$ other than b_0 or c_0 happens such that $(d_0 \cdots d_{2k})$ has a third representative $\bar{e}_k \cdots \bar{e}_1 0 e_1 \cdots e_k$ (besides $\bar{b}_k \cdots \bar{b}_1 0 b_1 \cdots b_k$ and $\bar{c}_k \cdots \bar{c}_1 0 c_1 \cdots c_k$). Thus, those two horizontal edges are produced solely from the feasible substrings $d_{i+1} \cdots d_{j-1}$ characterized above. □

To illustrate the ideas in the proof of Theorem 3, let $1 < h < n$ in $\mathbb{Z}$ be such that $\gcd(h, n) = 1$ and let the *h-interspersion* $\lambda_h : L_k/\pi \to L_k/\pi$ be given by $\lambda((a_0 a_1 \cdots a_n)) \to (a_0 a_h a_{2h} \cdots a_{n-2h} a_{n-h})$. For each h with $1 < h \leq k$, there is at least one h-feasible substring σ and a resulting associated vertex $\omega \in L_k/\pi$ as in the proof of the theorem. For example, applying h-interspersion repeatedly by starting at $\omega = (0^{k+1} 1^k) \in L_k/\pi$ produces a number of such vertices $\omega \in L_k/\pi$. If we assume $h = 2h'$ with $h' \in \mathbb{Z}$, then an h-feasible substring σ has the form $\sigma = \bar{a}_1 \cdots \bar{a}_{h'} a_{h'} \cdots a_1$, so there are at least $2^{h'} = 2^{\frac{h}{2}}$ such h-feasible substrings.

7 Quotient graphs under dihedral action

Given a graph G with an involution $\aleph : G \to G$, a *graph folding* of G is a graph H whose vertices are the pairs $\{v, \aleph(v)\}$, where $v \in V(G)$, and whose edges are the pairs $\{e, \aleph(e)\}$, where $e \in E(G)$. Here, e has end-vertices v and $\aleph(v)$ if and only if e yields a loop in H; otherwise, e yields a link in H ([3], pg. 3). Let us denote each pair $((B_A), \aleph_\pi((B_A))$ of M_k/π, horizontally represented in Section 6, via the notation $[B_A]$, where $|A| = k$.

A graph folding R_k of M_k/π is obtained whose vertices are the pairs $[B_A]$ and having
(1) an edge $[B_A][B_{A'}]$ per skew-edge pair $\{(B_A)\aleph_\pi((B_{A'})), (B_{A'})\aleph_\pi((B_A))\}$;
(2) a loop at $[B_A]$ per horizontal edge $(B_A)\aleph_\pi((B_A))$. By Theorem 3, there may be up to two loops at each vertex of R_k.

Theorem 4. *R_k is a quotient graph of M_k under an action $\Upsilon : D_{2n} \times M_k \to M_k$.*

Proof. To define Υ, recall that D_{2n} is the semidirect product $\mathbb{Z}_n \rtimes_\varrho \mathbb{Z}_2$ via the group homomorphism $\varrho : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_n)$ given by taking $\varrho(1)$ as the automorphism assigning $i \in \mathbb{Z}_n$ to $(n-i) \in \mathbb{Z}_n$, and $\varrho((0)$ as the identity. If $* : D_{2n} \times D_{2n} \to D_{2n}$ indicates multiplication and $i_1, i_2 \in \mathbb{Z}_n$, then $(i_1, 0)*(i_2, j) = (i_1+i_2, j)$, but $(i_1, 1)*(i_2, j) = (i_1-i_2, 1+j)$, for $j \in \mathbb{Z}_2$. Now, set $\Upsilon((i,j), v) = \Upsilon'(i, \aleph^j(v))$, for $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_2$, where Υ' was defined in display (3) of Section 5 above. It is easy to see that Υ is a well-defined action of D_{2n} on M_k. For example by writing $(i,j) \cdot v = \Upsilon((i,j), v)$ and $v = a_0 \cdots a_{2k}$, we have $(i, 0) \cdot v = a_{n-i+1} \cdots a_{2k} a_0 \cdots a_{n-i} = v'$ and $(0,1) \cdot v' = \bar{a}_{i-1} \cdots \bar{a}_0 \bar{a}_{2k} \cdots \bar{a}_i = (n-i, 1) \cdot v = ((0,1) * (i,0)) \cdot v$, leading to one instance of the compatibility condition $((i,j) * (i',j')) \cdot v = (i,j) \cdot ((i',j') \cdot v)$ that a group action must satisfy, (together with the identity condition) to fulfill its definition. □

