DIOPHANTINE REPRESENTATION OF THE FIBONACCI NUMBERS

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In the year 1202, the Italian mathematician Leonardo of Pisano, or *Fibonacci* as he is known today, gave the sequence $1, 1, 2, 3, 5, 8, 13, 21, \cdots,$ in his book *Liber Abacci.* The numbers occurred in connection with a problem concerning the number of offspring of a pair of rabbits. The sequence has many interesting properties, and has fascinated mathematicians for over 700 years. It is usually defined recursively by means of the equations

$$\phi_1 = 1, \quad \phi_2 = 1, \quad \text{and} \quad \phi_{n+2} = \phi_n + \phi_{n+1} \,.$$

These equations permit us to obtain the n^{th} Fibonacci number, $\phi_n,$ by computing all smaller Fibonacci numbers. Many formulas are known which permit calculation of the n^{th} Fibonacci number directly from n. J.P.M. Binet found [1] the well known formula

$$\phi_n = \frac{1}{\sqrt{5}}\left[\left[\frac{1+\sqrt{5}}{2}\right]^n - \left[\frac{1-\sqrt{5}}{2}\right]^n\right] \,.$$

E. Lucas [6] noticed that the Fibonacci numbers were the sums of the binomial coefficients on the "rising diagonals" of Pascal's triangle.

$$\phi_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots \,.$$

We shall prove here that the set of Fibonacci numbers is identical with the set of positive values of a polynomial of the fifth degree in two variables:

(1) $$2y^4x + y^3x^2 - 2y^2x^3 - y^5 - yx^4 + 2y \,.$$

To construct the polynomial (1), we shall need three lemmas. These lemmas assert that pairs of adjacent Fibonacci numbers, and only these, are to be found among the points with integer coordinates on the hyperbolas

$$y^2 - yx - x^2 = \pm 1 \,.$$

(L.E. Dickson [4] credits E. Lucas [7] and J. Wasteels [13] with this observation.)

Lemma 1. For any positive integer i,

$$\phi_{i+1}^2 - \phi_{i+1}\phi_i - \phi_i^2 = (-1)^i \,.$$

Proof. By induction on i. Plainly, the statement is true if $i = 1$. Suppose it holds for i. Then

$$\begin{aligned}\phi_{i+2}^2 - \phi_{i+2}\phi_{i+1} - \phi_{i+1}^2 &= (\phi_i + \phi_{i+1})^2 - (\phi_i + \phi_{i+1})\phi_{i+1} - \phi_{i+1}^2 \\ &= -(\phi_{i+1}^2 - \phi_{i+1}\phi_i - \phi_i^2) = -(-1)^i = (-1)^{i+1} \,.\end{aligned}$$

This completes the proof of the lemma.

Lemma 2. For any positive integers x and y, if $y^2 - yx - x^2 = 1$ then it is possible to find a positive integer i such that $x = \phi_{2i}$ and $y = \phi_{2i+1}$.

Proof. By induction on x. If $x = 1$ then necessarily $y = 2$. In this case we may take $i = 1$.

Suppose that x and y are numbers satisfying the equation of the lemma and that $1 < x$. Then $2 \leqslant y$. Assume that the statement of the lemma holds for all pairs, (x_0, y_0), of positive integers for which $x_0 < x$. Let us set $x_0 = 2x - y$, and $y_0 = y - x$. Since $2 \leqslant y$,

$$(x+1)^2 = x^2 + 2x + 1 \leqslant x^2 + yx + 1 = y^2 ,$$

hence $y > x$. And since $1 < x$,

$$y^2 = yx + x^2 + 1 < yx + x^2 + x = yx + (x+1)x \leqslant yx + yx = 2yx,$$

hence $y < 2x$. Therefore

(2) $$0 < x_0 < x, \quad \text{and} \quad 0 < y_0.$$

Furthermore,

(3) $$y_0^2 - y_0x_0 - x_0^2 = (y-x)^2 - (y-x)(2x-y) - (2x-y)^2 = y^2 - yx - x^2 = 1.$$

The induction hypothesis, together with (2) and (3) implies that it is possible to find a positive integer i such that $x_0 = \phi_{2i}$ and $y_0 = \phi_{2i+1}$. Then

$$x = x_0 + y_0 = \phi_{2i} + \phi_{2i+1} = \phi_{2(i+1)} \quad \text{and} \quad y = y_0 + x = \phi_{2i+1} + \phi_{2i+2} = \phi_{2(i+1)+1}.$$

This completes the proof of the lemma.

Lemma 3. For any positive integers x and y, if

$$y^2 - yx - x^2 = -1,$$

then it is possible to find a positive integer i such that $x = \phi_{2i-1}$ and $y = \phi_{2i}$.

