Ascent sequences avoiding pairs of patterns

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joint work with

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Ascents

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Definition

An ascent in the string $x_1 \cdots x_n$ is a position i such that $x_i < x_{i+1}$.

Example:

$\underline{01}024 \qquad 01\underline{02}4 \qquad 010\underline{24}$

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$$\underline{01}024 \qquad 01\underline{02}4 \qquad 010\underline{24}$$

Definition

$\mathrm{asc}(x_1 \cdots x_n)$ is the number of ascents of $x_1 \cdots x_n$.

Example: $\mathrm{asc}(01024) = 3$

Ascent Sequences

Definition

An ascent sequence is a string $x_1 \cdots x_n$ of non-negative integers such that:

- $x_1 = 0$
- $x_n \leq 1 + \mathrm{asc}(x_1 \cdots x_{n-1})$ for $n \geq 2$

$\mathcal{A}_n$ is the set of ascent sequences of length n

$\mathcal{A}_2 = \{00, 01\}$

$\mathcal{A}_3 = \{000, 001, 010, 011, 012\}$

More examples: 01234, 01013

Non-example: 01024

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More examples: 01234, 01013

Non-example: 01024

Theorem (Bousquet-Mélou, Claesson, Dukes, & Kitaev, 2010)

$|\mathcal{A}_n|$ is the nth Fishburn number (OEIS A022493).

$$\sum_{n\geq 0} |\mathcal{A}_n| x^n = \sum_{n \geq 0} \prod_{i=1}^{n} (1 - (1-x)^i)$$

Patterns

Definition

The reduction of $x = x_1 \cdots x_n$, $\mathrm{red}(x)$, is the string obtained by replacing the ith smallest digits of x with $i-1$.

Example: $\mathrm{red}(273772) = 021220$

Patterns

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Pattern containment/avoidance

$a = a_1 \cdots a_n$ contains $\sigma = \sigma_1 \cdots \sigma_m$ iff there exist $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $\mathrm{red}(a_{i_1} a_{i_2} \cdots a_{i_m}) = \sigma$.

$a_B(n) = |\{a \in \mathcal{A}_n \mid a \text{ avoids } B\}|$

001010345 contains 012, 000, 1102; avoids 210.

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Goal

Determine $a_B(n)$ for many of choices of B.

Previous Work

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- Duncan & Steingrímsson (2011)

Pattern σ	$\{a_\sigma(n)\}_{n\geq 1}$	OEIS
001, 010 011, 012	2^{n-1}	A000079
102 0102, 0112	$(3^{n-1}+1)/2$	A007051
101, 021 0101	$\frac{1}{n+1}\binom{2n}{n}$	A000108

- Mansour and Shattuck (2014)
  Callan, Mansour and Shattuck (2014)

Pattern σ	$\{a_\sigma(n)\}_{n\geq 1}$	OEIS
1012	$\sum_{k=0}^{n-1}\binom{n-1}{k}C_k$	A007317
0123	ogf: $\frac{1-4x+3x^2}{1-5x+6x^2-x^3}$	A080937
8 pairs of length 4 patterns	$\frac{1}{n+1}\binom{2n}{n}$	A000108

Overview

- 13 length 3 patterns
  6 permutations, 000, 001, 010, 100, 011, 101, 110
- $\binom{13}{2} = 78$ pairs
- at least 35 different sequences $a_{\sigma,\tau}(n)$
  16 sequences in OEIS
  - 3 sequences from Duncan/Steingrímsson
  - 1 eventually zero
  - 1 from pattern-avoiding set partitions
  - 3 from pattern-avoiding permutations
  - 1 sequence from Mansour/Shattuck
    (Duncan/Steingrímsson conjecture)

Unbalanced equivalences

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Theorem

$$a_{010,021}(n) = a_{010}(n) = a_{10}(n) = 2^{n-1}$$

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- 101-avoiders are restricted growth functions.
- If σ contains 201, then $a_{101,\sigma} = C_n$.

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Theorem

$a_{101,210}(n) = \frac{3^{n-1}+1}{2}$

- Proof sketch: bijection with ternary strings with even number of 2s
- (Duncan/Steingrímsson proof that $a_{102}(n) = \frac{3^{n-1}+1}{2}$ uses bijection with same strings.)