For example, the vertices in the 20-cycle to the right in Figure 3 of Section 11 below can be rewritten as follows in their shown disposition, where $a = 00011$ and $b = 11101$:

$\Upsilon((3,0),a)$ $\Upsilon((3,1),a)$ $\Upsilon((4,0),a)$ $\Upsilon((4,1),a)$ $\Upsilon((0,0),a)$ $\Upsilon((0,1),a)$ $\Upsilon((1,0),a)$ $\Upsilon((1,1),a)$ $\Upsilon((2,0),a)$ $\Upsilon((2,1),a)$
$\Upsilon((3,0),b)$ $\Upsilon((3,1),b)$ $\Upsilon((4,0),b)$ $\Upsilon((4,1),b)$ $\Upsilon((0,0),b)$ $\Upsilon((0,1),b)$ $\Upsilon((1,0),b)$ $\Upsilon((1,1),b)$ $\Upsilon((2,0),b)$ $\Upsilon((2,1),b)$

representing a Hamilton cycle in M_k, for $k = 2$, invariant under the action of D_{2n}.

Let the graph map $\gamma_k : M_k/\pi \to R_k$ be the corresponding projection, as represented for $k = 2$ in Figure 1. Then the canonical projection $\rho_{D_{2n}} : M_k \to R_k$ is the composition of the canonical projection $\rho_{\mathbb{Z}_k} : M_k \to M_k/\pi$ with γ_k. We remark that Υ is regular just when M_k is taken as a directed graph, because (undirected) edges of M_k leading to loops of R_k via $\rho_{D_{2n}}$ appear that are fixed by Υ. For example R_2, represented as the image of the graph map γ_2 depicted in Figure 1, contains two vertices and just one (vertical) edge between them, where each vertex is incident to two loops. The representation of M_2/π on its left has its edges indicated with colors 0,1,2. Here, the edge $\epsilon = (11000, 11100)$ is fixed via $\aleph$; not so for the each one of the two arcs composing ϵ.

In general, each vertex v of L_k/π will have its incident edges indicated with *lexical colors* $0, 1, \ldots, k$ obtained by the following procedure arising from [13], so that L_k/π admits a $(k+1)$-edge-coloring with color set $[k+1] = \{0, \ldots, k\}$.

8 Kierstead-Trotter lexical procedure

For each $v \in L_k/\pi$ there are $k+1$ n-vectors of the form $b_0b_1 \cdots b_{n-1} = 0b_1 \cdots b_{n-1}$ that represent v with $b_0 = 0$. For each such n-vector, we take a grid $\Gamma = P_{k+1}\Box P_{k+1}$ ([3], pg. 29), where P_{k+1} is the subgraph of the unit-distance graph of the real line $\mathbb{R}$ induced by the set $[k+1] \subset \mathbb{Z} \subset \mathbb{R}$. We trace the diagonal Δ of Γ from vertex $(0,0)$ to vertex (k,k). (For $k=2$, Δ is represented by a dashed line in the instances of Figure 2, analyzed up in Section 9). Recall that an *arc* of Γ is an ordered pair of adjacent vertices of Γ. Based on [13], we build a directed $2k$-path D in Γ from $w = (0,0)$ to $w' = (k,k)$ in $2k$ steps indexed from $i = 0$ to $i = 2k-1$, as follows. Initially, set $i = 0$ and $w = (0,0)$ and let D be the 0-path containing solely w. Repeat the following loop, formed subsequently by items (1)-(3), $2k$ times:

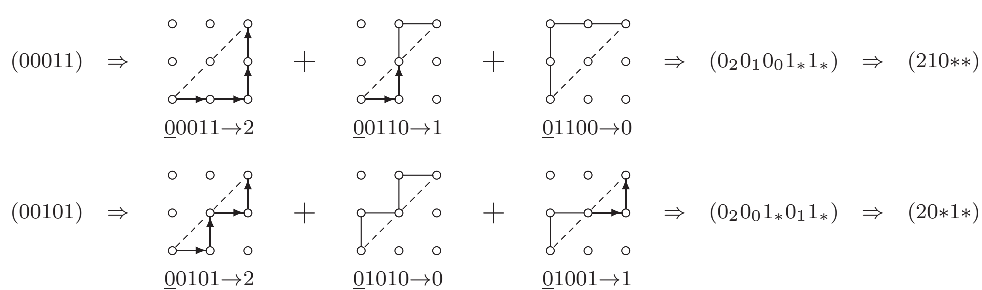

Figure 2: Representing the color assignment for $k = 2$

(1) If $b_i = 0$ (resp., $b_i = 1$), then set $w' := w + (1,0)$ (resp., $w' := w + (0,1)$).