Proof. Let x and y be numbers satisfying the conditions of the lemma. Then

$$(x+y)^2 - (x+y)(y) - y^2 = x^2 + 2xy + y^2 - xy - y^2 - y^2 = -(y^2 - xy - x^2) = -(-1) = 1.$$

According to Lemma 2 it is possible to find a positive integer i such that

$$y = \phi_{2i} \quad \text{and} \quad x + y = \phi_{2i+1}.$$

Hence

$$x = \phi_{2i+1} - \phi_{2i} = \phi_{2i-1} \quad \text{and} \quad y = \phi_{2i}.$$

This completes the proof of the lemma.

Lemmas 1, 2 and 3 imply that the set of all Fibonacci numbers has a very simple Diophantine defining equation. [A relation in positive integers is said to be *Diophantine* if it is equal to the set of values of parameters for which a polynomial equation is solvable in positive integers.]

Theorem 1. For any positive integer y, in order that y be a Fibonacci number, it is necessary and sufficient that there exist a positive integer x such that

(4) $$(y^2 - yx - x^2)^2 = 1.$$

Proof. We have only to use Lemmas 1, 2, and 3.

Lemma 4. If x and y are positive integers, then $y^2 - yx - x^2 \neq 0$.

Proof. Multiplying by 4 and completing the square, we find that

$$4y^2 - 4yx - 4x^2 = (2y-x)^2 - 5x^2.$$

If the right side of this expression were zero, for positive integers x and y, then $\sqrt{5}$ would be a rational number. The lemma is proved.

Theorem 2. The set of all Fibonacci numbers is identical with the set of positive values of the polynomial

(1) $$y(2 - (y^2 - yx - x^2)^2)$$

for $(x = 1, 2, \cdots, y = 1, 2, \cdots)$.

Proof. According to Theorem 1, if y is a Fibonacci number then a positive integer x may be found to satisfy equation (4). For such an x, (1) assumes the value y. Therefore all Fibonacci numbers are values of the polynomial (1).

To see that *only* Fibonacci numbers are assumed as values of (1), suppose that x, y and w are positive integers and that

(5) $$w = y(2 - (y^2 - yx - x^2)^2).$$

Then, since y and w are positive, we see that

(6) $$0 < (y^2 - yx - x^2)^2 < 2,$$

using Lemma 4, to obtain the lower inequality.

Since x and y are integers, (6) implies that equation (4) must hold. According to Theorem 1, y must be a Fibonacci number. Equations (4) and (5) imply that $w = y$. Therefore w is a Fibonacci number.

This completes the proof of the theorem. (Putnam's method [10] would produce a polynomial of degree 9.)

The polynomial (1), which represents the set of Fibonacci numbers, assumes in addition certain negative values such as -28 $(x = 2,\ y = 2)$. The appearance of non-Fibonacci numbers cannot be prevented, for we can prove

Theorem 3. The set of Fibonacci numbers is not the exact range of any polynomial.

Proof. We shall show that a polynomial $P(x_1, x_2, \cdots, x_k)$ which assumes only Fibonacci number values must be constant. The proof will be carried out by induction on the number k of variables.

If $k = 0$, there is nothing to prove. Let us assume that the result holds for k and consider a polynomial

$$P(x_1, x_2, \cdots, x_k, x_{k+1})$$

in $k+1$ variables. If this polynomial is not identically zero then we may write

$$P(x_1, x_2, \cdots, x_k, x_{k+1}) = \sum_{i=0}^{m} P_i(x_1, x_2, \cdots, x_k)x_{k+1}^i, \qquad P_m(x_1, x_2, \cdots, x_k) \not\equiv 0 .$$

If $m = 0$, then $P(x_1, x_2, \cdots, x_k, x_{k+1})$ is a polynomial in $x_1, x_2, \cdots, x_k$ only. If not we may find positive integers $a_1, a_2, \cdots, a_k$ for which the polynomial

$$Q(x) = P(a_1, a_2, \cdots, a_k, x)$$

is not constant. In this event we must have one or the other of two cases:

$$\text{(i)} \ \lim_{x \to +\infty} Q(x) = +\infty, \qquad \text{or (ii)} \ \lim_{x \to +\infty} Q(x) = -\infty .$$

Assuming there are no negative Fibonacci numbers (see remark following), we have only case (i) to deal with. Since $Q(x)$ is a polynomial, a positive integer b may be found such that

$$Q(b) < Q(b+1) < Q(b+2) < Q(b+3) < \cdots . \tag{7}$$

By assumption, $Q(x)$ assumes only Fibonacci number values. Choose a positive integer c such that $\phi_c = Q(b)$. Condition (7) implies that for each positive integer y

$$\phi_{c+y} \leqslant Q(b+y). \tag{8}$$

The formula of Binet may be used to prove that for each positive integer n,

$$\left| \phi_n - \frac{1}{\sqrt{5}} \left[\frac{1+\sqrt{5}}{2} \right]^n \right| < \frac{1}{2} . \tag{9}$$

Conditions (8) and (9) imply that for each positive integer y

$$\frac{1}{\sqrt{5}} \left[\frac{1+\sqrt{5}}{2} \right]^{(c+y)} \leqslant Q(b+y) + \frac{1}{2} . \tag{10}$$

Inequality (10) implies that the polynomial $Q(b+y) + \frac{1}{2}$ grows exponentially, which is, of course, impossible.