An Erdős-Szekeres-like Theorem

Theorem

$$a_{000,012}(n) = \begin{cases} |\mathcal{A}_n| & n \leq 2 \\ 3 & n = 3 \text{ or } n = 4 \\ 0 & n \geq 5 \end{cases}$$

$$\mathcal{A}_1(000,012) = \{0\}$$
$$\mathcal{A}_2(000,012) = \{00,01\}$$
$$\mathcal{A}_3(000,012) = \{001,010,011\}$$
$$\mathcal{A}_4(000,012) = \{0011,0101,0110\}$$

An Erdős-Szekeres-like Theorem

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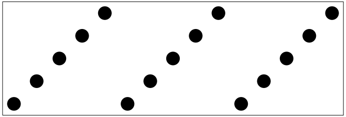
Theorem

$a_{0^a,012\cdots b}(n) = 0$ for $n \geq (a-1)((a-1)(b-2)+2)+1$

Proof:

- largest letter preceeded by at most $b-1$ smaller values
- at most $a-1$ copies of each value
- How to maximize number of ascents:



- $(a-1)(b-2)$ ascents before largest letter $\Rightarrow$ largest possible digit is $(a-1)(b-2)+1$
- Use all digits in $\{0,\ldots,(a-1)(b-2)+1\}$ each $a-1$ times.

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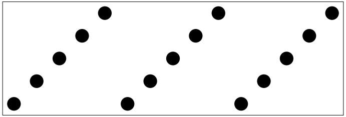

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- Use all digits in $\{0, \ldots, (a-1)(b-2)+1\}$ each $a-1$ times.
- Maximum avoider example: (a=3, b=5) 0123 0123 77665544

Other sequences

Patterns	OEIS	Formula
000,011	A000027	n
000,001	A000045	F_{n+1}
011,100	A000124	$\binom{n}{2}+1$
001,100	A000071	$F_{n+2}-1$
001,210	A000125	$\binom{n}{3}+n$
000,101	A001006	M_n
100,101	A025242	(Generalized Catalan)
021,102	A116702	$\lvert\mathcal{S}_n(123,3241)\rvert$
102,120	A005183	$\lvert\mathcal{S}_n(132,4312)\rvert$
101,120	A116703	$\lvert\mathcal{S}_n(231,4123)\rvert$
101,110	A001519	F_{2n-1}
201,210	A007317	$\sum_{k=0}^{n-1}\binom{n-1}{k}C_k$

Avoiding 100 and 101

Theorem

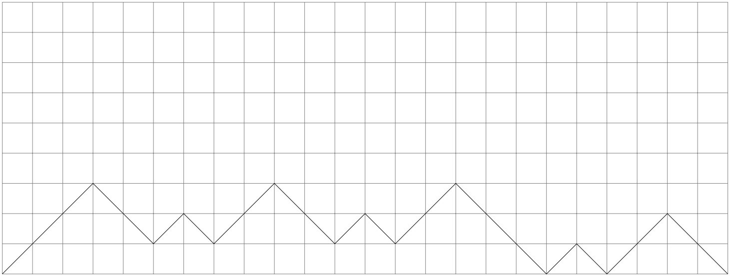
$a_{100,101}(n) = GC_n$, the nth generalized Catalan number

- $a_{100,101}(n) = a_{0100,0101}(n)$
- ascent sequences avoiding a subpattern of 01012 are restricted growth functions
- Mansour & Shattuck (2011): $1211, 1212$-avoiding set partitions are counted by GC_n
- used algebraic techniques
- known: GC_n counts DDUU-avoiding Dyck paths

New: bijective proof

Avoiding 100 and 101

Bijection from *DDUU*-avoiding Dyck paths to ascent sequences:



Heights of left sides of up steps:

012112112001

Avoiding 100 and 101

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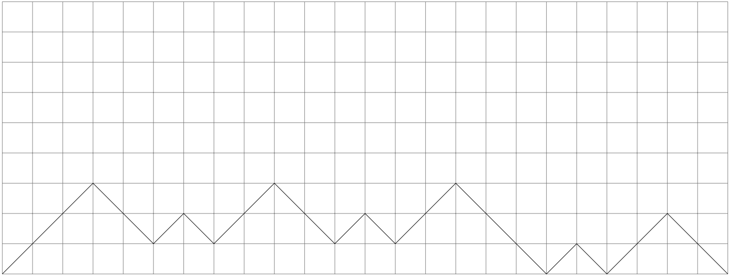