(2) Augment the i-path D by means of the arc (w, w') of Γ into an $(i+1)$-path, again denoted D; in other words, reset $D := D \cup (w, w')$; subsequently, set $i := i+1$ and $w := w'$.

(3) If $w \neq (k,k)$, or equivalently, if $i < 2k$, then go back to item (1).

(4) Set $\bar{v} \in L_{k+1}/\pi$ as a vertex of M_k/π adjacent to $v \in L_k/\pi$ and obtained from a representative $b_0b_1 \cdots\ b_{n-1} = 0b_1 \cdots b_{n-1}$ of v by replacing the entry b_0 of v by $\bar{b}_0 = 1$ in $\bar{v}$, keeping the entries b_i of v with $i > 0$ unchanged in $\bar{v}$.

(5) Set the *color* of the edge $v\bar{v}$ to be the number c of horizontal (alternatively vertical) arcs of D below the diagonal Δ of Γ.

We remark that [13] highlighted the number $k+1-c$, where c varies in $[k+1]$, instead of establishing a well-defined 1-1 correspondence between $[k+1]$ and the set of edges incident to v in L_k/π, as we do here. In fact, if addition and subtraction in $[n]$ are taken modulo n and we write $[y,x) = \{y, y+1, y+2, \ldots, x-1\}$, for $x, y \in [n]$, and $S^c = [n] \setminus S$, for $S = \{i \in [n] : b_i = 1\} \subseteq [n]$, then the cardinalities of the sets $\{y \in S^c \setminus x : |[y,x) \cap S| < |[y,x) \cap S^c|\}$ yield all the numbers $k+1-c$ in 1-1 correspondence with our colors c, where $x \in S^c$ varies.

As in [13], the lexical procedure (or LP for short) just presented yields 1-factorizations of R_k, M_k/π and M_k by means of the edge colors $c = 0, 1, \ldots, k$. This lexical approach is compatible with the graphs M_k/π and R_k, because each edge e of M_k has the same lexical color in $[k+1]$ for both arcs composing e.

9 Ascending castling

In what follows, a color notation $\delta(v)$ is set for each vertex v in L_k/π. In fact, there exists a unique k-word $\alpha = \alpha(v)$ with $[F(\alpha)] = \delta(v)$. We start by representing the lexical color assignment suggested in Figure 2 for $k = 2$, with the LP indicated by arrows "$\Rightarrow$" departing from $v = [00011]$ (top) and $v = [0010]$ (bottom) then going to the right via depiction of working sketches of $V(\Gamma)$ (separated by plus signs "+") for each of the three representatives $b_0 b_1 \cdots b_{n-1} = 0 b_1 \cdots b_{n-1}$ (shown as a subtitle for each sketch, with the entry $b_0 = 0$ underscored) in which to trace the arcs of $D \subset \Gamma$ below Δ, and finally pointing, via an arrow "$\rightarrow$" departing from the representative $b_0 b_1 \cdots b_{n-1} = 0 b_1 \cdots b_{n-1}$ in each sketch subtitle, the number of horizontal arcs of D below Δ. Only arcs of $D \subset \Gamma$ are traced on each sketch of $V(\Gamma)$, with those below Δ indicated as darts in bold trace, and the remaining ones as segments in thin trace.

In each of the two cases in Figure 2, to the right of the three sketches, an arrow "$\Rightarrow$" points to an unparenthesized modification of the notation $(b_0 b_1 \cdots b_{n-1})$ of v obtained by setting as a subindex of each entry 0 the color obtained from its corresponding sketch, and an asterisk "$*$" for each entry 1. Further to the right of this subindexed version of v, another arrow "$\Rightarrow$" points to the string of length n formed solely by the just established subindexes in the order they appear from left to right. This final notation is indicated by $\delta(v)$. For each such $\delta(v)$, there is a unique k-word $\alpha = \alpha(v)$ with $(F(\alpha)) = \delta(v)$, a fact whose proof depends on the inverse operation to descending castling in Section 3, that we may call *ascending castling* and that can be presented as follows:

