Weighted Catalan Numbers and Their Divisibility Properties

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Abstract

The weighted Catalan numbers, like the Catalan numbers, enumerate various mathematical objects. For example, the number of Morse links with n critical points is the n^{th} weighted Catalan number, L_n, with weights $1^2, 3^2, 5^2, \ldots, (2k+1)^2, \ldots$. This paper examines the conjecture made by Postnikov which involves the divisibility of L_n by powers of 3. An upper bound of $2 \cdot 3^{2r-7}$ on the period of L_n modulo 3^r is given, which supports Postnikov's conjecture that this period is $2 \cdot 3^{r-3}$.

The results are proven by representing L_n using combinatorial structures called Dyck paths. Dyck paths of length n are partitioned into sets according to the location of the steps corresponding to the weights divisible by 3^2. By noting the location of the steps corresponding to weights divisible by 3^2, an arbitrary Dyck path can be decomposed into several pieces called partial flat paths. Properties of partial flat paths within a certain structure are proven and lead to the main result. These ideas are extended to deal with partial 3-flat paths, which are slightly more complex, and general weighted Catalan numbers modulo p^r for a prime p. This paper shows that if the weights $B = b_0, b_1, b_2, \ldots$ satisfy certain algebraic properties, then the corresponding weighted Catalan numbers $C_n^{(B)}$ are periodic modulo p^r and gives an upper bound for the period.

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1 Introduction

Historically, mathematicians and scientists have been interested in objects with special properties which arise in their studies; for example, the subdivisions of a polygon into triangles. To better understand these objects, one of the first questions asked is: how many of these objects are there? This is one of the main questions in the mathematical area of combinatorics. For example, Euler asked: how many ways can an n-gon be divided into triangles [Gri12]? For $n = 5$, the possibilities are shown below in Figure 1.

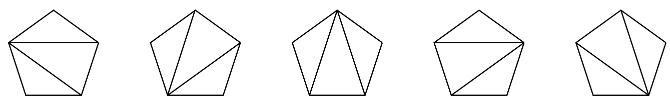

Figure 1: The five ways to divide a pentagon into triangles.

Also, Catalan wondered how many ways one could compute the product of n numbers [Gri12]. The possibilities for $n = 4$ are shown in Figure 2.

$$(ab)(cd)$$
$$(a(bc))d$$
$$((ab)c)d$$
$$a(b(cd))$$
$$a((bc)d)$$

Figure 2: The five ways to compute a product of four numbers.

As it turns out, both of these objects, along with more than one hundred others (see for example [Dav]), are given by the famous integer sequence, the Catalan numbers, $C_n = 1, 2, 5, 14, 42, \ldots$. See [OEI01a] for more details.

Recently, mathematicians have been interested in counting the number of kinds of *Morse links*, or sets of disjoint loops with critical points at different heights, with n critical points (see Figure 3). In [Pos00], Postnikov shows that the Morse links are counted by an integer sequence $L_n = 1, 10, 325, 22150, \ldots$, called the *weighted Catalan numbers*, which generalize the Catalan numbers [OEI01b].

The weighted Catalan numbers are an integer sequence which satisfies a recurrence similar to that of the Catalan numbers. They can be formed using any weights $B = b_0, b_1, b_2, \ldots$. In this paper, I primarily consider the weighted Catalan numbers, L_n, with the weights $1^2, 3^2, 5^2, \ldots, (2k+1)^2, \ldots$.

Divisibility properties concerning the number of objects of a given type illuminate patterns in the objects themselves, or provide ways to efficiently construct all the objects. Therefore, math-

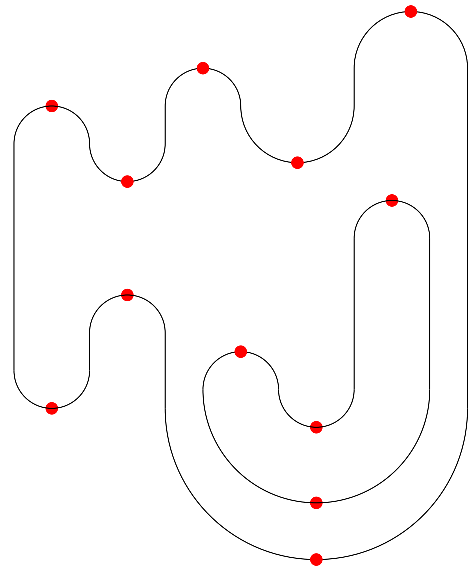

Figure 3: A morse link with 12 critical points

ematicians are interested in the divisibility properties of the weighted Catalan numbers L_n. In [Pos00], Postnikov and Sagan determine the highest power of 2 that divides L_n for each n, and based on this, suggest examining the divisibility properties of L_n with respect to powers of 3. Postnikov conjectures that L_n has a period of $2 \cdot 3^{r-3}$ modulo 3^r, meaning that $2 \cdot 3^{r-3}$ is the smallest positive integer such that $L_n - L_{n+2\cdot 3^{r-3}}$ is a multiple of 3^r for sufficiently large n.

This paper provides a theoretical result that supports Postnikov's conjecture. Namely, it proves that L_n is periodic modulo 3^r with its period dividing $2 \cdot 3^{2r-7}$. The proof also provides insight as to why L_n is periodic modulo 3^r.

In addition, this paper proves that the weights resulting from certain cases within L_n are periodic with a period of exactly $2 \cdot 3^{r-3}$, which is the overall period conjectured by Postnikov. This second result is a potential path to a complete resolution of Postnikov's conjecture.

Section 2 describes the definitions and methods that are used to prove these results. The main results are proven in Section 3 and several of these techniques are extended to more general cases of weighted Catalan numbers in Section 4. The implications of these results are then discussed in Section 5.

2 Concepts and Methods Used

We begin with a conceptual overview. First, certain combinatorial objects called Dyck paths are used to represent L_n. Then we show that Dyck paths can be decomposed into pieces called partial flat paths using a process called partial flat path decomposition. Using partial flat path decomposition, we obtain a skeleton for a path, and we can uniquely express a Dyck path in terms of partial flat paths and its skeleton. We prove divisibility properties about the number of Dyck paths with a given skeleton and partial flat paths using proof techniques from enumerative combinatorics such as generating functions, bijections, and Euler's Theorem. Then we can classify paths based on their skeletons and use results proven about these partial flat paths to prove the main results for L_n.

2.1 Dyck Paths

The weighted Catalan numbers can be represented by objects called Dyck paths. A *Dyck path* p of length n is a path starting and ending at height 0 formed from n "up-steps" of $(1, 1)$ and n "down-steps" of $(1, -1)$ such that the path does not cross the x-axis. A Dyck path of length 3 is shown below in Figure 4.