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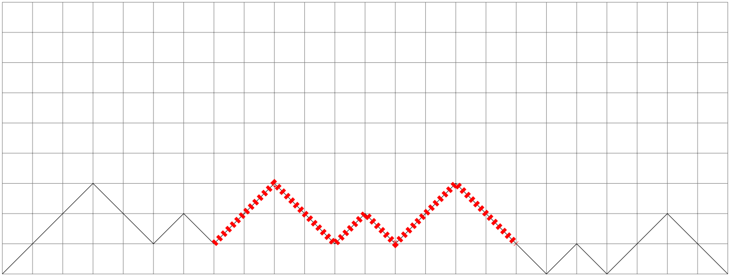

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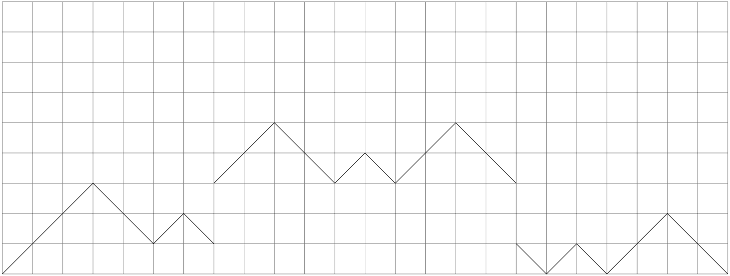

Heights of left sides of up steps:

01211<u>2112</u>001
0121<u>34334</u>001

Avoiding 100 and 101

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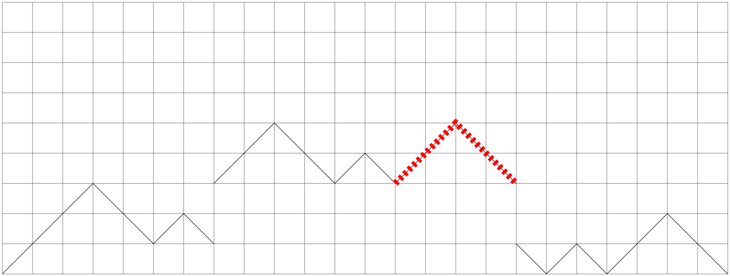

Heights of left sides of up steps:

012112112001
012134334001

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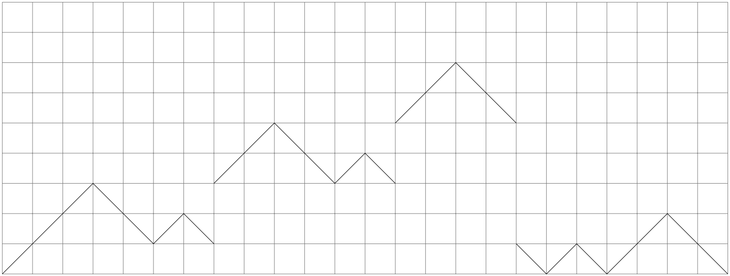
Bijection from *DDUU*-avoiding Dyck paths to ascent sequences:



Heights of left sides of up steps:

01211212001
01213433 4001
01213435 6001

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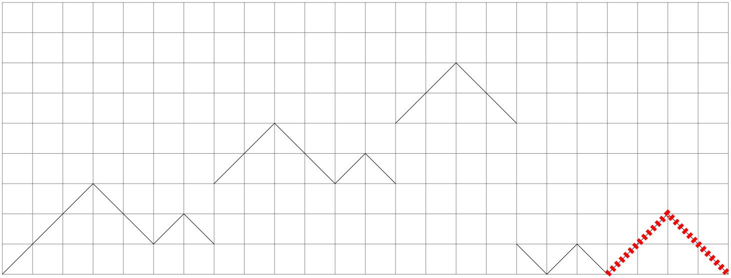

Heights of left sides of up steps:

01211212001
01213433400
01213435600

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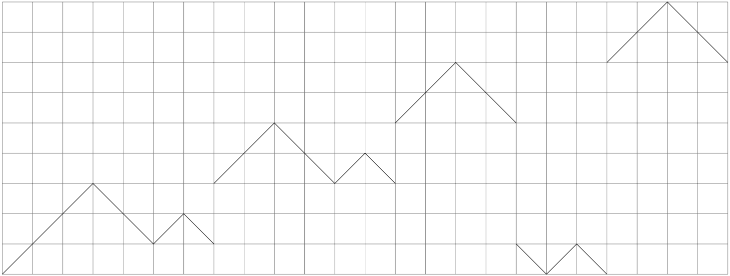
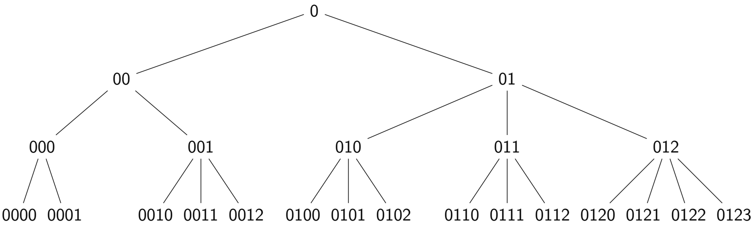


