On Complete and Size Balanced k-ary Tree Integer Sequences

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Abstract: Discovering new integer sequences and generalizing the existing ones are important and of great interest. In this article, various *balanced* k-ary trees are first studied and their taxonomy is built. In particular, two systematic *balanced* k-ary trees, whose nth tree is determined by a certain algorithm, are identified, i.e., *complete* and *size-balanced* k-ary trees. The integer sequences from the formal one is important to analyzing algorithms involving the popular *d-heap* data structures. Those derived from the later one is pervasive in analyzing *divide and conquer* algorithms. Numerous generalized and new formulae for existing and new integer sequences generated from the *complete* and *size-balanced* k-ary trees are given.

Key–Words: complete k-ary tree, integer sequence, null-balanced k-ary tree, size-balanced k-ary tree

1 Introduction

1.1 Preliminary Definition

Let n be the number of nodes in a tree, T which is the *size* of the tree, $n = |T| = \text{size}(T)$. A *rooted k-ary tree*, R_k can be defined recursively.

Definition 1 *A* rooted k-ary tree*, R_k is either empty or has a root node, t with a sequence of k children rooted k-ary sub-trees.*

$$R_k = \begin{cases} \varnothing, & \text{if } n = 0 \\ (t, < R_k^1, \cdots, R_k^k >), & \text{if } n > 0 \end{cases} \quad (1)$$

A node, t_i in a *rooted k-ary tree* is called either a *leaf* if it has no children or an *internal* node if it has up to k children nodes. In a k-ary tree, every node has exactly k children if we consider the *null* node as a child. Every node has a *parent* node except for one node which is called a *root* node. Figure 1 shows a ternary ($k = 3$) tree with ($n = 22$) nodes.

Definition 2 *The* level *of a node is the length of the path from the node to the root.*

Definition 3 *The* inclusive depth *of a node is the number of the levels from the node to the root inclusively. i.e.*

$$depth(t_i) = level(t_i) + 1 \quad (2)$$

Let $depth'(t_i)$ denote the exclusive version of the depth of a node which is identical to the *level* as defined simply as the depth in most literatures [1, 2, 3, 4, 5]. The inclusive and exclusive depths of the double circled node in Figure 1 are 3 and 2, respectively.

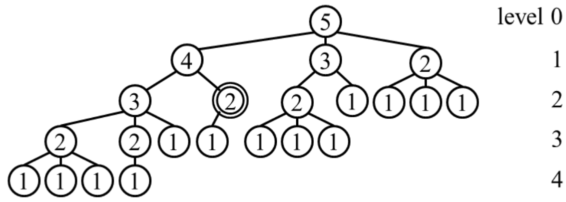


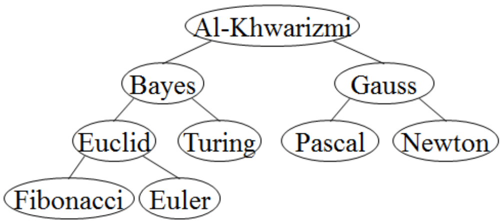
Figure 1: A ternary ($k = 3$) tree with ($n = 22$) nodes

Albeit there is no universally agreed-upon definition of the height of a rooted tree [3], it is defined as the length of the path from the root to the deepest node in the tree in most literatures [1, 2, 3, 4]. In other words, it is the exclusive number of nodes from the root to the deepest node. However, the inclusive version in the definition 4 shall be considered as well as the exclusive version.

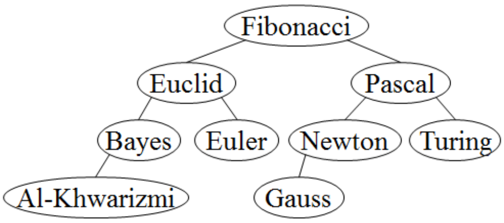
Definition 4 *The* height *of a node is the number of levels from the the node to the deepest leaf inclusively.*

$$height(t_i) = \begin{cases} 0, & \text{if } n = 0 \\ max(height(R_k^1), \cdots, height(R_k^k)) + 1, & \text{if } n > 0 \end{cases} \quad (3)$$

Every node, t_i can be considered to be a root of a sub-k-ary tree and the height(t_i) is the height of the sub-k-ary tree whose root is t_i. Note that the heights of a single node tree and an empty tree are 1 and



(a) complete binary tree (heap)



(b) size balanced tree (divide & conquer)

Figure 2: balanced binary tree examples

$height(\varnothing) = 0$ whereas they are 0 and -1 in the exclusive version of height. In figure 1, every node indicates their inclusive height information.

1.2 Integer Sequences

Let *sumh* and *sumd* be the sums of inclusive heights and inclusive depths of all nodes, respectively.

$$sumh(R_k(n)) = \sum_{i=1}^{n} height(t_i) \quad (4)$$

$$sumd(R_k(n)) = \sum_{i=1}^{n} depth(t_i) \quad (5)$$

Let *sumh'* and *sumd'* be the sums of exclusive heights and exclusive depths of all nodes, respectively. They can be derived from the inclusive versions as in the eqns (6) and (7).

$$sumh'(R_k(n)) = sumh(R_k(n)) - n \quad (6)$$

$$sumd'(R_k(n)) = sumd(R_k(n)) - n \quad (7)$$

Consider a unary ($k = 1$) tree of size n. The sum of each node's height provides an integer sequence generated by the eqn (8).

$$sumh(R_1(n)) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad (8)$$

When nth integer term is determined as a specific integer value by an explicit formula or algorithmically, the sequence is called an *integer sequence*. This integer sequence generated by the eqn (8) is the famous *triagular number sequence*. The On-Line Encyclopedia of Integer Sequences [6] contains over 200,000 integer sequences. Here numerous new and generalized integer sequences from balanced k-ary tree are discovered.