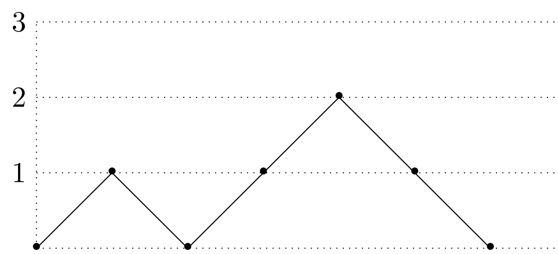

Figure 4: A Dyck path of length 3.

In order to obtain the weighted Catalan numbers, weights are assigned to each Dyck path. The weight of an up-step starting at height k is defined to be $(2k + 1)^2$ for L_n.

The *weight* $w(p)$ of a Dyck path p is the product of the weights assigned to the starting height of each up-step in the Dyck path. This is illustrated in Figure 5, where up-steps are shown in red.

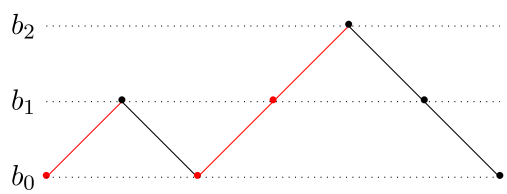

Figure 5: The weight of this Dyck path of length 3, where up-steps are shown in red, is given by $w(p) = b_0^2 b_1 = (1)^2 \cdot (9) = 9$.

The n^{th} *weighted Catalan number* with the weights $1^2, 3^2, 5^2, \ldots, (2k + 1)^2, \ldots$ is denoted L_n and is the sum of the weights $w(p)$ over all Dyck paths p of length n.

2.2 Partial Flat Path Decomposition

It is also useful for us to consider special Dyck paths which have a maximum height of at most 2. We call such a path a *flat path* of length n. We also consider *partial flat paths*, which are paths that start at height 0 or height 2 and end at height 0 or 2 with a maximum height of at most 2.

We can decompose each Dyck path by marking all up-steps or down-steps at heights $0, 1, 4, 7, 10, \ldots$. We note that the weights corresponding to these heights are equal to 1 or divisible by 3 (in fact, by 3^2). This breaks the Dyck path into partial flat paths. We call this process *partial flat path decomposition* of p, and we call the sequence of marked steps the *skeleton* of the Dyck path. An example of the skeleton of a Dyck path is shown in Figure 6.

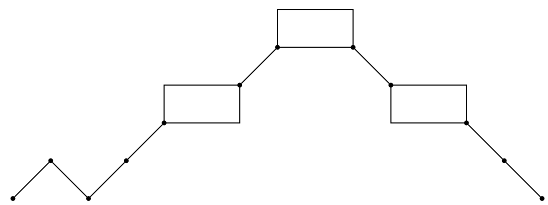

Figure 6: The skeleton of a Dyck path with partial flat paths in each box.

We can now think of each Dyck path as being uniquely formed from a series of partial flat paths and its skeleton, and we can also partition the Dyck paths into sets of paths with the same skeleton.

A special type of skeleton arises in our study: a Dyck path has a *simple skeleton* if its skeleton only contains steps that correspond only to the weight 1 and the weight 3^2. A Dyck path with a simple skeleton is shown in Figure 7.

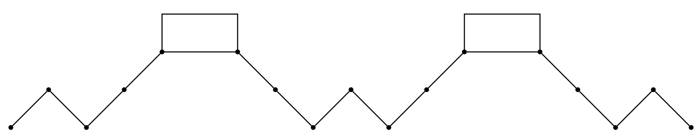

Figure 7: A Dyck path with a simple skeleton with partial flat paths in each box.

In Section 3 we will analyze the divisibility properties of the weights of partial flat paths, which allow us to prove results about the divisibility properties of L_n.

2.3 Mathematical Tools

In order to prove results about partial flat paths and L_n, we use mathematical tools from the area of enumerative combinatorics including generating functions, bijections, and Euler's Theorem.

First of all, we are concerned with the periodicity of the remainder of L_n when divided by 3^r. Two numbers a and b are *equivalent modulo* 3^r if their difference is divisible by 3^r and we write this as $a \equiv b \pmod{3^r}$. The sequence of numbers $S_n = s_0, s_1, s_2, \ldots$ is *periodic* modulo 3^r if there exists a positive integer p such that $s_k \equiv s_{k+p} \pmod{3^r}$ for all k sufficiently large. Additionally, we say that S_n has a period of p modulo 3^r if p is the smallest positive integer such that S_n is periodic.

According to Euler's Theorem [Gal13], if x is an integer which is relatively prime to 3^r, then the sequence $x, x^2, x^3, \ldots$ is periodic modulo 3^r with a period that divides $2 \cdot 3^{r-1}$. If the sequence corresponding to x has a period of exactly $2 \cdot 3^{r-1}$, then x is a *primitive root* modulo 3^r. For example, 2 is always a primitive root modulo 3^r [Gal13].

We are also interested in enumerating mathematical objects, such as flat paths. However, it may be hard to directly count these objects. Oftentimes, mathematicians take a set of objects that are harder to count and give a one-to-one correspondence to another set of objects which is easier to count. This correspondence is known as a *bijection* [Cam95]. These bijections make it easier for us to count certain objects by allowing us to count a set of different objects, which are often easier to count, with the same number of objects in both cases. In Lemma 3.1, we show that there is a one-to-one correspondence between flat paths and strings of binary digits.

Another very useful mathematical tool used is the generating function [Cam95]. The *generating function* for a sequence $S_n = s_0, s_1, s_2, \ldots$ is the formal infinite series $f(x) = s_0 + s_1x + s_2x^2 + \cdots + s_nx^n + \cdots$. Algebraic and analytic tools can be applied to the generating function to shed light on the sequence S_n. In Theorem 2, we study the generating function of the sequence S_n which enumerates the sum of the weights of the Dyck paths of length n with a certain skeleton. Using algebra, Euler's Theorem, and known results of the periodicity of binomial coefficients, we establish the main result.

3 Results

I prove two main results. The first one provides an upper bound for the period of L_n modulo 3^r, which, to the best of my knowledge and that of my mentor as well as a thorough literature review, seems to be the first theoretical result of its kind. It supports Postnikov's conjecture by showing that L_n is periodic modulo 3^r with a period dividing $2 \cdot 3^{2r-7}$. The second theorem proves that the sum of the weights of a certain class of Dyck paths has a period of exactly $2 \cdot 3^{r-3}$ modulo 3^r providing us with a potential way to prove Postnikov's conjecture.