Given an n-tuple v in L_k/π, let $W^i = k(k-1)\cdots(k-i)$ be the maximal initial $(i+1)$-substring of $\delta(v) = \delta^0(v)$, where $0 \leq i \leq k$. If $i = k$, then let $\alpha(v) = [00\cdots0]$ consist of $k-1$ zeros. Else, write $\delta^0(v) = [W^i|X|Y|Z^i]$, where Z^i is a j_0-substring with $j_0 = i+1$, and X, Y start respectively at two integers ℓ and $\ell+1 \leq k-i$. Let $\delta^1(v) = [W^i|Y|X|Z^]$). If $\delta^1(v) = [k(k-1)\cdots 10*\cdots*]$, then let $\alpha(v)$ differ from $a_{k-1}\cdots\alpha_1 = 00\cdots0$ just by unit incrementation of a_{j_0}. Else, repeat the procedure above starting at $\delta^1(v)$, and so on. In the end, we obtain a finite sequence $\delta^0(v), \delta^1(v), \ldots, \delta^s(v)$ of n-tuples in L_k/π with parameters $j_0 \geq j_1 \geq \ldots \geq j_s$ and k-words $\alpha(v^0), \alpha(v^1), \ldots, \alpha(v^s) = 00\cdots0$ and obtain $\alpha(v) = \alpha^0(v)$

from $\alpha^s(v) = 00\cdots 0 = a_{k-1}\cdots\alpha_1$ by unit incrementation of each a_{j_i}, for $i = 0,\dots,s$, with each such incrementation yielding the corresponding $\alpha(v^i)$. Now, observe that F is a bijection between the set of k-words and the set L_k/π, both of cardinality C_k. This shows that in order to work with $V(R_k)$ is enough to deal with the set of k-words, a fact useful in interpreting Theorem 5 below.

Take for example $\delta^0(v) = [40*1*2*3*] = [4|0*|1*2*3|*]$. Then,

$$
\begin{array}{llcccccc}
j_0{=}0, & \delta^1(v) & = & [4|1{*}2{*}3|0{*}|{*}] & = & [41{*}2{*}30{*}{*}] & = & [4|1{*}|2{*}30{*}|{*}], \\
j_1{=}0, & \delta^2(v) & = & [4|2{*}30{*}|1{*}|{*}] & = & [42{*}30{*}1{*}{*}] & = & [4|2{*}|30{*}1{*}|{*}], \\
j_2{=}0, & \delta^3(v) & = & [4|30{*}1{*}|2{*}|{*}] & = & [430{*}1{*}2{*}{*}] & = & [43|0{*}|1{*}2|{*}{*}], \\
j_3{=}1, & \delta^4(v) & = & [43|1{*}2|0{*}|{*}{*}] & = & [431{*}20{*}{*}{*}] & = & [43|1{*}|20{*}|{*}{*}], \\
j_4{=}1, & \delta^5(v) & = & [43|20{*}|1{*}|{*}{*}] & = & [4320{*}1{*}{*}{*}] & = & [432|0{*}|1|{*}{*}], \\
j_5{=}2, & \delta^6(v) & = & [432|1|0{*}|{*}{*}{*}] & = & [43210{*}{*}{*}{*}], & &
\end{array}
$$

yielding $\alpha(v) = \alpha(v^0) = 123$, with the $\alpha(v^i)$ s corresponding to the formed rows being $\alpha(v^1) = [122]$, $\alpha(v^2) = [121]$, $\alpha(v^3) = [120]$, $\alpha(v^4) = [110]$, $\alpha(v^5) = [100]$ and $\alpha(v^6) = \alpha(v^s) = [000]$.

Thus, the function F that sends the k-words onto their corresponding n-tuples $F(\alpha)$ in Section 3 happens to provide a backbone relating the succeeding applications of the LP to the elements of L_k/π, finally covering all of L_k/π. A pair of skew edges $(B_A)\aleph_\pi((B_{A'}))$ and $(B_{A'})\aleph((B_A))$ in M_k/π is said to be a *skew reflective edge pair*. This provides a color notation for any $v \in L_{k+1}/\pi$ such that in each particular edge class mod π:

(1) each edge receives the same color regardless of the endpoint on which the LP or its modification for $v \in L_{k+1}/\pi$ is applied;

(2) each skew reflective edge pair in M_k/π is assigned a sole color in $[k+1]$.

The modification in item (1) consists in replacing in Figure 2 each v by $\aleph_\pi(v)$ so that on the left we have now instead (00111) (top) and (01011) (bottom) with respective sketch subtitles

$$
\begin{array}{lll}
0011\underline{1}{\to}2, & 1001\underline{1}{\to}1, & 1100\underline{1}{\to}0, \\
0101\underline{1}{\to}2, & 1010\underline{1}{\to}0, & 0110\underline{1}{\to}1,
\end{array}
$$

resulting in similar sketches when the rules of the LP are taken with right-to-left reading-and-processing of the entries on the left side of the subtitles (before the arrows "$\to$"), where now the values of each b_i must be taken complemented.