Two systematic k-ary trees whose nth tree is determined, are studied, i.e., a *complete* and *size-balanced* k-ary trees. Adding heights or depths of every node in a *complete* k-ary tree as shown in Figure 2 (a) produces an integer sequence. These are important sequences in analyzing the popular algorithms involving *d-heap* data structures. Adding heights of a *size-balanced* k-ary trees as shown in Figure 2 (b) also produce new integer sequences. These sequences are very popular in numerous algorithm analysis involving the famous *divide-and conquer* paradigm.

1.3 Organization

This article is an extended version of the conference proceeding article in [7] and the rest of the paper is organized as follows. Since the terminologies in *Trees*, especially the *balanced* k-ary tree, are still in flux, the section 2 provides formal definitions and the relationships and taxonomy of various *balanced* k-ary trees are studied. In section 3 provides the general formulae for the *sumh* and *sumd* integer sequences derived from the *complete* k-ary trees. Furthermore, new integer sequneces derived from the *size balanced* k-ary trees are also given. Finally, the section 4 concludes this work.

2 Taxonomy of k-ary Trees

Balanced trees can be defined in various ways. Rosen defined the balanceness of a tree in terms of their leave node locations as in Definition 5 [2].

Definition 5 *A tree is called a* leaf balanced *k-ary tree, L_k if all leaves are at levels $h-1$ or $h-2$.*

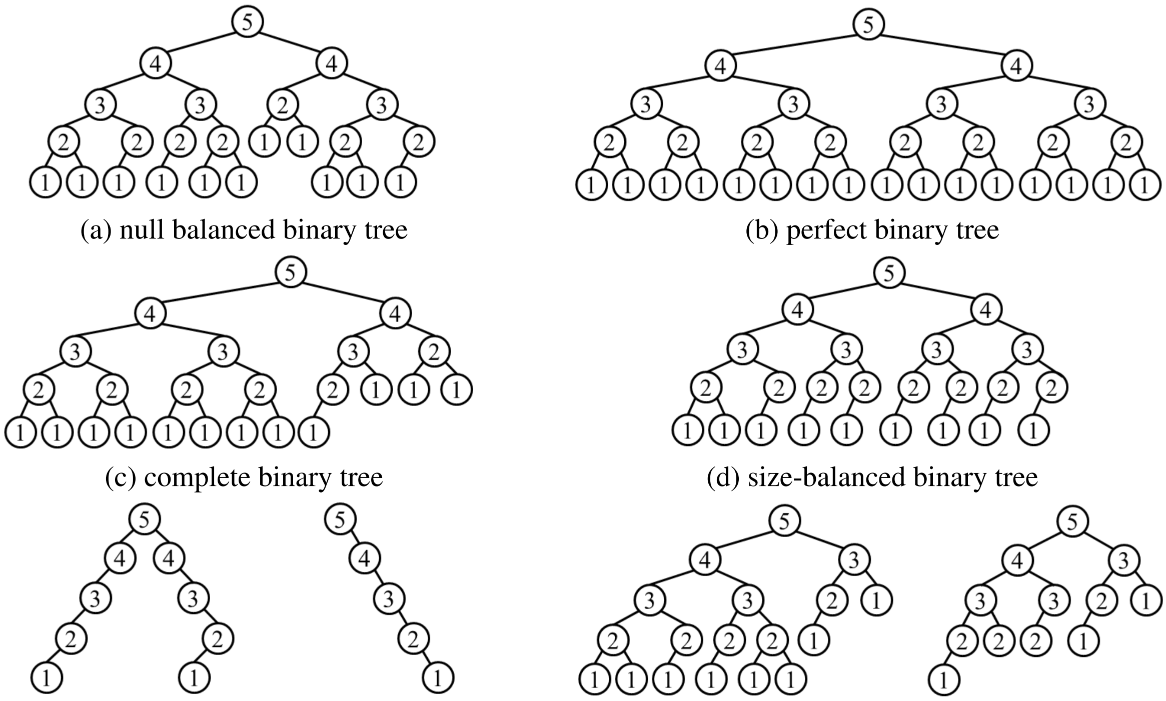
All binary ($k = 2$) trees in Figures 3 (a∼e) are *leaf balanced* binary trees except for those in Figure 3 (f).

A *balanced tree* is defined in terms of heights of sub-trees as in Definition 6.

Definition 6 *A tree is called a* height balanced *k-ary tree, H_k if the eqn (9) is satisfied for every node t_i and for every sub-tree pair (H_k^x, H_k^y) of t_i.*

$$|height(H_k^x) - height(H_k^y)| \leq 1 \quad (9)$$

All trees in Figures 3 except for (e) are *height balanced* binary trees. A height balanced binary search tree is known as the *AVL* tree [1, 4] and the definition 6 is a generalized for any k version of the balanced binary tree defined in [1, 4].



(a) null balanced binary tree (b) perfect binary tree

(c) complete binary tree (d) size-balanced binary tree

(e) Leaf balanced binary tree (f) Height balanced binary tree

Figure 3: balanced binary tree examples

A different definition of a *balanced* k-ary tree is given and used in this article. It is a slight vicissitude of Definition 5.

Definition 7 *A tree is called a* null balanced k*-ary tree,* N_k *if all null nodes are at levels* h *or* $h-1$.

All binary trees in Figure 3 (a∼d) are *null balanced* binary trees while those trees in Figure 3 (e) and (f) are not.