This completes the proof of the theorem.

REMARK. The sequence of Fibonacci numbers is sometimes continued into the negative:

$$\cdots, -55, 34, -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots .$$

The assertion of Theorem 3 remains correct for this enlarged set. We need only modify the proof to deal with case (ii) as was done with case (i). Also, it is not difficult to see that the number of variables in the polynomial (1) cannot be further decreased. Thus Theorem 2 is best possible.

THE RELATION $v = \phi_u$

In 1970 Ju. V. Matijasevič made ingenious use of the Fibonacci numbers to solve Hilbert's tenth problem. In his famous address of 1900 [5], David Hilbert posed the problem of finding an algorithm to decide of an arbitrary polynomial equation, in several variables, with integer coefficients, whether or not the equation was solvable in integers.

Matijasevič [8], [9] showed that no such algorithm exists. He proved this by proving that *every recursively enumerable set is Diophantine.*

The Fibonacci numbers were important in Matijasevič's proof, because the sequence of Fibonacci numbers grows exponentially. Martin Davis, Julia Robinson and Hilary Putnam [3] had nearly solved Hilbert's tenth problem in 1961, when they succeeded in proving that the stated result would follow from the existence of a single Diophantine predicate with exponential growth. Matijasevič completed the solution of Hilbert's tenth problem by proving that the relation $v = \phi_{2u}$ is Diophantine.

In [8], [9], Matijasevič gives an explicit system of ten Diophantine equations such that, for any given positive integers u and v, the equations are solvable in the other variables if and only if $v = \phi_{2u}$. Of course it follows from the central result of [8], [9] that the relation $v = \phi_u$ is also Diophantine. However, an explicit system of equations for this relation is not written out in [9].

We shall give here an explicit system of Diophantine equations for the relation $v = \phi_u$. Our equations may conveniently be based upon Lemmas 1 and 2 and the equations of Matijasevič [9].

Theorem 4. For any positive integers t and w, in order that $w = \phi_t$, it is necessary and sufficient that there exist positive integers $a, b, c, d, e, g, h, l, m, p, r, u, v, x, y, z$ such that

$$u + a = l, \tag{11}$$

$$v + b = l, \tag{12}$$

$$l^2 - lz - z^2 = 1, \tag{13}$$

$$g^2 - gh - h^2 = 1, \tag{14}$$

$$l^2 c = g, \tag{15}$$

$$ld = m - 2, \tag{16}$$

$$(2h + g)e = m - 3, \tag{17}$$

$$x^2 - mxy + y^2 = 1, \tag{18}$$

$$l(p - 1) = x - u, \tag{19}$$

$$(2h + g)(r - 1) = x - v, \tag{20}$$

$$((2u - t)^2 + (w - v)^2)((2u + 1 - t)^2 + (w^2 - wv - v^2 - 1)^2) = 0. \tag{21}$$

Proof. For the proof we refer the reader to [9], proof of Theorem 1. There it is shown that equations (11)–(20) are solvable in positive integers if and only if $v = \phi_{2u}$. (In the necessity part of this proof we find that $3 < m$ and also $u \leqslant v \leqslant x$, so that conditions (40), (41), (43) and (44) there, may be replaced by equations (16), (17), (19) and (20) above.) When $v = \phi_{2u}$, Lemma 2 implies that the condition $w^2 - wv - v^2 = 1$ is equivalent to $w = \phi_{2u+1}$. Thus equation (21) holds if and only if

$$t = 2u \quad \text{and} \quad w = \phi_{2u}, \qquad \text{or} \qquad t = 2u + 1 \quad \text{and} \quad w = \phi_{2u+1}$$

Thus equations (11)–(21) are solvable if and only if $w = \phi_t$.

Theorem 4 makes it possible to give a polynomial formula for the t^{th} Fibonacci number, ϕ_t. We shall prove

Theorem 5. There exists a polynomial $P(t, x_1, \cdots, x_{12})$, of degree 13, with the property that, for any positive integers t and s,

$$\phi_t = s \Leftrightarrow (\exists x_1, \cdots, x_{12})[P(t, x_1, \cdots, x_{12}) = s].$$

Proof. The variables l, g, m and x are easily eliminated from the system (11)–(21) by means of Eqs. (11), (15), (16) and (19). Also, the variables b and c may be replaced by a single variable. (We need only use the fact that when α and β are positive integers, and γ is any integer, $\alpha | \beta$ and $0 < \gamma$ is equivalent to $(\exists\lambda)[\alpha\beta\gamma = \beta + \lambda\alpha]$.) If we now transpose all terms in the equations to the left side and sum the squares of the equations, we obtain the polynomial $Q(t, w, a, \cdots, z)$ with the property that $\phi_t = w$ if and only if $Q(t, w, a, \cdots, z) = 0$ for some positive integers $a, \cdots, z$. Q will be a polynomial of the 12^{th} degree. For P we may take the polynomial $w(1 - Q(t, w, a, \cdots, z))$.

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