Before we can prove the main results, we first establish some properties about flat paths.

Lemma 3.1. *The number of flat paths of length n is 2^{n-1}.*

Proof. We construct a bijection between binary strings of length $n-1$ and flat paths of length n as follows: Given a flat path of length n, we take two steps. If the height is 0, then the corresponding binary digit is 0. Otherwise, the corresponding binary digit is 1. This will give us a binary string of length n ending in 0, since Dyck paths end at height 0. Therefore, we can remove the final zero, and we have a correspondence between flat paths of length n and binary strings of length $n-1$.

Using the fact that Dyck paths start and end at height 0, we can see in Figure 3 that the binary digits uniquely determine the two corresponding steps in the path.

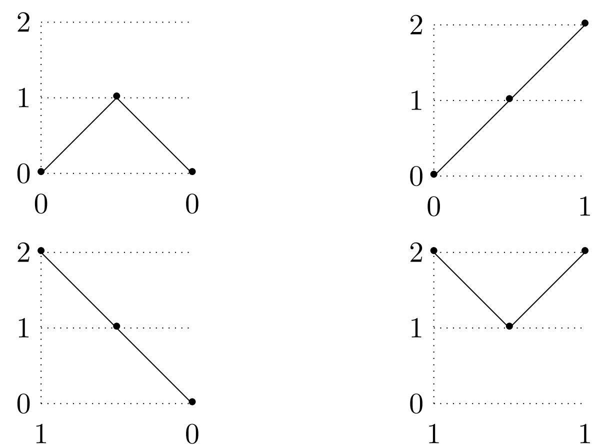

Figure 8: This indicates the two steps in a flat path that must be taken given any two consecutive binary digits in its corresponding binary string.

Therefore, the number of flat paths of length n is equal to the number of binary strings of length $n-1$, so there are 2^{n-1} flat paths of length n. $\square$

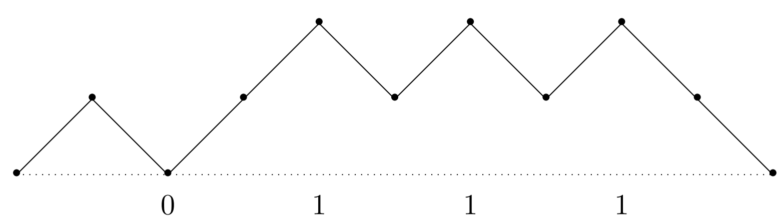

Figure 9: A flat path of length 5 corresponding to the binary string 0111.

Next we will establish a connection between the weight of a flat path and its corresponding binary string.

Lemma 3.2. *A flat path p of length n with weights $B = b_0, b_1$ with a corresponding binary string of $s_1\, s_2\, s_3\, \cdots s_{n-1}$ has the weight $b_0 b_{s_1} b_{s_2} \cdots b_{s_{n-1}}$.*

Proof. We prove this by induction on n.

We can see that for $n = 1$ we have a binary string of length zero and a path with weight b_0.

Assume that this statement holds for a flat path p of length n with a corresponding binary string of length $n-1$ and proceed by induction. We insert a digit at the beginning of the string and obtain two cases:

Case 1: 0 is inserted at the beginning of the string.

This is shown in Figure 10.

The weight of this flat path of length $n+1$ is $b_0 \cdot w(p)$ as desired.

Case 2: 1 is inserted at the beginning of the string.

This is shown in Figure 11.

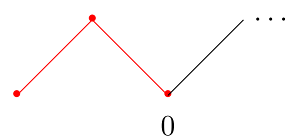

Figure 10: Inserting 0 in the binary string for a flat path.

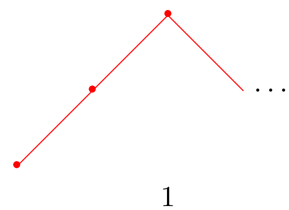

Figure 11: Inserting 1 in the binary string for a flat path.

We can view this as replacing the first step of a flat path of length n, which has a weight of b_0, with the three steps in red pictured above. Therefore, we get that the weight of this flat path of length $n+1$ is $b_0 b_1 \cdot \frac{w(p)}{b_0} = b_1 \cdot w(p)$ as desired.
Thus by induction, the lemma holds for all n. $\square$

We now combine the results from Lemmas 3.1 and 3.2 to obtain an expression for the sum of the weights for all flat paths of length n.

Corollary 3.3. *The sum of the weights $T_n^{(B)}$ for all flat paths of length n with weights $B = b_0, b_1$ is given by: $T_0^{(B)} = 1$ and $T_n^{(B)} = b_0(b_0 + b_1)^{n-1}$ for $n \geq 1$.*

Proof. From Lemmas 3.1 and 3.2 we have that the sum of the weights of all flat paths of length n is

$$\sum b_0 b_{s_1} b_{s_2} \dots b_{s_{n-1}}$$

for all possible binary strings $s_1 s_2 s_3 \dots s_{n-1}$.

Therefore, this sum is equal to $b_0(b_0 + b_1)(b_0 + b_1) \cdots (b_0 + b_1) = b_0(b_0 + b_1)^{n-1}$. $\square$

We extend these ideas to partial flat paths and obtain a similar expression for the sum of the weights for all partial flat paths of length n.

Proposition 3.4. *The sum of the weights of all partial flat paths of length $n \geq 1$ with the same starting and ending heights is:*

- $b_0(b_0 + b_1)^{n-1}$ *for paths starting and ending at height* 0,
- $b_1(b_1 + b_0)^{n-1}$ *for paths starting and ending at height* 2,
- $(b_0 + b_1)^n$ *for paths starting at height* 2 *and ending at height* 0,

- *$b_0 b_1 (b_0 + b_1)^{n-2}$ for paths starting at height 0 and ending at height 2.*

Proof. We examine partial flat paths by cases.

Case 1: The path starts and ends at height 0.

Corollary 3.3 handles this case.

Case 2: The path starts and ends at height 2.

If we have a partial flat path with length n that starts at height 2 and ends at height 2, then we can consider this as an upside down flat path of length n with the weights b_1, b_0. This gives that the sum of all such partial flat paths of length n is $b_1(b_1 + b_0)^{n-1}$ by Corollary 3.3. Note, for $n = 0$ this sum is defined to be 1.

Case 3: The path starts at height 2 and ends at height 0.

If we have a partial flat path of length n that starts at height 2 and ends at height 0, then we consider this as steps 3 through $2n + 2$ of a path of length $n + 1$ with the first digit of its corresponding binary string equal to 1. The sum of the weights of these paths is equal to $(b_0 + b_1)^n$.

Case 4: The path starts at height 0 and ends at height 2.