Fact 1 *The height of the* null balanced k*-ary tree is*

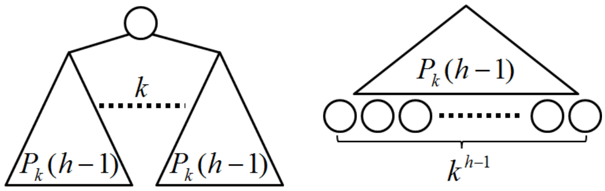
$$height(N_k(n)) = \lceil \log_k (n(k-1)+1) \rceil \qquad (10)$$

Definition 8 *A tree is called a* perfect k*-ary tree,* P_k *if all internal nodes have exactly* k *children and all leaves lie at the same depth,* h.

The *perfect* k-ary tree is often called a *full* k-ary tree such as in [5]. However, the *full* k-ary tree is defined differently in [2] as a tree whose internal nodes have exactly k children but leaves may not lie at the same depth. In other words, a node in a *full* k-ary tree is either a leaf or has exactly k number of non-empty *full* k-ary sub trees.

Let $size(P_k(h))$ be the size of the hth height *perfect* k-ary tree and it can be computed using the following closed formula in the eqn (11).

$$size(P_k(h)) = \sum_{i=1}^{h} k^{i-1} = \frac{k^h - 1}{k-1} = n. \qquad (11)$$



(a) k children recursion (b) Non-leaf level recursion

Figure 4: Recursive relations of *Perfect* k-ary trees

In case that $k = 2$ in Figure 3 (b), the perfect binary trees are possible only for $n = 1, 3, 7, 15, \cdots, 2^h - 1$. The integer sequences of sizes of some *perfect* k-ary trees are given in Table 1.

Albeit straightforward, the size of a *perfect* k-ary tree has two simple recursive relations. First, a root node has k number of *sub perfect* k-ary trees whose height is $h-1$ as shown in Figure 4 (a). Hence, $size(P_k(h))$ can be computed and defined recursively as in the eqn (12).

$$size(P_k(h)) = \begin{cases} 1, & \text{if } h = 0 \\ k \times size(P_k(h-1)) + 1, & \text{otherwise} \end{cases} \qquad (12)$$

Next, the sub-tree which excludes the leaf level nodes also forms a *perfect* k-ary tree of height, $h-1$ as illustrated in Figure 4 (b). There are exactly k^{h-1} number of leaf nodes at the $h-1$th level. Hence, a non-leaf

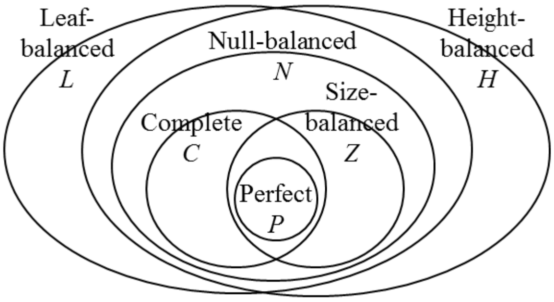


Figure 6: Venn Diagram of *balanced* k-ary trees

level recursive relation for $size(P_k(h))$ is defined as in the eqn (13).

$$size(P_k(h)) = \begin{cases} 1, & \text{if } h = 0 \\ size(P_k(h-1)) + k^{h-1}, & \text{otherwise} \end{cases} \quad (13)$$

These simple recursive relations in eqns (12) and (13) shall shed light on other definitions in balanced k-ary trees in section 3.

The *null-balanced* k-ary tree can be defined in terms of the *perfect* k-ary tree.

Definition 9 *A* null-balanced *k-ary tree, N_k has a* perfect *k-ary tree whose height is $h-1$ and the remaining $n - size(P_k(h-1))$ number of nodes are at the depth h.*

There are several systematic ways to make a null balanced tree of size, n where a unique nth tree is determined algorithmically. Here a couple of them are considered. The first one is the *complete* k-ary tree where a node is added in the *breadth first order* as shown in Figure 3 (c). Figure 5 (a) and (b) demonstrate the first few *complete binary* and ternary trees.

Definition 10 *A tree is called a* complete *k-ary tree, $C_k(n)$ if it has a* pefect *k-ary tree of height $h-1$ and the remaining nodes are added from left to right order.*

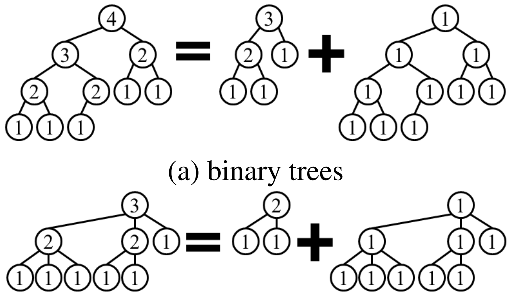
A tree can be balanced by sizes of sub-trees.

Definition 11 *A tree is called a* size balanced *k-ary tree, Z_k if the eqn (14) is satisfied for every node t_i and for every sub-tree pair (Z_k^x, Z_k^y) of t_i.*

$$|size(Z_k^x) - size(Z_k^y)| \leq 1 \quad (14)$$

If the eqn (15) is added as a constraint, the *size balanced* k-ary tree becomes systematic. Figure 5 (c) and (d) demonstrate the first few *size-balanced binary* and ternary trees.

$$size(Z_k^x) \leq size(Z_k^y) \text{ if } x < y \leq k \quad (15)$$



(a) binary trees

(b) ternary trees

Figure 7: Illustration of computing *sumh*$(C_k(n))$

Only trees in Figures 3 (b) and (d) are *size balanced* binary trees. The sizes of k-sub trees follow the integer partition into k balanced parts defined in the eqn (16).

$$\textit{BIP}(m,k) = \Big(\overbrace{\underbrace{\left\lceil \frac{m}{k} \right\rceil, \ldots, \left\lceil \frac{m}{k} \right\rceil}_{\tilde{k}=m\%k}, \left\lfloor \frac{m}{k} \right\rfloor, \ldots, \left\lfloor \frac{m}{k} \right\rfloor}^{k}\Big) \quad (16)$$

For examples, *BIP*$(51,2) = (26,25)$, *BIP*$(32,3) = (11,11,10)$, and *BIP*$(23,4) = (6,6,6,5)$.