Since a skew reflective edge pair in M_k determines a unique edge ϵ of R_k (and vice versa), the color received by this pair can be attributed to ϵ, too. Clearly, each vertex of M_k or M_k/π or R_k defines a bijection between its incident edges and the color set $[k+1]$. The edges obtained via $\aleph$ or $\aleph_\pi$ from these edges have the same corresponding colors because of the LP.

Theorem 5. *A* 1*-factorization of* M_k/π *formed by the edge colors* $0,1,\dots,k$ *is obtained via the LP. This* 1*-factorization can be lifted to a covering* 1*-factorization of* M_k *and can further be collapsed onto a folding* 1*-factorization of* R_k *which induces a color notation* $\delta(v)$ *on each of its vertices* v*. Moreover, for each* $v \in V(R_k)$ *and induced notation* $\delta(v)$*, there is a unique* k*-word* $\alpha = \alpha(v)$ *such that* $[F(\alpha)] = \delta(v)$*.*

Proof. As pointed out in item (2) above, each skew reflective edge pair in M_k/π has its edges with the same color in $[k+1]$. Thus, the $[k+1]$-coloring of M_k/π induces a well-defined

$[k+1]$-coloring of R_k. This yields the claimed collapsing to a folding 1-factorization of R_k. The lifting to a covering 1-factorization in M_k is immediate. The arguments above in this section and from Section 3 determine that the collapsing 1-factorization in R_k induces the k-word $\alpha(v)$ claimed in the statement. ☐

Corollary 6. *L_k/π and L_{k+1}/π can be represented respectively by the resulting classes $(F(A))$ and $(\aleph(F(A)))$.*

Proof. The corollary follows from Theorem 5 and its preceding discussion. ☐

10 Lexically colored adjacency tables

From now on and justified by Theorem 5, we use the color notation $\delta(v)$ for the vertices v of R_k with no enclosures in parentheses or brackets as above. Furthermore, we consider a lexically colored adjacency table for R_k in which the vertices $F(\alpha)$ of R_k are expressed via their notation α, and with the order of such α s taken stair-wise, as agreed before Observation 1. According to this, we view R_k as the graph whose vertices are the k-words α and whose adjacency is inherited from that of their δ-notation in R_k via pullback by F^{-1} (namely, via ascending castling). In writing elements of R_k, we avoid now parentheses or brackets.

TABLE III

α	$F(\alpha)$	$F^0(\alpha)$	$F^1(\alpha)$	$F^2(\alpha)$	$F^3(\alpha)$	α^0	α^1	α^2	α^3
0	$210**$	$210**$	$20*1*$	$10**2$	$-$	0	1	0	$-$
1	$20*1*$	$1*20*$	$210**$	$0*1*2$	$-$	1	0	1	$-$
00	$3210***$	$3210***$	$320*1**$	$310**2*$	$210***3$	00	10	01	00
01	$310**2*$	$2*310**$	$2*30*1*$	$3210***$	$1*20**3$	01	12	00	11
10	$320*1**$	$31*20**$	$3210***$	$30*1*2*$	$20*1**3$	11	00	12	10
11	$31*20**$	$320*1**$	$20**31*$	$31*20**$	$10**2*3$	10	11	11	01
12	$30*1*2*$	$1*2*30*$	$2*310**$	$320*1**$	$0*1*2*3$	12	01	10	12

In Table III, examples of such disposition are shown for $k=2$ and 3. Notice that the neighbors of each $F(\alpha)$ in the second column are presented as $F^0(\alpha)$, $F^1(\alpha)$, …, $F^k(\alpha)$ respectively for the colors $0,1,\ldots,k$ of the edges incident to them, where the notation is given via the direct effect of the function $\aleph$. The last four columns yield the k-words α^0, α^1, …, α^k associated via F^{-1} respectively with the listed neighbor vertices $F^0(\alpha)$, $F^1(\alpha)$, …, $F^k(\alpha)$ of $F(\alpha)$ in R_k.

For $k=4$, observe in Table IV the resulting stair-wise adjacency disposition. In general, for any $k>1$, the columns α^i of the stair-wise adjacency table preserve their respective j-th entries (taken from right to left, so $j=k,k-1,\ldots,2,1$) in the following way: $j(\alpha^0)=k,\ j(\alpha^2)=k,\ j(\alpha^3)=k-1,\ \ldots,\ j(\alpha^{k-1})=2,\ j(\alpha^k)=1$, while we do not have such a simple entry invariance rule for column α^1. A further analysis of the relation between α and each of $\alpha^1,\ldots,\alpha^{k-1}$ proceeds in [8], Sections 2-7, in terms of k-words, as in Table IV.