Finally, if we have a partial flat path of length n that starts at height 0 and ends at height 2, then we consider this as all but the last two steps of a flat path of length n with a last digit of 1. This gives the sum of the weights of these paths is $b_0 b_1 (b_0 + b_1)^{n-2}$. $\square$

We now prove our main result.

Theorem 3.5. *The period of L_n modulo 3^r divides $2 \cdot 3^{2r-7}$.*

Proof. We partition the Dyck paths of length n into sets of Dyck paths which have the same skeleton.

Consider a skeleton containing f steps corresponding to weights divisible by 3^2. If $f > r$, then the weight of any Dyck path with this skeleton will be divisible by 3^f, and it will be equivalent to 0 modulo 3^r. Thus, we will only consider skeletons where $f < r$.

Note that a skeleton with f steps corresponding to weights divisible by 3^2 divides the path into $f - 1$ partial flat paths.

We have that the weight of all paths of length n with this skeleton is equal to

$$C_1 \cdot 3^f \sum T_{i_1}^{(B_1)} T_{i_2}^{(B_2)} T_{i_3}^{(B_3)} \cdots T_{i_{f-1}}^{(B_{f-1})}$$

for some constant C_1 and with $i_1 + i_2 + i_3 + \cdots + i_{f-1} = m$ where $m = n - d$, and d is a positive integer dependent on the skeleton.

By Proposition 3.4, this is equal to

$$C \cdot 3^f \sum z_1^{n_1} z_2^{n_2} z_3^{n_3} \cdots z_{f-1}^{n_{f-1}}$$

for a constant C where z_j is the sum of the first two weights of B_j and $n_j = i_j - 1$.

Next we consider the generating function g corresponding to S_n, the sum of the weights of all paths of length n with this skeleton. Note that:

$$g(x) = C \cdot 3^f (1 + z_1 x + z_1^2 x^2 + \cdots)(1 + z_2 x + z_2^2 x^2 + \cdots) \cdots (1 + z_{f-1} x + z_{f-1}^2 x^2 + \cdots).$$

Considering $g(x)$ modulo 3^r is equivalent to examining

$$h(x) = C(1 + z_1 x + z_1^2 x^2 + \cdots)(1 + z_2 x + z_2^2 x^2 + \cdots) \cdots (1 + z_{f-1} x + z_{f-1}^2 x^2 + \cdots)$$

modulo 3^{r-f}.

By Euler's theorem, we know that for each z_i, the sequence $z_i, z_i^2, z_i^3, \ldots$ is periodic modulo 3^{r-f} with a period dividing $k = 2 \cdot 3^{r-f-1}$. Therefore, $z_i^a \equiv z_i^{a+k} \pmod{3^{r-f}}$ for all a so

$$(1 + z_i x + z_i^2 x^2 + \cdots) \equiv \sum_t x^{kt}(1 + z_i x + z_i^2 x^2 + \cdots + z_i^{k-1} x^{k-1}) \pmod{3^{r-f}}.$$

Therefore,

$$h(x) \equiv C(\sum_t x^{kt})^{f-1} \prod_{i=1}^{f-1} (1 + z_i x + z_i^2 x^2 + \cdots + z_i^{k-1} x^{k-1}) \pmod{3^{r-f}}. \tag{1}$$

Let $q(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{(f-1)(k-1)} x^{(f-1)(k-1)}$ be the polynomial equal to

$$\prod_{i=1}^{f-1} \left(1 + z_i x + z_i^2 x^2 + \cdots + z_i^{k-1} x^{k-1}\right).$$

We can write the coefficient of x^{kt} in $(1 + x^k + x^{2k} + \cdots)^{f-1}$ as the number of ways to partition t into $f-1$ pieces, or

$$\binom{f+t-2}{f-2}.$$

Thus,

$$h(x) \equiv \sum_t \binom{f+t-2}{f-2} x^{kt} \cdot q(x) \pmod{3^{r-f}}.$$

If $f \leq 2$, $\binom{f+t-2}{f-2}$ is a constant. Otherwise, by [Kwo89], we know that $\binom{f+t-2}{f-2}$ is periodic modulo 3^{r-f} with period $3^{r-f+\lfloor \log_3(f-2) \rfloor}$.

Thus, when we multiply out $h(x)$, we get a sum of polynomials where each polynomial is periodic modulo 3^{r-f} with a period of $k \cdot 3^{r-f+\lfloor \log_3(f-2) \rfloor}$. Therefore, $h(x)$ is periodic modulo 3^{r-f} with a period dividing $k \cdot 3^{r-f+\lfloor \log_3(f-2) \rfloor}$. Since $k = 2 \cdot 3^{r-f-1}$, we get that this period is $2 \cdot 3^{2r-2f+\lfloor \log_3(f-2) \rfloor - 1}$. We note that for $f \geq 3$ we have $2f - \lfloor \log_3(f-2) \rfloor + 1 \geq 7$. Therefore, $2 \cdot 3^{2r-2f+\lfloor \log_3(f-2) \rfloor - 1}$ is at most $2 \cdot 3^{2r-7}$.

Therefore L_n is periodic modulo 3^r with a period that divides $2 \cdot 3^{2r-7}$. $\square$

Now we study the algebraic properties of these z_i in the generating function above so that we can obtain an exact expression for the period of the sum of the weights of Dyck paths with certain skeletons.

Each z_i corresponds to the sum of the weights b_0 and b_1 corresponding to a partial flat path in the Dyck path. By the construction of the skeleton, b_0 and b_1 are consecutive weights that are not divisible by 3. Therefore, $z_i = (6j-1)^2 + (6j+1)^2$ for some integer j, so $z_i \equiv 2 \pmod 9$. We will use this to show that z_i is a primitive root modulo 3^r.

Proposition 3.6. *If $z = (6j-1)^2 + (6j+1)^2$ for some integer j, then $z^{2\cdot 3^{r-1}} \equiv 1 + 3^r \pmod{3^{r+1}}$.*

Proof. The proof is by induction on r.

Since $z \equiv 2 \pmod 9$, we see the statement is true for $r = 1$.

Now assume that $z^{2\cdot 3^{k-1}} \equiv 1 + 3^k \pmod{3^{k+1}}$.

Then $z^{2\cdot 3^{k-1}} = 1 + 3^k + 3^{k+1}m$ for some integer m. Cubing this expression gives $z^{2\cdot 3^k} = 1 + 3^{k+1} + 3^{k+2}M$ for some integer M. Therefore, $z^{2\cdot 3^k} \equiv 1 + 3^{k+1} \pmod{3^{k+2}}$ and by induction the proposition is true for all r. □

Proposition 3.7. *Let $z = (6j-1)^2 + (6j+1)^2$ for some integer j. Then the sequence $Z = z, z^2, z^3, \ldots$ has period $2 \cdot 3^{r-1}$ modulo 3^r.*

Proof. We prove this by induction on r.