Figure 6 gives the venn diagram of *balanced* k-ary trees defined in this section.

3 k-ary Tree Integer Sequences

Consider the first 15 and 13 sequences of *complete binary* and *ternary* trees in Figure 5 (a) and (b), respectively. The sum of all nodes' heights in a *complete* k-ary tree, *sumh*$(C_k(n))$, can be computed recursively as defined in the eqn (17) as depicted in Figure 7,

$$\textit{sumh}(C_k(n)) = \begin{cases} n, & \text{if } n \leq 1 \\ \textit{sumh}(\lceil C_k(\lceil \frac{n-1}{k} \rceil) \rceil) + n, & \text{otherwise} \end{cases} \quad (17)$$

The first one hundred sequences for the *complete unary, binary, ternary, quaternary*, and *quinary* trees are listed in the Table 2

The sum of exclusive heights, *sumh*$'(C_k(n))$ can be computed using the eqn (6), i.e., *sumh*$'(C_k(n)) =$ *sumh*$(C_k(n)) - n$ and their integer sequences are given in Table 3.

Theorem 1 *sumh*$(C_k(n)) = \Theta(n)$.

Proof: Let *sumh*$(C_k(n)) = f(n)$. Then $f(n) = f(n/k) + n$ approximately. According to the *Master Theorem* [1], this recursive function belongs to the case 3 and thus $\Theta(n)$. □

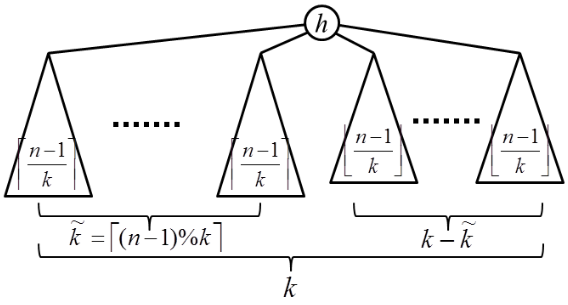


Figure 8: Illustration of computing *sumh*$(Z_k(n))$

When $k = 2$, the integer sequence *sumh*$(C_2(n))$ was studied in differential topology [8]. Both *sumh*$(C_2(n))$ and *sumh*$(C'_2(n))$ integer sequences are found in the *OEIS* A005187 and A011371, respectively and their relationships to the number of 1's that appear in the binary expansion of n are described in [6]. However, only *sumh*$(C_3(n))$ but not *sumh*$(C'_3(n))$ is found in the *OEIS* A127427 for the complete ternary trees. Hence, the eqn (17) is the generalized version of the complete k-ary tree for any k.

Consider the first 16 and 15 sequences of *size-balanced binary* and *ternary* trees in Figure 5 (c) and (d), respectively. The sum of all nodes' heights in a *size-balanced* k-ary tree, *sumh*$(Z(n))$, can be defined recursively as in the eqn (18) by slightly modifying the k *children resursion* defined in the eqn (12).

$$\mathit{sumh}(Z(n)) = \begin{cases} n, & \text{if } n \le 1 \\ \begin{array}{l} h + \tilde{k} \times \mathit{sumh}\left(Z(\lceil \frac{n-1}{k} \rceil)\right) \\ +(k-\tilde{k}) \times \mathit{sumh}\left(Z(\lfloor \frac{n-1}{k} \rfloor)\right) \end{array}, & \text{otherwise} \end{cases}$$
$$\text{where } \tilde{k} = (n-1) \mod k \tag{18}$$

If $n-1$ is not divisible by k, then there are only two different sized children where one group has the size of $\lceil(\frac{n-1}{k})\rceil$ and the other group has the size of $\lfloor(\frac{n-1}{k})\rfloor$. Exactly $\tilde{k} = (n-1) \mod k$ number of children's size must be $\lceil(\frac{n-1}{k})\rceil$ and the other $(k-\tilde{k})$ number of children's size must be $\lfloor(\frac{n-1}{k})\rfloor$.

Albeit the sum of exclusive heights can be computed by the eqn (6), it can be also defined recursively as in the eqn (19).

$$\mathit{sumh}'(Z_k(n)) = \begin{cases} 0, & \text{if } n \le 1 \\ \begin{array}{l} h-1 \\ +\tilde{k} \times \mathit{sumh}'\left(Z(\lceil \frac{n-1}{k} \rceil)\right) \\ +(k-\tilde{k}) \times \mathit{sumh}'\left(Z(\lfloor \frac{n-1}{k} \rfloor)\right) \end{array}, & \text{otherwise} \end{cases}$$
$$\text{where } \tilde{k} = (n-1)\%k \tag{19}$$

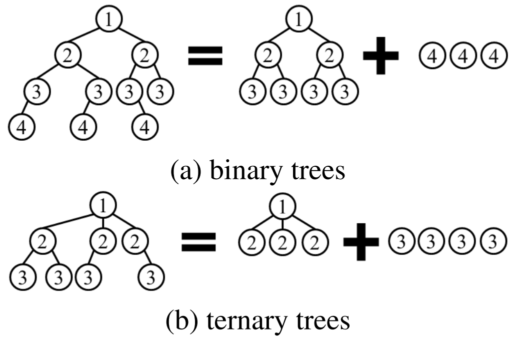


Figure 9: Illustration of computing *sumd*$(N_k(n))$

Surprisingly, neither *sumh*$(Z_k(n))$ nor *sumh*$'(Z_k(n))$ integer sequence for any k appears in the *OEIS*.

