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CATALAN TRANSFORM OF THE *k*-FIBONACCI SEQUENCE

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Abstract. In this paper we apply the Catalan transform to the k-Fibonacci sequence finding different integer sequences, some of which are indexed in OEIS and others not. After we apply the Hankel transform to the Catalan transform of the k-Fibonacci sequence and obtain an unusual property.

1. Introduction

The classical Fibonacci numbers have been very used in as different sciences as the biology, demography or economy [7]. Recently they have been applied even in the high-energy physics [10, 11, 12, 13]. But there exist generalizations of these numbers given by researches as Horadam [8] and recently by we ourselves [3, 4, 5]. Now we present this last generalization, so called the k-Fibonacci numbers.

1.1. *k*-Fibonacci numbers

For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$, is defined recurrently by

$$F_{k,n+1} = k\, F_{k,n} + F_{k,n-1} \text{ for } n \geq 1 \tag{1}$$

with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$.

For $k = 1$, the classical Fibonacci sequence is obtained and for $k = 2$, the Pell sequence appears.

The well-known Binet's formula in the *Fibonacci numbers theory* [3, 8, 16] allows us to express the k-Fibonacci numbers in function of the roots σ_1 and σ_2 of the characteristic equation, associated to the recurrence relation $r^2 = k\,r+1$:

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \tag{2}$$

If σ denotes the positive root of the characteristic equation, $\sigma = \frac{k+\sqrt{k^2+4}}{2}$, the general term may be written in the form $F_{k,n} = \frac{\sigma^n-(-\sigma)^{-n}}{\sigma+\sigma^{-1}}$, and it is

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verified the limit of the quotient of two terms is:

$$\lim_{n\to\infty} \frac{F_{k,n+r}}{F_{k,n}} = \sigma^r \tag{3}$$

In particular, if $k=1$, then σ is the Golden Ratio, $\phi = \frac{1+\sqrt{5}}{2}$. In addition, the general term of the k-Fibonacci sequence may be obtained by the formula:

$$F_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} k^{n-2i-1}(k^2+4)^i \tag{4}$$

or, equivalently, by:

$$F_{k,n} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} k^{n-1-2i} \tag{5}$$

1.2. Catalan numbers

Catalan numbers are described by [1]

$$C_n = \frac{1}{n+1}\binom{2n}{n} \tag{6}$$

and the ordinary generating function of the corresponding Catalan sequence is given by

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}. \tag{7}$$

Formula (6) can be written as $C_n = \frac{(2n)!}{(n+1)!n!}$.

Finally, a recurrence relation for $C(n)$ is obtained from $\frac{C_{n+1}}{C_n} = \frac{2(2n+1)}{n+2}$ [17].

The first few Catalan numbers, for $n = 0, 1, 2, \ldots$, are

$$\{1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots\}$$

cited in [15], from now on OEIS, as A000108.

2. Catalan transform of the k-Fibonacci sequence

Following [1], we define the Catalan transform of the k-Fibonacci sequence $\{F_{k,n}\}$ as

$$CF_{k,n} = \sum_{i=0}^{n} \frac{i}{2n-i}\binom{2n-i}{n-i} F_{k,i} \quad \text{for } n \geq 1 \tag{8}$$

with $CF_{k,0} = 0$.

The first members of these sequence, that is to say, Catalan transform of the first k-Fibonacci numbers, are the polynomials in k:

- $CF_{k,1} = \sum_{1}^{1} \frac{i}{2-i}\binom{2-i}{1-i} F_{k,i} = 1$

- $CF_{k,2} = \sum_{1}^{2} \frac{i}{4-i}\binom{4-i}{2-i}F_{k,i} = k+1$
- $CF_{k,3} = \sum_{1}^{3} \frac{i}{6-i}\binom{6-i}{3-i}F_{k,i} = k^2+2k+3$
- $CF_{k,4} = k^3+3k^2+7k+8$
- $CF_{k,5} = k^4+4k^3+12k^2+22k+24$
- $CF_{k,6} = k^5+5k^4+18k^3+43k^2+73k+75$

We can write equation (8) as the product of the lower triangular matrix C and the $n\times 1$ matrix F_k as we indicate next:

$$\begin{pmatrix} CF_{k,1}\\ CF_{k,2}\\ CF_{k,3}\\ CF_{k,4}\\ CF_{k,5}\\ CF_{k,6}\\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 2 & 2 & 1 & & & & \\ 5 & 5 & 3 & 1 & & & \\ 14 & 14 & 9 & 4 & 1 & & \\ 42 & 42 & 28 & 14 & 5 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} F_{k,1}\\ F_{k,2}\\ F_{k,3}\\ F_{k,4}\\ F_{k,5}\\ F_{k,6}\\ \vdots \end{pmatrix}$$

The entries of the matrix C verify the recurrence relation $C_{i,j}=\sum_{r=j-1}^{i-1} C_{i-1,r}$ and the first column equals the second for $i>1$ and they are the Catalan numbers.

The lower triangular matrix $C_{n,n-i}$ is known as the Catalan triangle and its elements verify the formula $C_{n,n-i} = \frac{(2n-i)!(i+1)}{(n-i)!(n+1)!}$ with $0 \le i \le n$.

With the coefficients of the Catalan transform of the k-Fibonacci sequence we will form the following triangle:

Table 1. Catalan triangle of the k-Fibonacci sequence

CF_1	1						
CF_2	1	1					
CF_3	1	2	3				
CF_4	1	3	7	8			
CF_5	1	4	12	22	24		
CF_6	1	5	18	43	73	75	
CF_7	1	6	25	72	156	246	243
...	...	...	...	...	...	...	...