TABLE IV

α	α^0	α^1	α^2	α^3	α^4
−−	−−	−−	−−	−−	−−
000	000	100	010	001	000
001	001	101	012	000	011
010	011	121	000	112	110
011	010	120	011	111	001
012	012	123	001	110	122
100	110	000	120	101	100
101	112	001	123	100	121
110	100	111	110	012	010
111	111	110	122	011	111
112	101	122	112	010	112
120	122	011	100	123	120
121	121	010	121	122	101
122	120	112	111	121	012
123	123	012	101	120	123
−−	−−	−−	−−	−−	−−
	3∗∗	∗∗∗	3∗∗	∗2∗	∗∗1

For every $k > 1$, each color defines an involution, as displayed in Table V, where fixed points are enclosed in parentheses, the remaining cycles are all transpositions and such fixed points and transpositions between columns α and α^i in the tables above are presented after a corresponding header "i:".

TABLE V

k=2	0:(0)(1); 1:(0 1); 2:(0)(1);	k=3	0:(00)(01)(12)(10 11); 1:(11)(00 10)(01 12); 2:(11)(00 01)(10 12); 3:(00)(10)(12)(01 11);
k=4	0:(000)(001)(012)(111)(121)(123)((010)(011)(100 110)(101 112)(120 122); 1:(000)(100)(001)(101)(010 121)(011 120)(012 123)(110 111)(112 122); 2:(011)(110)(112)(121(000 010)(001)(012)(100 120)(101 123)(111 122); 3:(000)(001)(010)(112)(011 111)(012 110)(100 101)(120 123)(121 122); 4:(000)(100)(111)(112)(120)(123)((001 011)(010 110)(012 122)(101 121).		

11 On sorting Shields-Savage Hamilton paths

Inspired in the construction technique considered by the author and his students in [6, 7], Shields and Savage showed in their Lemma 3 [18] that a Hamilton path ξ_k in R_k starting at $[F(0^{k-1})] = [0^{k+1}1^k]$ and ending at $[F(12\dots k)] = [0(01)^k]$ exists that determines a Hamilton cycle η_k in M_k invariant under Υ (Theorem 3 in Section 7), thus constituting a Hamilton cycle with dihedral symmetry of M_k, meaning it is invariant under the dihedral group action.

First, we pull back ξ_k via the inverse image γ_k^{-1} (of γ_k in Section 7) onto a Hamilton cycle ζ_k in M_k/π invariant under Υ' (Section 5), where a loop at each end of ξ_k lifts onto its corresponding parallel edge in M_k/π. Second, we pull ζ_k back via $\rho_{\mathbb{Z}_n}^{-1}$ onto a η_k in M_k invariant under Υ.

For example, the reflection about ℓ on the left of Figure 3 is used to transform ξ_2 first into a Hamilton cycle ζ_2 of M_2/π invariant under the action of $\mathbb{Z}_2$ induced by $\aleph$, represented on the figure, and then into a path of length $2|V(R_2)| = 4$ starting at $00101 = x^2 + x^4$ and ending at $01010 = x + x^3$, in the same class mod $1 + x^5$, that can be repeated five times to form a Hamilton cycle η_2 invariant under Υ, represented on the rest of the figure.

In the same way and again due to Lemma 3 [18], a Hamilton cycle η_k in M_k invariant under the action of D_{2n} is guaranteed by the determination of a Hamilton path ξ_k in R_k from vertex $\delta(0^{k+1}1^k) = k(k-1)\cdots 21*\cdots*$ to vertex $\delta(0(10)^k) = k0*1*2*\cdots*(k-1)*$.

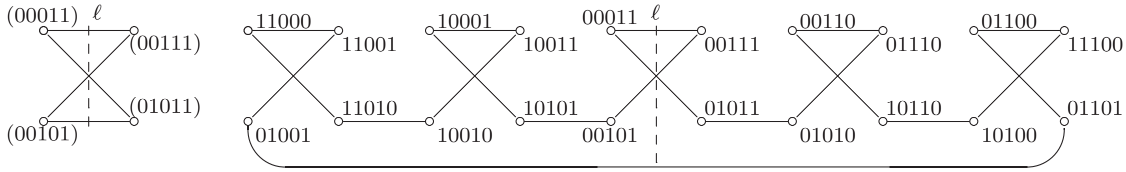

Figure 3: Hamilton cycles in M_2/π and M_2

A Shields-Savage Hamilton path ξ_k offers the finite sequence of colors of successive edges in ξ_k as a succinct code for the Hamilton cycle η_k. We can describe ξ_k via the sequence of its edge colors, or *admissible color sequence*. Table VI exemplifies in detail such sequences for $k = 2, 3$, namely: $c^2_{1_0} = 1$ and $c^3_{1_0}c^3_{2_0}c^3_{3_0}c^3_{4_0} = 1031$, where i_0 indicates the order of presentation in which the color $c^k_{i_0}$ is selected.