Theorem 2 *sumh*$(Z_k(n)) = \Theta(n)$.

Proof: Let *sumh*$(Z_k(n)) = f(n)$. Then $f(n) = kf(n/k) + \log_k n$ simply. According to the *Master Theorem* [1], this recursive function belongs to the case 1 and thus $\Theta(n)$. □

Finally, other integer sequences can be derived from aforementioned systematic k-ary trees if we add the depths instead of heights as exemplified in Figures 5 (e) and (f). The sum of depths in a *complete* k-ary tree is the same as that in a *size-balanced* k-ary tree. In other words, any *null-balanced* k-ary tree of size n, *sumd*$(N_k(n))$ has the same sum of depths of all nodes as defined in the eqn (20).

$$\mathit{sumd}(N_k(n)) = \begin{array}{l} h \times (n - \mathit{size}(P_k(h-1))) \\ + \sum_{i=1}^{h-1}(i \times k^{i-1}) \end{array} \tag{20}$$

While the eqn (18) is extended from the k *children recursion* defined in the eqn (12), the *non-leaf level recursion* can be utilized to constitue the eqn (20). A *null-balanced* k-ary tree has a *perfect* k-ary tree up to $h-1$ depth. The second term of the eqn (20) is adding its depth times the number of nodes in the respective depth in a *perfect* k-ary tree. And the remaining $n-$ *size*$(P_k(h-1))$ number of nodes has the value h as depicted in Figure 9.

Theorem 3 *sumd*$(N_k(n)) = \Theta(n \log_k n)$.

Proof: Let *sumd*$(N_k(n)) = f(n)$. Then $f(n) = f(n/k) + n \log_k n$ simply. According to the *Master Theorem* [1], this recursive function belongs to the case 3 and thus $\Theta(n \log_k n)$. □

sumd$(N_k(n))$ is asymptotically equivalent to the $\Theta(n \log_k n)$ which are called *linearithmic*, *loglinear*, or *quasilinear*. The integer sequence *sumd*$(N_2(n))$ was widely studied and espeically to count the maximal number of comparisons for sorting n elements by binary insertion [9] and appears in *OEIS* A001855 [6].

Albeit the sum of exclusive depths can be computed by the eqn (6), it can be also defined as in the eqn (21).

$$\mathit{sumd}(N_k(n)) = (h-1) \times (n - \mathit{size}(P_k(h-1))) + \sum_{i=1}^{h-1}((i-1) \times k^{i-1}) \quad (21)$$

The sum of depths in a *null-balanced binary* tree, $\mathit{sumd}(N'_2(n))$, appears in the *OEIS* A061168 [6, 10]. However, no integer sequences were found for both $\mathit{sumd}(N_k(n))$ and $\mathit{sumd}(N'_k(n))$ when $k > 2$.

4 Conclusion

In this paper, several different definitions of a *balanced* k-ary tree and their relationships were presented. Two kinds of special *null-balanced* k-ary trees where nth tree is determined were also presented, i.e., *complete* and *size-balanced* k-ary trees.

Explicit formulae were given to generate numerous integer sequences related to the *complete* and *size-balanced* k-ary trees. Some integer sequences are already in *OEIS* but this article provided a generalized k-ary tree version formulae. The sum of height or depth integer sequences from *complete ternary* trees are not found but only the sum of inclusive height appears.

One of the most notable findings in this paper is discovering the sum of height integer sequences from *size-balanced* k-ary trees. These sequences appear very often in certain types of the famous *divide-and conquere* algorithm analysis.

Numerous integer sequences that are related to *divide and conquer* appear in [6]. However, n is only the number of external nodes (leaves) in most of existing integer sequences whereas n is the number of both internal and external nodes in integer sequences $\mathit{sumh}(Z_k(n))$ and $\mathit{sumh}'(Z_k(n))$. That may be the reason why these popular integer sequences do not appear in OEIS.

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Table 1: size of *perfect* k-ary trees.

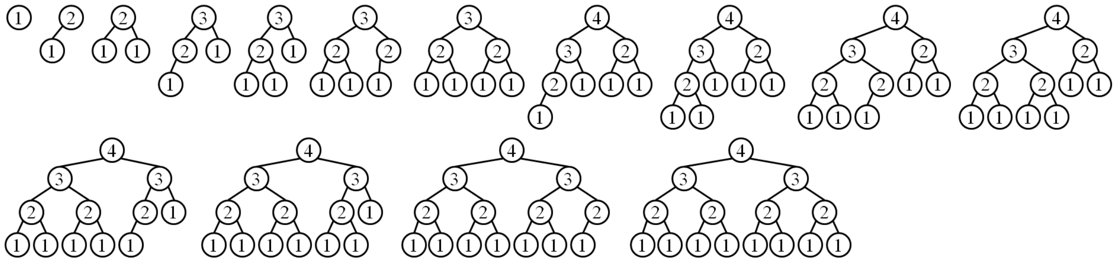
$k\backslash h$	1	2	3	4	5	6	7	8	9	10	11	12	13	OEIS
1	1	2	3	4	5	6	7	8	9	10	11	12	13	A000027
2	1	3	7	15	31	63	127	255	511	1023	2047	4095	8191	A000225
3	1	4	13	40	121	364	1093	3280	9841	29524	88573	265720	797161	A003462
4	1	5	21	85	341	1365	5461	21845	87381	349525	1398101	5592405	22369621	A002450