For $r = 1$ we see that $z \equiv 2 \pmod 3$ and has period 2.

Now assume that $z, z^2, z^3, \ldots$ has period $2 \cdot 3^{k-1}$ modulo 3^k. This implies that $2 \cdot 3^{k-1}$ divides the period of Z modulo 3^{k+1}. Additionally, by Euler's theorem, the period of Z modulo 3^{k+1} must also divide $2 \cdot 3^k$. Therefore, the period is either $2 \cdot 3^{k-1}$ or $2 \cdot 3^k$. By Proposition 3.6, we get that the period must be $2 \cdot 3^k$, so by induction Proposition 3.7 holds. □

We can now prove that for a certain class of Dyck paths, the sum of their weights is periodic modulo 3^r with a period of exactly $2 \cdot 3^{r-3}$.

Theorem 3.8. *The sum of the weights of all paths of length n with simple skeletons is periodic modulo 3^r with period $2 \cdot 3^{r-3}$.*

Proof. As in Theorem 3.5, the generating function corresponding to the weights of all Dyck paths with a simple skeleton with f steps corresponding to weights divisible by 3^2 has form

$$g(x) = C \cdot 3^f (1 + z_1 x + z_1^2 x^2 + \cdots)(1 + z_2 x + z_2^2 x^2 + \cdots) \cdots (1 + z_{f-1} x + z_{f-1}^2 x^2 + \cdots).$$

Since this is a simple skeleton, all of these z_i are equal and additionally each z_i is equal to the sum of the second and third weights, namely $z_i = 5^2 + 7^2 = 74$. Therefore,

$$g(x) = C \cdot 3^f (1 + 74x + 74^2 x^2 + \cdots)^{f-1}.$$

As in the proof of Theorem 3.5, we can simplify this to examining $h(x)$ modulo 3^{r-f} where

$$h(x) = C\sum_t \binom{f+t-2}{f-2} 74^t x^t.$$

From [Kwo89], we know that $\binom{f+t-2}{f-2}$ is constant for $f \leq 2$ and is periodic modulo 3^{r-f} with period $3^{r-f+\lfloor \log_3(f-2) \rfloor}$ for $f \geq 3$. By Proposition 3.7, we know that 74 is a primitive root modulo 3^{r-f} so 74^t has a period of $2 \cdot 3^{r-f-1}$ modulo 3^{r-f}. Therefore, the coefficients of $h(x)$ are periodic modulo 3^{r-f} with a period dividing $2 \cdot 3^{r-f+\lfloor \log_3(f-2) \rfloor}$. Since f is certainly even, we see that for $f > 2$, this period that properly divides $2 \cdot 3^{r-3}$.

For the case $f = 2$, we can construct a path of length n by taking a path of maximum height 1 of length $n-1-i$ and insert one step of weight 3^2 and a flat path of length i, p_i. Therefore the weight of all of these paths is:

$$F_n = 3^2 \sum_{i=0}^{n-2} (n-1-i) T_i^{(B)},$$

where $T_i^{(B)}$ is the sum of the weights of all flat paths of length n with the weights $B = 5^2, 7^2$ as in Corollary 3.3.

This expression is equivalent to

$$F_n = 9\left[n - 1 + 25\sum_{i=1}^{n-2}(n-1-i) \cdot 74^{i-1}\right]. \tag{2}$$

Let $A = \sum_{i=1}^{n-2}(n-1-i) \cdot 74^{i-1}$. Multiplying by 74 and subtracting gives us:

$$73A = 1 - n + 1 + 74 + 74^2 + \cdots + 74^{n-2},$$

which implies that

$$A = \frac{1-n}{73} + \frac{74^{n-1}-1}{73^2}.$$

Substituting A into (2) gives

$$\begin{aligned} F_n &= 9(n - 1 + \tfrac{25(1-n)}{73} + \tfrac{25(74^{n-1}-1)}{73^2}) \\ &= 9(\tfrac{48(n-1)}{73} + \tfrac{25(74^{n-1}-1)}{73^2}) \\ &= 9(\tfrac{48 \cdot 73(n-1) + 25(74^{n-1}-1)}{73^2}). \end{aligned}$$

Since $3 \nmid 73$, we know 73^2 and 3^r are relatively prime, and thus we know that there is an integer a such that $a \cdot 73^2 \equiv 1 \pmod{3^r}$. Therefore,

$$F_n \equiv 9 \cdot a(48 \cdot 73(n-1) + 25(74^{n-1} - 1)) \pmod{3^r}.$$

We know $n-1$ has period 3^k modulo 3^k, so $9 \cdot 48(n-1)$ has period 3^{r-3}. By Proposition 3.7, we know that 74^{n-1} has period $2 \cdot 3^{k-1}$ modulo 3^k, so $9 \cdot (25(74^{n-1} - 1))$ has period $2 \cdot 3^{r-3}$.

Therefore, F_n has period $2 \cdot 3^{r-3}$ modulo 3^r. This means that the weights of all paths of length n with simple skeletons is periodic modulo 3^r with period $2 \cdot 3^{r-3}$. □

4 Extensions

In the previous sections, we derived results for L_n modulo 3^r using flat paths. In this section, we derive some results that extend the notion of flat paths, and we also consider the periodicity of other weighted Catalan numbers modulo p^r for other primes p.

In this section, we prove results about 3-*flat paths*, which are Dyck paths with a maximum height of at most 3.

We will proceed by showing that a sequence satisfying a certain two-term recurrence is periodic modulo p^r for a prime p and then show that the sum of the weights of all 3-flat paths and partial 3-flat paths of length n satisfy an appropriate two-term recurrence. We will combine these results to prove a result similar to the main result but for more general cases.

Theorem 4.1. *Let p be an odd prime and let S_n be an integer sequence satisfying the recurrence $S_n = aS_{n-1} + bS_{n-2}$ for $n \geq 2$ with S_0 and S_1 given. If $p \nmid a$, $p \nmid b$, and $p \nmid a^2 + 4b$, then S_n is periodic modulo p^r with a period dividing $k = (p-1)(p+1) \cdot p^{2r-1}$.*

Proof. Let us consider the ring $R = \mathbb{Z}_{p^r}$ of integers modulo p^r. Assume within the ring $S_n = s^n$, or equivalently $S_n \equiv s^n \pmod{p^r}$ for some ring element s.

