A Short Note on Unsigned Stirling Numbers

Dennis Walsh
Middle Tennessee State University

The unsigned Stirling numbers $|s(n,k)|$, the absolute values of Stirling numbers of the first kind, are well known to represent the number of permutations on n elements with exactly k cycles. For example, $|s(8,\ 6)|\ = 322$ since there are 322 permutations of $\{1,2,...,8\}$ that have exactly 6 cycles. Interestingly, if one takes each size-2 subset of $\{1,2,...,7\}$, multiplies the two elements, and then sums the products, the resulting sum is also 322. The table below illustrates this result.

Subset elements	1,2	1,3	1,4	1,5	1,6	1,7	2,3	2,4	2,5	2,6	2,7
product	2	3	4	5	6	7	6	8	10	12	14
Subset elements	3,4	3,5	3,6	3,7	4,5	4,6	4,7	5,6	5,7	6,7	$\sum$products $\downarrow$
product	12	15	18	21	20	24	28	30	35	42	**322**

The two routes to the number 322 above suggests a generalization. In fact, for $n > k \geq 1$, if one takes each size $(n-k)$ subset of $\{1,2,...,n-1\}$, multiplies all the elements, and then sums the products, the resulting sum is equal to the unsigned Stirling number $|s(n,k)|$. The following theorem formalizes this result.

Theorem. For $1 \leq k < n$, let $|s(n,k)|$ denote an unsigned Stirling number of the first kind, and let $A = \{a_1, a_2, ..., a_{n-k}\}$ denote a size $(n-k)$ subset of $\{1,2,...,n-1\}$. Then

$$|s(n,k)| = \sum_A (a_1 a_2 \cdot\ \cdot\ \cdot a_{n-k})$$

where the sum is over all $\binom{n-1}{n-k}$ subsets A.

Proof. Using the well-known fact that the generating function of the unsigned Stirling numbers $|s(n,k)|$, when n is fixed, is given by

$$t(t+1)(t+2)\cdot\ \cdot\ \cdot(t+n-1) = \sum_{k=1}^{n} |s(n,k)|\, t^k.$$

Upon expanding $t(t+1)(t+2)\cdot\ \cdot\ \cdot(t+n-1)$, the coefficient of t^k is equal to the sum of $\binom{n-1}{n-k}$ products, each product consisting of $(n-k)$ different factors from $\{1, ...,\ n-1\}$ and k factors of one (the coefficients of the t's). Hence $|s(n,k)| = \sum_A (a_1 a_2 \cdot\ \cdot\ \cdot a_{n-k})$ where the sum is over all $\binom{n-1}{n-k}$ size $(n-k)$ subsets A of $\{1,2,...,n-1\}$. $\square$