Table 2: sum of inclusive heights of *complete* k-ary trees: *sumh*$(C_k(n))$

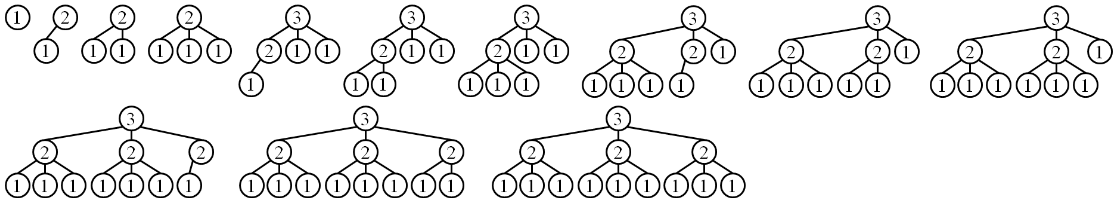
k	Integer sequence for $n = 1, \ldots, 100$	$n = 1000$	OEIS
1	1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225, 1275, 1275, $\cdots$	500500	A000217
2	1, 3, 4, 7, 8, 10, 11, 15, 16, 18, 19, 22, 23, 25, 26, 31, 32, 34, 35, 38, 39, 41, 42, 46, 47, 49, 50, 53, 54, 56, 57, 63, 64, 66, 67, 70, 71, 73, 74, 78, 79, 81, 82, 85, 86, 88, 89, 94, 95, 97, 98, 101, 102, 104, 105, 109, 110, 112, 113, 116, 117, 119, 120, 127, 128, 130, 131, 134, 135, 137, 138, 142, 143, 145, 146, 149, 150, 152, 153, 158, 159, 161, 162, 165, 166, 168, 169, 173, 174, 176, 177, 180, 181, 183, 184, 190, 191, 193, 194, 197, $\cdots$	1994	A005187
3	1, 3, 4, 5, 8, 9, 10, 12, 13, 14, 16, 17, 18, 22, 23, 24, 26, 27, 28, 30, 31, 32, 35, 36, 37, 39, 40, 41, 43, 44, 45, 48, 49, 50, 52, 53, 54, 56, 57, 58, 63, 64, 65, 67, 68, 69, 71, 72, 73, 76, 77, 78, 80, 81, 82, 84, 85, 86, 89, 90, 91, 93, 94, 95, 97, 98, 99, 103, 104, 105, 107, 108, 109, 111, 112, 113, 116, 117, 118, 120, 121, 122, 124, 125, 126, 129, 130, 131, 133, 134, 135, 137, 138, 139, 143, 144, 145, 147, 148, 149, $\cdots$	1498	A127427
4	1, 3, 4, 5, 6, 9, 10, 11, 12, 14, 15, 16, 17, 19, 20, 21, 22, 24, 25, 26, 27, 31, 32, 33, 34, 36, 37, 38, 39, 41, 42, 43, 44, 46, 47, 48, 49, 52, 53, 54, 55, 57, 58, 59, 60, 62, 63, 64, 65, 67, 68, 69, 70, 73, 74, 75, 76, 78, 79, 80, 81, 83, 84, 85, 86, 88, 89, 90, 91, 94, 95, 96, 97, 99, 100, 101, 102, 104, 105, 106, 107, 109, 110, 111, 112, 117, 118, 119, 120, 122, 123, 124, 125, 127, 128, 129, 130, 132, 133, 134, $\cdots$	1334	-
5	1, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 42, 43, 44, 45, 46, 48, 49, 50, 51, 52, 54, 55, 56, 57, 58, 60, 61, 62, 63, 64, 66, 67, 68, 69, 70, 73, 74, 75, 76, 77, 79, 80, 81, 82, 83, 85, 86, 87, 88, 89, 91, 92, 93, 94, 95, 97, 98, 99, 100, 101, 104, 105, 106, 107, 108, 110, 111, 112, 113, 114, 116, 117, 118, 119, 120, 122, 123, 124, 125, $\cdots$	1251	-