From the recurrence, we have s must satisfy $s^2 = as + b$. We would expect from our knowledge of the reals that $s = \frac{a \pm \sqrt{a^2+4b}}{2}$ from the quadratic equation. However, since we are working in the ring R, we must be more careful, but we can still let this notion guide our thinking.

Note: In R, if d is relatively prime to p, then d has a multiplicative inverse d^{-1} which we write as $\frac{1}{d}$. We also say that d is invertible.

Now consider two cases.

Case 1: There is a ring element x such that $x^2 = a^2 + 4b$.

This means we have two solutions $\lambda_1 = \frac{a+x}{2}$ and $\lambda_2 = \frac{a-x}{2}$. Since p is an odd prime, 2 and p are relatively prime, so 2 has a multiplicative inverse.

This gives us for all c and d in R, $c\lambda_1^n + d\lambda_2^n$ satisfies the recurrence.

Now we show that we can find c and d such that $S_0 = c + d$ and $S_1 = c\lambda_1 + d\lambda_2$. Solving for c and d using Cramer's rule gives $c = \frac{S_0\lambda_2 - S_1}{\lambda_2 - \lambda_1}$ and $d = \frac{S_1 - S_0\lambda_1}{\lambda_2 - \lambda_1}$. We know $\lambda_2 - \lambda_1 = -x$ and since $p \nmid a^2 + 4b$, $-x$ is invertible so these solutions are valid.

We can see that $\lambda_1\lambda_2 = \frac{a^2 - (a^2+4b)}{4} = -b$, and since $p \nmid b$, this is invertible, so therefore λ_1 and λ_2 are invertible.

Since λ_1 and λ_2 are invertible elements of R, from Euler's Theorem, we know the period of λ_1 and the period of λ_2 divide $(p-1) \cdot p^{r-1}$.

Case 2: There is no ring element whose square is $a^2 + 4b$.

Let's consider a new ring $R[x] = \{\alpha + \beta x | \alpha, \beta \in R, x^2 = a^2 + 4b\}$ where addition and multiplication are as before with the understanding that $x^2 = a^2 + 4b$.

This gives us two solutions in $R[x]$ $\lambda_1 = \frac{a+x}{2}$ and $\lambda_2 = \frac{a-x}{2}$.

Also, for all c and d in $R[x]$, $c\lambda_1^n + d\lambda_2^n$ satisfies the recurrence.

Now we show that there are c and d such that $S_0 = c + d$ and $S_1 = c\lambda_1 + d\lambda_2$. Using Cramer's rule gives $c = \frac{S_0\lambda_2 - S_1}{\lambda_2 - \lambda_1}$ and $d = \frac{S_1 - S_0\lambda_1}{\lambda_2 - \lambda_1}$. We know $\lambda_2 - \lambda_1 = -x$ and since $p \nmid a^2 + 4b$, $-x$ is invertible so these solutions are valid.

As before, λ_1 and λ_2 are invertible.

By Fermat's Little Theorem [Gal13], the period of λ_1 and λ_2 divides the order of $R[x]^*$, the group of elements of $R[x]$ with multiplicative inverses.

Consider an element $e + fx$ of $R[x]$. We know $e + fx$ is invertible if and only if we can find $g + hx \in R[x]$ such that $(e + fx)(g + hx) = 1$. This means $eg + fhx^2 + (eh + fg)x = 1$ in $R[x]$, so we must solve the equations $eg + fhx^2 = 1$ in R and $eh + fg = 0$ in R for g and h. This gives $g = \frac{e}{e^2 - f^2x^2}$ and $h = \frac{-f}{e^2 - f^2x^2}$, meaning we must show $(e^2 - f^2x^2)u = e$ and $(e^2 - f^2x^2)v = f$ have solutions in R.

By [Gal13], these equations both have solutions in R if and only if $\gcd(p^r, e^2 - f^2x^2) | e$ and $\gcd(p^r, e^2 - f^2x^2) | f$.

We now have two subcases.

Case 2.1: Suppose $p \nmid e$.

Then $e + fx$ will be invertible if and only if $p \nmid e^2 - f^2x^2$.

If $p | f$, we get $e^2 - f^2x^2 \equiv e^2 \not\equiv 0 \pmod{p}$ since $p \nmid e$. Therefore, $p \nmid e^2 - f^2x^2$ so $e + fx$ is invertible.

If $p | f$, then $e + fx$ will be invertible unless $e^2 - f^2x^2 \equiv 0 \pmod{p}$.

This would mean $\frac{e^2}{f^2} \equiv x^2 \pmod{p}$. Since $p \nmid f$, this means that $\frac{e}{f} \in R$ and $(\frac{e}{f})^2 \equiv x^2 \pmod{p}$. However, this is a contradiction since we assumed there was no element of R whose square was x^2.

Therefore, $e + fx$ is invertible for $(p-1) \cdot p^{r-1}$ choices of e and for all p^r choices of f.

This gives us $(p-1) \cdot p^{r-1} \cdot p^r$ such e and f where $e + fx$ is invertible.

Case 2.2: Suppose $p^k || e$ and $p^\ell || f$ where $1 \leq k < r$ and $0 \leq \ell < r$.

We know that $\gcd(p^r, e^2 - f^2x^2) = p^{\min\{2k, 2\ell, r\}}$ and $e + fx$ is invertible if and only if $\gcd(p^r, e^2 - f^2x^2) | e, f$.

If $2k$ is the minimum, we would need $p^{2k} | p^k$ for $1 \leq k < r$ which is a contradiction.

If 2ℓ is the minimum, the only ℓ such that $p^{2\ell} | e, f$ is $\ell = 0$. There are $(p-1) \cdot p^{r-1}$ such f.

Since k and ℓ are both less than r, that means $p^r \nmid e, f$, so r cannot be the minimum.

This gives us $p^{r-1} \cdot (p-1) \cdot p^{r-1}$ such e and f where $e + fx$ is invertible.

Therefore we have a total of $(p-1)(p+1) \cdot p^{2r-2}$ invertible elements of $R[x]$, thus the period of λ_1 and λ_2 divides $(p-1)(p+1) \cdot p^{2r-2}$.

Therefore S_n is periodic with a period dividing $(p-1)(p+1) \cdot p^{2r-1}$. □

Lemma 4.2. *The sum $W_n^{(B)}$ of the weights of all 3-flat paths of length n with weights $B = b_0, b_1, b_2$ satisfies the recurrence $W_n^{(B)} = (b_0 + b_1 + b_2)W_{n-1}^{(B)} - b_0 b_2 W_{n-2}^{(B)}$ with $W_0 = 1$ and $W_1 = b_0$.*

Proof. First, we note that we can remove the first consecutive up-step and down-step in a 3-flat path of length n to obtain a 3-flat path of length $n-1$. Therefore, we can generate all 3-flat paths of length n by inserting a consecutive up-step and down-step at heights 0, 1 and 2 to each 3-flat path of length $n-1$ at the first point the path reaches height 0, 1, and 2 respectively. The sum of the weights of these paths is $(b_0 + b_1 + b_2)W_{n-1}^{(B)}$.