Heights of left sides of up steps:

01211211 2001
0121343334001
0121343560 01
01213435607 8

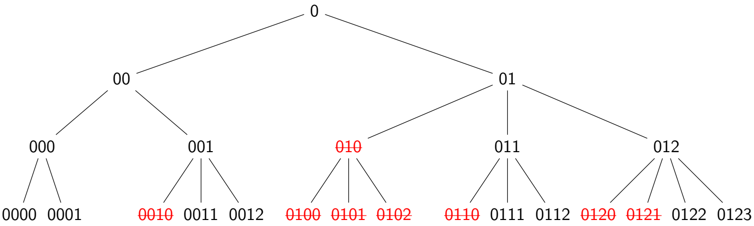
Generating trees

$\mathcal{A}_n$:



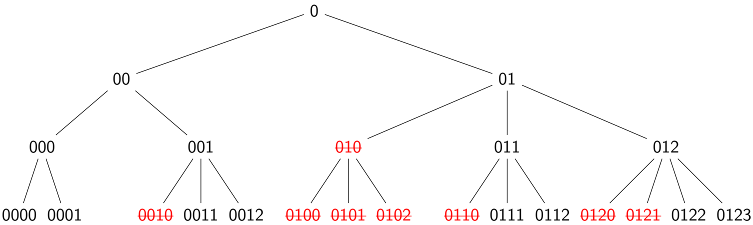
Generating trees

$\mathcal{A}_n(10)$:



Generating trees

$\mathcal{A}_n(10)$:



Root: (2)
Rule: $(2) \rightsquigarrow (2)(2)$

$$|\mathcal{A}_{10}(n)| = 2^{n-1}$$

Permutations

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Theorem

$a_{102,120}(n) = |\mathcal{S}_n(132, 4312)|$

Proof: Isomorphic generating tree

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$a_{101,120}(n) = |\mathcal{S}_n(231, 4123)|$

Proof: Isomorphic generating tree

Theorem

$a_{021,102}(n) = |\mathcal{S}_n(123, 3241)|$

Proof: Generating trees...
Ascent sequences $\rightarrow$ 5 labels.
Permutations $\rightarrow$ 8 labels. (Vatter, FINLABEL, 2006)
Transfer matrix method gives same enumeration, bijective proof open.

Avoiding 201 and 210

Theorem

$$a_{201,210}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

Proof scribble:
generating tree $\rightarrow$ recurrence $\rightarrow$ system of functional equations $\rightarrow$ experimental solution $\rightarrow$ plug in for catalytic variables

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Conjecture (Duncan & Steingrímsson)

$$a_{0021}(n) = a_{1012}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

Note: Proving this would complete Wilf classification of 4 patterns.

A familiar sequence...

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Conjecture (Duncan & Steingrímsson)

$$a_{0021}(n) = a_{1012}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

Theorem (Mansour & Shattuck)

$$a_{1012}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

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Theorem

$$a_{0021}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

Proof: Similar technique to $a_{201,210}(n)$.

Summary and Future work

- 16 pairs of 3-patterns appear in OEIS.
- Erdős-Szekeres analog for ascent sequences.
- New bijective proof connecting 100,101-avoiders to Dyck paths.
- Completed Wilf classification of 4-patterns.
- Open:
  - 19 sequences from pairs of 3-patterns not in OEIS.
  - Bijective explanation that $a_{021,102}(n) = |\mathcal{S}_n(123, 3241)|$.

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Forthcoming:

- Enumeration schemes for pattern-avoiding ascent sequences
- Details on $a_{201,210}(n)$ and $a_{0021}(n)$
- More bijections with other combinatorial objects?

References

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- D. Callan, T. Mansour, and M. Shattuck. Restricted ascent sequences and Catalan numbers. `arXiv:1403.6933`, March 2014.
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- T. Mansour and M. Shattuck. Some enumerative results related to ascent sequences. *Discrete Mathematics* **315-316** (2014), 29–41.
- V. Vatter. Finitely labeled generating trees and restricted permutations, J. Symb. Comput. 41 (2006), 559–572.

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Thanks for listening!