Table 3: sum of exclusive heights of *complete* k-ary trees: *sumh*$'(C_k(n))$

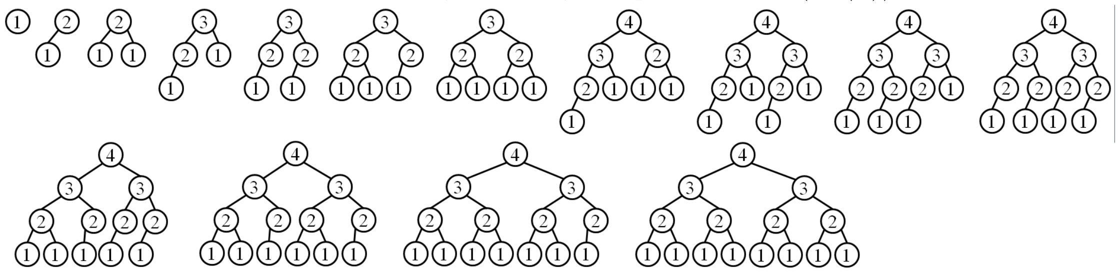
k	Integer sequence for $n = 1, \ldots, 100$	$n = 1000$	OEIS
1	0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225, 1275, $\cdots$	499500	A000217
2	0, 1, 1, 3, 3, 4, 4, 7, 7, 8, 8, 10, 10, 11, 11, 15, 15, 16, 16, 18, 18, 19, 19, 22, 22, 23, 23, 25, 25, 26, 26, 31, 31, 32, 32, 34, 34, 35, 35, 38, 38, 39, 39, 41, 41, 42, 42, 46, 46, 47, 47, 49, 49, 50, 50, 53, 53, 54, 54, 56, 56, 57, 57, 63, 63, 64, 64, 66, 66, 67, 67, 70, 70, 71, 71, 73, 73, 74, 74, 78, 78, 79, 79, 81, 81, 82, 82, 85, 85, 86, 86, 88, 88, 89, 89, 94, 94, 95, 95, 97, $\cdots$	994	A011371
3	0, 1, 1, 1, 3, 3, 3, 4, 4, 4, 5, 5, 5, 8, 8, 8, 9, 9, 9, 10, 10, 10, 12, 12, 12, 13, 13, 13, 14, 14, 14, 16, 16, 16, 17, 17, 17, 18, 18, 18, 22, 22, 22, 23, 23, 23, 24, 24, 24, 26, 26, 26, 27, 27, 27, 28, 28, 28, 30, 30, 30, 31, 31, 31, 32, 32, 32, 35, 35, 35, 36, 36, 36, 37, 37, 37, 39, 39, 39, 40, 40, 40, 41, 41, 41, 43, 43, 43, 44, 44, 44, 45, 45, 45, 48, 48, 48, 49, 49, 49, $\cdots$	498	-
4	0, 1, 1, 1, 1, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 9, 9, 9, 9, 10, 10, 10, 10, 11, 11, 11, 11, 12, 12, 12, 12, 14, 14, 14, 14, 15, 15, 15, 15, 16, 16, 16, 16, 17, 17, 17, 17, 19, 19, 19, 19, 20, 20, 20, 20, 21, 21, 21, 21, 22, 22, 22, 22, 24, 24, 24, 24, 25, 25, 25, 25, 26, 26, 26, 26, 27, 27, 27, 27, 31, 31, 31, 31, 32, 32, 32, 32, 33, 33, 33, 33, 34, 34, 34, $\cdots$	334	-



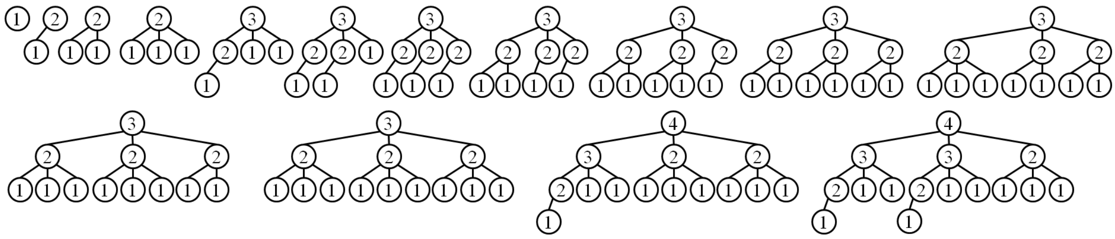
(a) *complete* binary tree integer sequence, $sumh(C_2(n))$



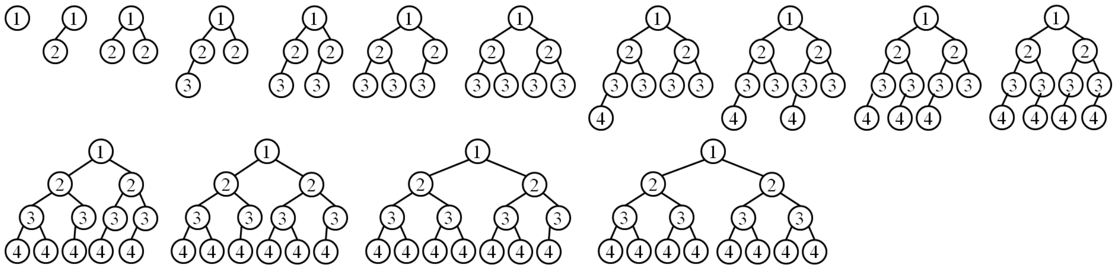
(b) *complete* ternary tree integer sequence, $sumh(C_3(n))$



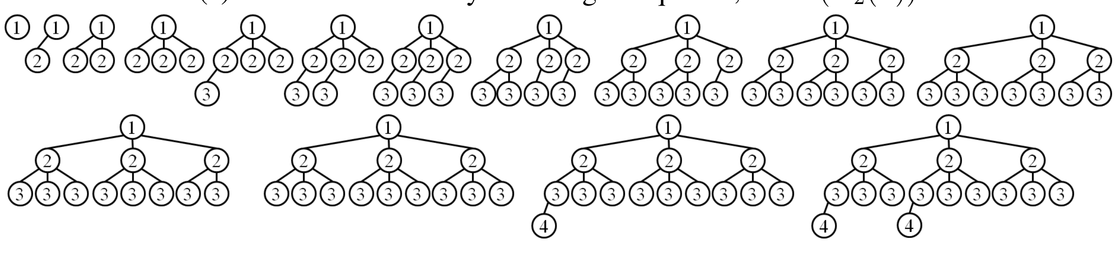
(c) *size-balanced* binary tree integer sequence, $sumh(Z_2(n))$