The first diagonal sequence $\{1, 1, 3, 8, 24, 75, 243, \dots\}$: A000958, is the Catalan transform of the Fine sequence $\{1, 0, 1, 2, 6, 18, 57, 186, \dots\}$: A000957 (see [1]).

The second diagonal sequence, A114495, is the self-convolution of the first sequence, A000958. From this point on, each diagonal sequence is the convolution of its preceding diagonal sequence and the first one.

Finally, for $k = 1, 2, 3, \ldots$ we obtain the following Catalan transform of the k-Fibonacci sequence:

- $CF_1 = \{0, 1, 2, 6, 19, 63, 215, 749, 2650, 9490, \ldots\}$, indexed in OEIS as A109262
- $CF_2 = \{0, 1, 3, 11, 42, 164, 649, 2591, 10408, 41998, \ldots\}$: A143464
- $CF_3 = \{0, 1, 4, 18, 83, 387, 1815, 8541, 40276, 190182, \ldots\}$
- $CF_4 = \{0, 1, 5, 27, 148, 816, 4511, 24971, 138328, \ldots\}$
- $CF_5 = \{0, 1, 6, 38, 243, 1559, 10015, 64373, 413878, \ldots\}$

2.1. Generating function

The function $f_k(x) = \frac{x}{1-k\,x-x^2}$ is the generating function of the k-Fibonacci polynomials [3] whereas the generating function of the Catalan numbers is $c(x) = \frac{1-\sqrt{1-4x}}{2x}$.

In [1] it is proved that if $c(x)$ is the generating function of the sequence of the Catalan numbers $\{C_n\}$ and $A(x)$ is the generating function of the sequence $\{a_n\}$, then $A(x{*}c(x))$ is the generating function of the Catalan transform of this last sequence. Consequently, the generating function of the Catalan transform of the k-Fibonacci sequence $\{F_{k,n}\}$ is

$$cf_k(x) = f_k(x * c(x)) = \frac{1-\sqrt{1-4x}}{1-k+(k+1)\sqrt{1-4x}+2x}. \tag{9}$$

3. Hankel transform

Let $A = \{a_0, a_1, a_2, \ldots\}$ be a sequence of real numbers [2, 9].
The Hankel transform of the sequence A is the sequence of determinants $H_n = \mathrm{Det}[a_{i+j-2}]$, i.e.,

$$H_n = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}.$$

The Hankel determinant of order n of the sequence A is the upper-left $n \times n$ subdeterminant of H_n.

The Hankel transform of the Catalan sequence is the sequence $\{1, 1, 1, \ldots\}$ [19] and the Hankel transform of the sum of consecutive generalized Catalan numbers is the bisection of classical Fibonacci sequence [14]. On the other hand, the element of order $n \geq 3$ of the Hankel transform of the k-Fibonacci sequence is null because each row of the determinant is a linear combination of the two preceding rows (1), according to the definition of k-Fibonacci numbers. By this reason, the Hankel transform of the k-Fibonacci sequence $\{F_{k,n}\}$ lacks interest. However, it is very interesting the study of the Catalan transform of this sequence, as we will see in the sequel.

Taking into account the Catalan transform of the k-Fibonacci sequence of the preceding subsection, we find out:

$HCF_1 = \text{Det}[1] = 1$

$$HCF_2 = \begin{vmatrix} 1 & k+1 \\ k+1 & k^2+2k+3 \end{vmatrix} = 2$$

$$HCF_3 = \begin{vmatrix} 1 & k+1 & k^2+2k+3 \\ k+1 & k^2+2k+3 & k^3+3k^2+7k+8 \\ k^2+2k+3 & k^3+3k^2+7k+8 & k^4+4k^3+12k^2+22k+24 \end{vmatrix} = 5$$

We can continue in this form and then we will find that the Hankel transform of the Catalan transform of the k-Fibonacci sequence $\{F_{k,n}\}$ is the sequence $\{1, 2, 5, \dots\}$, A001519 in OEIS.

There is a wonderful property that we will study next.

Main Theorem 1. *The Hankel transform of the Catalan transform of the k-Fibonacci sequence is the bisection of the classical Fibonacci sequence* $\{1, 2, 5, 13, 34, 89, \dots\}$ (A001519)*, independently of the value of $k \in N$. That is,* $\{HCF_{k,n}\} = \{F_{2n+1}\}$.

Proof. Because the determinant $H_n \neq 0$, its matrix is nonsingular and it may factored as the product $L_n \cdot U_n$ with L_n is a lower triangular matrix which main diagonal is $\{1, 1, 1, \dots\}$ and the first column $\{CF_1, CF_2, CF_3, \dots\}$ and U_n is an upper triangular matrix which the first row is $\{CF_1, CF_2, CF_3, \dots\}$ and the main diagonal $\{1, \frac{2}{1}, \frac{5}{2}, \frac{13}{5}, \frac{34}{13}, \dots\}$. Then, as $H_n = \text{Det}(L_n) \cdot \text{Det}(U_n) = \text{Det}(U_n) = a_{11} \cdot a_{22} \cdots a_{nn}$, the sequence of determinants is A001519.

Finally, to make notice that this sequence is the complementary sequence of A001906 in the Fibonacci sequence, obtained as the Hankel transform of the sequence of sums of two adjacent Catalan numbers A005807 [2]. □

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