TABLE VI

i_0	α_{i_0}	$F(\alpha_{i_0})$	$\psi(\alpha_{i_0})$	$\bar{\psi}(\alpha_{i_0})$	$\aleph(\bar{\psi}(\alpha_{i_0}))$	$\aleph(\psi(\alpha_{i_0}))$	$c^k_{i_0}$
1_0	0	210∗∗	00011	$0_2 0_1 0_0 1_* 1_*$	$0_* 0_* 1_0 1_1 1_2$	00111	1
2_0	1	20∗1∗	01011	$0_* 1_1 0_* 1_0 1_2$	$0_2 0_0 1_* 0_1 1_*$	00101	
1_0	00	3210∗∗∗	0000111	$0_3 0_2 0_1 0_0 1_* 1_* 1_*$	$0_* 0_* 0_* 1_0 1_1 1_2 1_3$	0001111	1
2_0	10	320∗1∗∗	0010111	$0_* 0_* 1_1 0_* 1_0 1_2 1_3$	$0_3 0_2 0_0 1_* 0_1 1_* 1_*$	0001011	0
3_0	11	31∗20∗∗	0010011	$0_3 0_1 1_* 0_2 0_0 1_* 1_*$	$0_* 0_* 1_0 1_2 0_* 1_1 1_3$	0011011	3
4_0	01	310∗∗2∗	1010011	$1_3 0_* 1_0 0_* 0_* 1_0 1_1$	$0_1 0_0 1_* 1_* 0_1 1_* 0_3$	0011010	1
5_0	12	30∗1∗2∗	1010010	$1_* 0_2 1_* 0_3 0_0 1_* 0_1$	$1_1 0_* 1_0 1_3 0_* 1_2 0_*$	1011010	

In fact, the feasible admissible color sequences for $k = 2, 3, 4$ are *lexically sorted*, that is ordered according to the Kierstead-Trotter 1-factorization, from top to bottom and then from left to right, as shown in Table VII.

TABLE VII

k=2:	1±	k=4:	1013101314121	1032103132313	1201041413421	1203130143131
−−−−	−−−−		1013103130121	1032141312313	1201314143131	1203403413421+
k=3:	1031+		1213141310121	1214103132131	1214231213142	1303132132413
	2302−		1213143134121	1214141312131	1214231231302	1341312132413
			2140102013421	2140123423023	2304302013421	2304323423023
			2140121021421	2140142431023	2304321021421	2304342431023
			3103132312031	3230213213142	3230303132313	3231301310323
			3141312312031	3230213231302	3230341312313	3231303134323
			3241041031023	3243130301313	3412303132131	3431341314323−
			3241314301313	3243403031023	3412341312131	3431343130323

Indicated in the rows of Table VII with a sign ± to the right are their admissible color sequences for which the corresponding sequences of succeeding k-words form a maximum (+) and a minimum (−) according to the stair-wise order (or α ordering) of the k-words involved in the sequences. This illustrates the notions of stair-wise sorting and extremality for the admissible color sequences, to be compared with the lexical sorting and corresponding extremality in the previous paragraph. For $k = 3, 4$, the corresponding α-ordering extrema are as follows:

00 (1) 10 (0) 11 (3) 01 (1) 12 +
00 (2) 01 (3) 11 (0) 10 (2) 12 −
000 (1) 100 (2) 120 (0) 122 (3) 121 (4) 101 (0) 112 (3) 010 (4) 110 (1) 111 (3) 011 (4) 001 (2) 012 (1) 123 +
000 (3) 001 (4) 011 (3) 111 (1) 110 (3) 012 (4) 122 (1) 112 (3) 010 (1) 121 (4) 101 (3) 100 (2) 120 (3) 123 −

where succeeding k-words are separated by adjacency colors expressed between parentheses, so that by concatenating from left to right the contents of those parentheses yields the corresponding admissible color sequence. On the other hand, the minimum and maximum of the admissible color sequence for $k = 5$ are

1010201012010104301012010342131310101010121;
4545354543545451254543545213424245454545434.

Lexical extremality as above yields for $k = 5$ the contents of Table VIII.

