NNTDM 17 (2011), 4, 9–13

Some recurrence relations for binary sequence matrices

A. G. Shannon

Faculty of Engineering & IT, University of Technology
Sydney, NSW 2007, Australia
e-mails: `tshannon38@gmail.com`, `anthony.shannon@uts.edu.au`

Abstract: This note compares and contrasts some properties of binary sequences with matrices and associated recurrence relations in order to stimulate some enrichment exercises and pattern puzzles.

Keywords: Binary, Matrix, Sequence, Recurrence relation, Kronecker delta, Repunit, Fibonacci numbers, Lucas numbers, Pell numbers, Pascal's triangle.

AMS Classification: 1101, 11B39, 11C20

1 Introduction

Binary numbers are reasonably well known as in Table 1.

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{B}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \tilde{B}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \tilde{B}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Table 1. Binary matrices (arrays)

These binary matrices can be generated by the first order recurrence relation

$$B_{n+1} = \begin{bmatrix} 0_{2^n \times 1} & B_n \\ 1_{2^n \times 1} & B_n \end{bmatrix}, \text{ n>1}$$

with initial term

$$B_1 = G_1$$

where

$$G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and $0_{2^n \times 1}$ is a column vector (or matrix with one column) and 2^n rows each with a zero in it, and B_{n+1} has 2^{n+1} rows and $n + 1$ columns.

2 Gray codes

A Gray code is also made up of binary numbers as shown in the arrays of Gray matrices (Table 2).

$$G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \; \tilde{G}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \; \tilde{G}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \; \tilde{G}_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Table 2. Gray matrices

If we define the reverse of the unit $n \times n$ matrix, I, by

$$I^R_{n\times n} = [\delta_{i,n-j+1}]$$

in which $\delta_{i,j}$, the element in the i-th row and j-th column, is the Kronecker delta:

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0 & i \neq j. \end{cases}$$

Thus, for example,

$$I^R_{2\times 2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The reverse of any $n \times m$ matrix, A, can then be defined as

$$A^R_{n\times m} = I^R_{n\times n} A;$$

For instance,

$$G^R_n = I_{2^n \times 2^n} G_n ,$$

so that

$$G^R_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It can next be observed that in general G_{n+1} is a block matrix which satisfies the matrix recurrence relation

$$G_{n+1} = \begin{bmatrix} 0_{2^n\times 1} & G_n \\ 1_{2^n\times 1} & G^R_n \end{bmatrix},$$

where $0_{2^n\times 1}$ and $1_{2^n\times 1}$ are column matrices containing zeros or ones, respectively.

Gray Codes are clearly binary in nature, so it is tempting to contrast them with the formal binary numbers (Table 3).

Decimal order	Binary	Gray	Gray decimal
0	000000	000000	0
1	000001	000001	1
2	000010	000011	3
3	000011	000010	2
4	000100	000110	6
5	000101	000111	7
6	000110	000101	5
7	000111	000100	4
8	001000	001100	12
9	001001	001101	13
10	001010	001111	15

Table 3. Gray and binary numbers

We can use Gray arrays to define third-order recursive sequences $\{u_n\}$, $\{v_n\}$, which are coupled or lacunary (have missing terms), with initial terms a, b, c and d, e, f, respectively, and eight pairs of recurrence relations defined by

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} \tilde{g}_{i\bullet} & g_{i\bullet} \\ g_{i\bullet} & \tilde{g}_{i\bullet} \end{bmatrix} [u_{k-3}, u_{k-2}, u_{k-1} v_{k-3}, v_{k-2}, v_{k-1}]^T \tag{2.1}$$

in which $g_{i\bullet}$ and $\tilde{g}_{i\bullet}$ are the i-th rows of G_3 and $\tilde{G}_3$, respectively [1]. Try it! Once again, there is a two-fold idea of finding patterns and getting the notation to suggest new ideas [5].