These paths are unique except for two cases. Firstly, there is one path of length $n-1$ that never reaches height 2, so we cannot add a consecutive up-step and down-step at height 2 to this path. Secondly, a path of the same form as the one in Figure 12 can be generated in two ways: by inserting a consecutive up-step and down-step at height 0 or by inserting a consecutive up-step and down-step at height 2 as shown below.

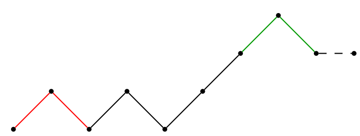

Figure 12: A 3-flat path of length n that can be generated in two ways.

Therefore, we must subtract the weight of all such paths formed by inserting a consecutive up-step and down-step at height 0 and at height 2 to a 3-flat path of length $n-2$ as well as the weight of inserting a consecutive up-step and down-step at height 2 to the path that never reaches height 2, or subtract $b_0 b_2 W_{n-2}^{(B)}$.

This gives the sum of the weights of all 3-flat paths of length n satisfies

$$W_n^{(B)} = (b_0 + b_1 + b_2)W_{n-1}^{(B)} - b_0 b_2 W_{n-2}^{(B)}.$$

□

We call a path with n up-steps that starts at height 0 or 3 and ends at height 0 or 3 and never reaches a height greater than 3 a *partial* 3*-flat path* of length n.

Lemma 4.3. *The sum of the weights of all partial 3-flat paths of length n with weights $B = b_0, b_1, b_2$ with the same starting and ending point, $W_n^{(B)}$, satisfies the recurrence $(b_0+b_1+b_2)W_{n-1}^{(B)} - b_0 b_2 W_{n-2}^{(B)}$.*

Proof. We examine the partial 3-flat paths by cases.

Case 1: The path starts and ends at height 0.

Lemma 4.2 gives us the recurrence for 3-flat paths starting and ending at height 0.

Case 2: The path starts and ends at height 3.

The weight of a partial 3-flat path starting and ending at height 3 with the weights b_0, b_1, b_2 is the same as the weight of a partial 3-flat path starting and ending at height 0 with the weights b_2, b_1, b_0.

Therefore the sum of the weights of all partial 3-flat paths of length n starting and ending at height 3 satisfies the recurrence $W_n^{(B)} = (b_0 + b_1 + b_2)W_{n-1}^{(B)} - b_0 b_2 W_{n-2}^{(B)}$.

Case 3: The path starts at height 0 and ends at height 3.

We note that by removing the first consecutive up-step and down-step of a partial 3-flat path of length n starting at height 0 and ending at height 3 creates a partial 3-flat path of length $n-1$ starting at height 0 and ending at height 3.

Therefore, we can generate all partial 3-flat paths of length n starting at height 0 and ending at height 3 by inserting a consecutive up-step and down-step at heights 0, 1, and 2 at the first point where the path reaches height 0, 1, and 2 respectively as in Lemma 4.2.

Once again, paths of the same form as the path in Figure 12 can be created by inserting a consecutive up-step and down-step at height 0 or at height 2, so we must subtract off the weights of these paths.

This yields the recurrence $W_n^{(B)} = (b_0 + b_1 + b_2)W_{n-1}^{(B)} - b_0 b_2 W_{n-2}^{(B)}$.

Case 4: The path starts at height 3 and ends at height 0.

We note that by removing the last consecutive up-step and down-step of a partial 3-flat path of length n starting at height 3 and ending at height 0 creates a partial 3-flat path of length $n-1$ starting at height 3 and ending at height 0.

Therefore, we can generate all partial 3-flat paths of length n starting at height 3 and ending at height 0 by inserting a consecutive up-step and down-step at heights 0, 1, and 2 at the last point where the path reaches height 0, 1, and 2 respectively as in Lemma 4.2.

Once again, paths of the same form as the path in Figure 12 can be created by inserting a consecutive up-step and down-step at height 0 or at height 2, so we must subtract off the weights of these paths.

This yields the recurrence $W_n^{(B)} = (b_0 + b_1 + b_2)W_{n-1}^{(B)} - b_0 b_2 W_{n-2}^{(B)}$. □

We will now apply the results of this section to weighted Catalan numbers.

Theorem 4.4. *Let a string of weights $B = b_0, b_1, b_2, \ldots$ satisfy the following properties:*

- *There are no four consecutive weights $b_i, b_{i+1}, b_{i+2}, b_{i+3}$ satisfying $p \nmid b_i, b_{i+1}, b_{i+2}, b_{i+3}$ for a prime p,*
- *For any three consecutive weights b_j, b_{j+1}, b_{j+2} where $p \nmid b_j, b_{j+1}, b_{j+2}$, we have $p \nmid b_j + b_{j+1} + b_{j+2}$ and $p \nmid (b_j + b_{j+1} + b_{j+2})^2 - 4b_j b_{j+2}$.*

Then the corresponding weighted Catalan numbers $C_n^{(B)}$ are periodic modulo p^r with a period dividing $(p-1)\cdot(p+1)\cdot p^{3r-1}$.

Proof. We can create the skeleton of a Dyck path of length n by marking every step corresponding to a weight divisible by p.

We partition the Dyck paths of length n into sets of Dyck paths that have the same skeleton.

Consider a skeleton containing $2f$ steps, and therefore f up-steps, corresponding to weights divisible by p. If $f > r$, then the weight of any Dyck path with this skeleton will be divisible by p^f and will equal 0 modulo p^r. Therefore, we will only consider skeletons where $f < r$.

Note that a skeleton with $2f$ steps corresponding to weights divisible by p divides the path into $2f+1$ partial 3-flat paths, partial flat paths, and paths with maximum height 1.

Noting that the weight of a path of maximum height 1 of length ℓ with the weight b_j is b_j^ℓ, we know that the weight of all paths of length n with this skeleton is equal to

$$C_1 \cdot p^f \sum b_1^{i_1} b_2^{i_2} \cdots b_q^{i_q} T_{i_{q+1}}^{(B_{q+1})} T_{i_{q+2}}^{(B_{q+2})} \cdots T_{i_r}^{(B_r)} W_{i_{r+1}}^{(B_{r+1})} W_{i_{r+2}}^{(B_{r+2})} \cdots W_{i_{2f+1}}^{(B_{2f+1})}$$

in $R[x]$ for some constant C_1 with q paths with maximum height 1, $r-q$ partial flat paths and $2f+1-r$ partial 3-flat paths with $i_1 + i_2 + i_3 + \cdots + i_{2f+1} = m$ where $m = n-d$ where d is a positive integer dependent on the skeleton.