(d) *size-balanced* ternary tree integer sequence, $sumh(Z_3(n))$



(e) *null-balanced* binary tree integer sequence, $sumd(N_2(n))$



(f) *null-balanced* ternary tree integer sequence, $sumd(N_3(n))$

Figure 5: various *balanced* *k*-ary tree Integer Sequences

Table 4: sum of inclusive heights of *size balanced* k-ary trees: *sumh*$(Z_k(n))$

k	Integer sequence for $n = 1, \ldots, 100$	$n = 1000$	OEIS
2	1, 3, 4, 7, 9, 10, 11, 15, 18, 20, 22, 23, 24, 25, 26, 31, 35, 38, 41, 43, 45, 47, 49, 50, 51, 52, 53, 54, 55, 56, 57, 63, 68, 72, 76, 79, 82, 85, 88, 90, 92, 94, 96, 98, 100, 102, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 127, 133, 138, 143, 147, 151, 155, 159, 162, 165, 168, 171, 174, 177, 180, 183, 185, 187, 189, 191, 193, 195, 197, 199, 201, 203, 205, 207, 209, 211, 213, 215, 216, 217, 218, 219, 220, $\cdots$	2013	-
3	1, 3, 4, 5, 8, 10, 12, 13, 14, 15, 16, 17, 18, 22, 25, 28, 30, 32, 34, 36, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 63, 67, 71, 74, 77, 80, 83, 86, 89, 91, 93, 95, 97, 99, 101, 103, 105, 107, 109, 111, 113, 115, 117, 119, 121, 123, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154, 155, 156, 157, 158, $\cdots$	1543	-

Table 5: sum of exclusive heights of *size balanced* k-ary trees: *sumh*$'(Z_k(n))$

k	Integer sequence for $n = 1, \ldots, 100$	$n = 1000$	OEIS
2	0, 1, 1, 3, 4, 4, 4, 7, 9, 10, 11, 11, 11, 11, 11, 15, 18, 20, 22, 23, 24, 25, 26, 26, 26, 26, 26, 26, 26, 26, 26, 31, 35, 38, 41, 43, 45, 47, 49, 50, 51, 52, 53, 54, 55, 56, 57, 57, 57, 57, 57, 57, 57, 57, 57, 57, 57, 57, 57, 57, 57, 57, 57, 63, 68, 72, 76, 79, 82, 85, 88, 90, 92, 94, 96, 98, 100, 102, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 120, 120, 120, 120, 120, $\cdots$	1013	-
3	0, 1, 1, 1, 3, 4, 5, 5, 5, 5, 5, 5, 5, 8, 10, 12, 13, 14, 15, 16, 17, 18, 18, 18, 18, 18, 18, 18, 18, 18, 18, 18, 18, 18, 18, 18, 18, 18, 18, 18, 22, 25, 28, 30, 32, 34, 36, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, 58, $\cdots$	543	-

Table 6: sum of inclusive depths of *null balanced* k-ary trees: *sumd*$(N_k(n))$

k	Integer sequence for $n = 1, \ldots, 100$	$n = 1000$	OEIS
2	1, 3, 5, 8, 11, 14, 17, 21, 25, 29, 33, 37, 41, 45, 49, 54, 59, 64, 69, 74, 79, 84, 89, 94, 99, 104, 109, 114, 119, 124, 129, 135, 141, 147, 153, 159, 165, 171, 177, 183, 189, 195, 201, 207, 213, 219, 225, 231, 237, 243, 249, 255, 261, 267, 273, 279, 285, 291, 297, 303, 309, 315, 321, 328, 335, 342, 349, 356, 363, 370, 377, 384, 391, 398, 405, 412, 419, 426, 433, 440, 447, 454, 461, 468, 475, 482, 489, 496, 503, 510, 517, 524, 531, 538, 545, 552, 559, 566, 573, 580, $\cdots$	8987	A001855
3	1, 3, 5, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98, 102, 106, 110, 114, 118, 122, 126, 130, 134, 138, 142, 147, 152, 157, 162, 167, 172, 177, 182, 187, 192, 197, 202, 207, 212, 217, 222, 227, 232, 237, 242, 247, 252, 257, 262, 267, 272, 277, 282, 287, 292, 297, 302, 307, 312, 317, 322, 327, 332, 337, 342, 347, 352, 357, 362, 367, 372, 377, 382, 387, 392, 397, 402, 407, 412, 417, 422, 427, 432, 437, 442, $\cdots$	6457	-

Table 7: sum of exclusive depths of *null balanced* k-ary trees: *sumd*$'(N_k(n))$

k	Integer sequence for $n = 1, \ldots, 100$	$n = 1000$	OEIS
2	0, 1, 2, 4, 6, 8, 10, 13, 16, 19, 22, 25, 28, 31, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98, 103, 108, 113, 118, 123, 128, 133, 138, 143, 148, 153, 158, 163, 168, 173, 178, 183, 188, 193, 198, 203, 208, 213, 218, 223, 228, 233, 238, 243, 248, 253, 258, 264, 270, 276, 282, 288, 294, 300, 306, 312, 318, 324, 330, 336, 342, 348, 354, 360, 366, 372, 378, 384, 390, 396, 402, 408, 414, 420, 426, 432, 438, 444, 450, 456, 462, 468, 474, 480, $\cdots$	7987	A061168
3	0, 1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69, 72, 75, 78, 81, 84, 87, 90, 93, 96, 99, 102, 106, 110, 114, 118, 122, 126, 130, 134, 138, 142, 146, 150, 154, 158, 162, 166, 170, 174, 178, 182, 186, 190, 194, 198, 202, 206, 210, 214, 218, 222, 226, 230, 234, 238, 242, 246, 250, 254, 258, 262, 266, 270, 274, 278, 282, 286, 290, 294, 298, 302, 306, 310, 314, 318, 322, 326, 330, 334, 338, 342, $\cdots$	5457	-