3 Good sequences

Austin and Guy [2] define a ‘good’ sequence as an ordered set of zeros and ones in which each ‘one’ in it has a neighbouring ‘one’; that is, if there is a ‘one’ in the set, then it has a ‘one’ next to it. a_n is then defined as the number of good sequences of length n (Table 4).

n	‘Good’ sequences of length n	a_n
1	0	1
2	00,11	2
3	000,011,110,111	4
4	0000,0011,0110,1100,0111,1110,1111	7
5	00000,00011,00110,01100,11000,00111,01110,11100,01111,11110,11011,11111	12

Table 4. ‘Good’ sequences

The ‘good’ numbers, a_n, can be shown to satisfy the fourth order recurrence relation:

$$a_n = a_{n-1} + a_{n-2} + a_{n-4} \, . \tag{3.1}$$

What about investigating ‘better’ sequences, $\{b_n\}$ in which the requirement is that each ‘one’ (if present) be accompanied by two other ‘ones’; that is, in blocks of length three? (This is actually using the mathematical principle of exclusion-inclusion.) Or, as Austin and Guy consider the binary sequences of length n in which the 'ones' occur only in blocks of length at least k. The elements of this are designated by $a_n^{(k)}$, so that the 'good' sequence is $\{a_n^{(2)}\}$, and we again gradually begin to see how notation becomes a tool of thought rather than an artificial burden [5]. Can you justify that

$$a_{k+n}^{(k)} = 1 + \tfrac{1}{2}(n+1)(n+2),\ 0 \le n \le k\ ? \qquad (3.2)$$

These are the central polygonal numbers and a solution to the 'Lazy caterer's problem' [4, 6, 7].

4 Some other recursive sequences

Sburlati [9] used a recursive sequence $\{k_n\}$ defined as a repunit [3] by

$$k_n = \tfrac{1}{3}(4^n - 1) \qquad (4.1)$$

It satisfies the second order homogenous linear recurrence relation

$$k_n = 5k_{n-1} - 4k_{n-2}, n \ge 2, k_1 = 1, k_2 = 5, \qquad (4.2)$$

which is a generalization of the well-known Fibonacci recurrence relation

$$F_n = F_{n-1} + F_{n-2}, n \ge 2, F_1 = 1, F_2 = 1, \qquad (4.3)$$

As a means of comparison, the first ten elements of the Fibonacci, Lucas and Pell sequences as well as those from [9] are printed in Table 5.

n	Fibonacci	Lucas	Pell	Sburlati
1	1	1	1	1
2	1	3	2	5
3	2	4	5	21
4	3	7	12	85
5	5	11	29	341
6	8	18	70	1365
7	13	29	169	5461
8	21	47	408	21845
9	34	76	985	87381
10	55	123	2378	1398101

Table 5. Some second order sequences

Representations of numbers in the last column of Table 5 are disguised in Table 6.

				1				
			1	0	1			
		1	0	1	0	1		
	1	0	1	0	1	0	1	
1	0	1	0	1	0	1	0	1

1				
1	4			
1	4	16		
1	4	16	64	
1	4	16	64	256

Table 6. A curious binary triangle and a 'decimal relative'

Can you find the sequence {1, 5, 21, 85, 341, 1365, 5461,…} in the two triangles of Table 6? Can you add along the various slopes (diagonals) and rows in Table 6 and come up with sequences? What are their recurrence relations? Can you develop other binary triangles?

5 Concluding Comments

The next challenge for the interested reader is to develop triangles associated with the other sequences and to play with other combinations of these sequences either directly or by treating the triangles as upper and lower triangular matrices, and by adding along the diagonals of the triangles, including Pascal's triangle, as well as across rows and down columns (partial sums).

References

[1] Atanassov, K., V. Atanassova, A. Shannon, J. Turner. *New Visual Perspectives on Fibonacci Numbers.* New Jersey: World Scientific, 2002.

[2] Austin, R., R. Guy. Binary Sequences without Isolated Ones. *The Fibonacci Quarterly*. Vol. 16, 1978, 84–87, 96.

[3] Beiler, A. H. *Recreations in the Theory of Numbers: The Queen of Mathematics Entertains.* New York: Dover, 1966.

[4] Gardner, M. *Knotted Doughnuts and Other Mathematical Entertainment.* New York: Freeman, 1986.

[5] Iverson, K. E. Notation as a Tool of Thought. *Communications of the Association for Computing Machinery.* Vol. 23, 1980, 444–465.

[6] Kimberley, C. Polynomials Defined by a Second Order Recurrence, Interlacing Zeros, and Gray Codes. *The Fibonacci Quarterly*. Vol. 48, 2010, 209–218.

[7] Shannon, A. G. Shrewd Guessing in Problem-solving. *International Journal of Mathematical Education in Science and Technology.* Vol. 22, 1991, 144–147.

[8] Sloane, N.J.A., S. Plouffe. *The Encyclopedia of Integer Sequences.* San Diego: Academic Press, 1995.

[9] Sburlati, G. Generalized Fibonacci Sequences and Linear Congruences. *The Fibonacci Quarterly.* Vol. 40, 2002, 446–452.