From Proposition 3.4 and Lemma 4.3 we know that this is equivalent modulo p^r to a sum of expressions of the form

$$C \cdot p^f \sum z_1^{i_1} z_2^{i_2} z_3^{i_3} \cdots z_{2f+1}^{i_{2f+1}}.$$

Here, each z_i^n is a solution to the corresponding recurrence. Now we consider the generating function g corresponding to

$$C \cdot p^f \sum z_1^{i_1} z_2^{i_2} z_3^{i_3} \cdots z_{2f+1}^{i_{2f+1}}.$$

We have

$$g(x) \equiv C \cdot p^f (1 + z_1 x + z_1^2 x^2 + \cdots)(1 + z_2 x + z_2^2 x^2 + \cdots) \cdots (1 + z_{2f+1} x + z_{2f+1}^2 x^2 + \cdots) \pmod{p^r}.$$

Considering $g(x)$ modulo p^r is equivalent to examining

$$h(x) \equiv C(1 + z_1 x + z_1^2 x^2 + \cdots)(1 + z_2 x + z_2^2 x^2 + \cdots) \cdots (1 + z_{2f+1} x + z_{2f+1}^2 x^2 + \cdots) \pmod{p^{r-f}}.$$

By Theorem 4.1 and Euler's Theorem, we know that for each z_i, the sequence $z_i, z_i^2, z_i^3, \ldots$ is periodic modulo p^{r-f} with a period dividing $k = (p-1) \cdot (p+1) \cdot p^{2(r-f)-1}$. Therefore, $z_i^a = z_i^{a+k}$ for all a so $(1 + z_i x + z_i^2 x^2 + \cdots) = \sum_t x^{kt}(1 + z_i x + z_i^2 x^2 + \cdots + z_i^{k-1} x^{k-1})$.

Therefore,

$$h(x) \equiv C(\sum_t x^{kt})^{2f+1} \prod_{i=1}^{2f+1} (1 + z_i x + z_i^2 x^2 + \cdots + z_i^{k-1} x^{k-1}) \pmod{p^{r-f}}. \tag{3}$$

Let $q(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{(k-1)(2f+1)} x^{(k-1)(2f+1)}$ be the polynomial

$$\prod_{i=1}^{2f+1} (1 + z_i x + z_i^2 x^2 + \cdots + z_i^{k-1} x^{k-1}).$$

We can write the coefficient of x^{kt} in $(1+x^k+x^{2k}+\cdots)^{2f+1}$ as the number of ways to partition t into $2f+1$ pieces, or

$$\binom{2f+t}{2f}.$$

Thus:

$$h(x) \equiv C\sum_t \binom{2f+t}{2f} x^{kt} \cdot q(x) \pmod{p^{r-k}}.$$

By [Kwo89], we know that $\binom{2f+t}{2f}$ is periodic modulo p^{r-f} with period $p^{r-f+\lfloor \log_p(2f)\rfloor}$.

Thus $h(x)$ is periodic with a period dividing $k \cdot p^{r-f+\lfloor \log_p(2f)\rfloor}$. Since $k = (p-1)\cdot(p+1)\cdot p^{2(r-f)-1}$, we get that this period is $(p-1)\cdot(p+1)\cdot p^{3(r-f)+\lfloor \log_p(2f)\rfloor-1}$. We note that we have $3f-\lfloor \log_p(2f)\rfloor+1 \geq 1$. Therefore, $(p-1)\cdot(p+1)\cdot p^{3(r-f)+\lfloor \log_p(2f)\rfloor-1}$ is at most $(p-1)\cdot(p+1)\cdot p^{3r-1}$.

Therefore $C_n^{(B)}$ is periodic modulo p^r with a period that divides $(p-1)\cdot(p+1)\cdot p^{3r-1}$. $\square$

5 Discussion of Results & Future Research

This paper provides what seems to be the first theoretical result regarding the periodicity of L_n modulo 3^r. In particular, Theorem 3.5 shows the period of L_n modulo 3^r divides $2\cdot 3^{2r-7}$. Additionally, Theorem 3.8 could potentially be used to completely resolve Postnikov's conjecture; i.e.; that the period is exactly $2\cdot 3^{r-3}$. If we can prove that the weights of all paths with non-simple skeletons have a period less than $2\cdot 3^{r-3}$, then we will have proven Postnikov's conjecture! However, if we cannot show this, then we could try to cleverly group together these paths with certain skeletons so that the overall period is less than $2\cdot 3^{r-3}$.

Although we only considered L_n, we have extended the proof techniques to weighted Catalan numbers using other weights. In these proofs, we use the fact that we can decompose any given Dyck path into a skeleton using partial flat path decomposition. Therefore, if we have weights such that at least one of any three consecutive weights is divisible by a prime p, then we can use partial flat path decomposition and mark the steps corresponding to weights divisible by p to obtain a skeleton. In particular, we can use the same proof techniques to prove that the weighted Catalan numbers with such weights are periodic modulo p^r and provide an upper bound on its period.

We also focused on flat paths, or paths with a maximum height of at most 2 and the behavior of L_n modulo 3^r. A next natural step is to examine the behavior of L_n modulo 5^r. There are recurrences for the number of paths of maximum height of at most 3, or of at most 4. We have found and used recurrences for the sum of the weights of Dyck paths with a maximum height of at most 3 and have also proven results about the periodicity of sequences with a two-term recurrence. This means that we could use a process similar to partial flat path decomposition to break a Dyck path into pieces with a maximum height of at most 4 by marking the steps corresponding to the weight 1 or weights divisible by 5^2 and extend the ideas used in these proofs to investigate L_n modulo 5^r.

Another subject that we could potentially investigate is the matter of pure periodicity. While the theorems in this paper prove that certain weighted Catalan numbers are periodic modulo p^r, we have not proven that they are purely periodic. We have shown that $L_{n+k} \equiv L_n \pmod{3^r}$ for a period k and n sufficiently large, but we have no results that tell us whether $L_{n+k} \equiv L_n \pmod{3^r}$ for small n. In other words, we have shown that certain weighted Catalan numbers are eventually periodic, and it seems from data gathered that the weighted Catalan numbers, in particular L_n, are purely periodic. A next step would be to investigate this further and prove whether or not L_n and other weighted Catalan numbers are purely periodic.

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