TABLE VIII

0000 (1)	1000 (2)	1200 (3)	1230 (0)	1233 (4)	1231 (0)	1232 (3)
1221 (5)	1211 (0)	1222 (1)	1120 (4)	1123 (5)	1223 (3)	1212 (4)
1010 (5)	1210 (3)	1220 (1)	1122 (4)	1121 (3)	0121 (2)	0010 (1)
1011 (5)	1001 (2)	1201 (1)	0112 (3)	1112 (4)	1110 (0)	1111 (3)
0122 (0)	0120 (3)	1100 (4)	0101 (2)	0001 (5)	0011 (4)	0111 (2)
0110 (0)	0100 (4)	1101 (3)	0123 (2)	0012 (1)	1012 (2)	1234 +
0000 (4)	0001 (5)	0011 (0)	0010 (5)	0110 (4)	0012 (5)	0122 (2)
0112 (0)	0101 (4)	1100 (5)	0100 (4)	1101 (5)	0121 (3)	1121 (0)
1010 (3)	1000 (4)	1001 (5)	1011 (0)	1120 (4)	1123 (0)	1012 (4)
1210 (2)	1211 (0)	1222 (5)	1112 (0)	1122 (3)	0111 (5)	1111 (0)
1110 (2)	1221 (3)	1232 (2)	1212 (3)	1223 (4)	0120 (5)	1220 (4)
0123 (5)	1233 (4)	1231 (5)	1201 (4)	1200 (3)	1230 (4)	1234 −

References

[1] J. Ammerlaan, T. Vassilev, Properties of the binary hypercube and middle level graph, *Applied Mathematics*, **3** (2013), 20–26.

[2] J. Arndt, *Matters Computational: Ideas, Algorithms, Source Code*, Springer, 2011.

[3] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, 2008.

[4] M. Buck, D. Wiedemann, Gray codes with restricted density. *Discrete Math.*, **48** (1984), 163-171.

[5] G. Cantor, Über einfache Zahlensysteme, *Zeitschrift für Mathematik und Physik*, **14** (1869), 121–128.

[6] I. J. Dejter, J. Cordova, J. Quintana, Two Hamilton cycles in bipartite reflective Kneser graphs, *Discrete Math.*, **72** (1988), 63–70,

[7] I. J. Dejter, W. Cedeño, V. Jauregui, A note on Frucht diagrams, Boolean graphs and Hamilton cycles, *Discrete Math.*, **114** (1993), 131–135,

[8] I. J. Dejter, Dihedral-symmetry middle-levels problem via a Catalan system of numeration, II, preprint 2015, http://home.coqui.net/dejterij/acson-ii.pdf.

[9] A. S. Fraenkel, Systems of numeration, *IEEE Symposium on Computer Arithmetic*, (1983), 37–42.

[10] A. S. Fraenkel, The use and usefulness of numeration systems, *Inf. Comput.*, **81(1)**, (1989), 46–61.

[11] C. Godsil, G. Royle, *Algebraic Graph Theory*, Springer-Verlag, 2001.

[12] I. Hável, Semipaths in directed cubes, in: M. Fiedler (Ed.), *Graphs and other Combinatorial Topics*, Teubner-Texte Math., Teubner, Leipzig, (1983), 101–108.

[13] H. A. Kierstead, W. T. Trotter, Explicit matchings in the middle levels of the boolean lattice, *Order*, **5** (1988), 163–171.

[14] D. E. Knuth, *The Art of Computer Programming, Vol.* 2*: Seminumerical Algorithms*, third edition, Addison-Wesley, 1977.

[15] D. E. Knuth, *The Art of Computer Programming, Vol.* 3*: Sorting and Searching*, third edition, Addison-Wesley, 1977.

[16] T. Mütze, Proof of the middle-levels conjecture, preprint, `http://arxiv.org/abs/1404.4442`, August 2014.

[17] F. Ruskey: Simple combinatorial Gray codes constructed by reversing sublists, *Lecture Notes in Computer Science*, **762** (1993), 201–208.

[18] I. Shields, C. Savage, A Hamilton path heuristics with applications to the middle two levels problem, *Congr. Num.*, **140** (1999), 161–178.

[19] I. Shields, B. Shields, C. Savage. An update on the middle levels problem. *Discrete Math.*, **309(17)** (2009), 52715277.

[20] M. Shimada, K. Amano, A note on the middle levels conjecture, `http://arxiv.org./abs/0912.4564`, September 2011.

[21] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/`.

[22] R. P. Stanley, *Enumerative Combinatorics, Vol.* 2, Cambridge, 1999.