ENUMERATION FORMULÆ FOR PATTERN RESTRICTED STIRLING PERMUTATIONS

MARKUS KUBA AND ALOIS PANHOLZER

ABSTRACT. We classify k-Stirling permutations avoiding a set of ordered patterns of length three according to Wilf-equivalence. Moreover, we derive enumeration formulæ for all of the classes using a variety of techniques such as the kernel method, a bijection related to a classical result of Simion and Schmidt, and also structural decompositions of k-Stirling permutations via the so-called block decomposition, or via bijections with families of trees.

1. INTRODUCTION

A Stirling permutation is a permutation of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ such that, for each i, $1 \le i \le n$, the elements occurring between the two occurrences of i are larger than i. E.g., 1122, 1221 and 2211 are Stirling permutations, whereas the permutations 1212 and 2112 of $\{1, 1, 2, 2\}$ are not. These combinatorial objects have been introduced by Gessel and Stanley [8] in the context of finding combinatorial interpretations of the coefficients of certain polynomials, where the Stirling numbers appear.

The notion of Stirling permutations has been generalized by Park [14] to permutations of the multiset $\{1^k, 2^k, \ldots, n^k\}$, with an integer $k \ge 1$ (here and throughout this work we use in this context $j^l := \underbrace{j, \ldots, j}_{l}$, for $l \ge 1$). As in [12] we call a permutation of the multiset $\{1^k, 2^k, \ldots, n^k\}$ a *k-Stirling permutation* of order n (Park [14] used for these objects the name k-multipermutations), if for each i, $1 \le i \le n$, the elements occurring between two occurrences of i are at least i. (Alternatively, one might say that the elements occurring between two consecutive occurrences of i are larger than i.) We denote the combinatorial family of k-Stirling permutations of order n by $\mathcal{Q}_{n,k}$; note that $k = 2$ yields exactly Stirling permutations as defined by Gessel and Stanley [8], whereas $k = 1$ gives just ordinary permutations.

The previous definition can be extended further in a straightforward way to permutations of a general multiset $\{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$, with $k_i \in \mathbb{N}$ for $1 \le i \le n$. The permutations of the multiset $\{1^{k_1}, 2^{k_2}, \ldots, n^{k_n}\}$ which satisfy, for each i, $1 \le i \le n$, that the elements occurring between two occurrences of i are at least i, are, as in [12], called *generalized Stirling permutations*; these objects have been considered previously by Brenti [5, 6].

The focus of recent studies [4, 11, 12] on (generalized) Stirling permutations has been given to an analysis of various permutation statistics as the number of ascents, descents and plateaux or the number of left-to-right maxima and minima. One interest in studying these combinatorial objects arises from the fact that there are close connections to various important so-called increasing tree families, see, e.g., [12].

In this work we are dealing with enumerative questions arising in the context of the avoidance of patterns in generalized Stirling permutations, where we mainly focus on the most interesting case of

---

Date: September 28, 2010.

Key words and phrases. Stirling permutations, Pattern avoidance, Enumeration formulæ.

The authors were supported by the Austrian Science Foundation FWF, grant S9608-N23.

k-Stirling permutations. Starting with the work [17] of Simion and Schmidt the enumerative study of permutations, which avoid certain ordered patterns, has obtained a lot of attention in combinatorics and lead to remarkable results as, e.g., a proof of the Stanley-Wilf conjecture (see, e.g., [3]). If $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_n)$ are two sequences of numbers, then the sequence α is said to be contained in β as a pattern if there is a subsequence $\beta_{i_1}, \dots, \beta_{i_m}$ of β, with $1 \le i_1 < i_2 < \cdots < i_m \le n$, which is order isomorphic to α, i.e., it holds $\alpha_p \le \alpha_q$ if and only if $\beta_{i_p} \le \beta_{i_q}$. If β does not contain α one says that β avoids the pattern α. In contrast to the many studies treating questions for pattern avoiding permutations of a set there has been done relatively little work for permutations of a multiset; such exceptions are the papers [1] and [10], both dealing with the avoidance of patterns of length three.

Using the notion of pattern avoidance generalized Stirling permutations can simply be characterized as permutations of a multiset, which avoid the pattern 212; in particular, k-Stirling permutations of order n are exactly the 212-avoiding permutations of $\{1^k, 2^k, \dots, n^k\}$. Of course, for $k = 1$ this gives no restriction and one gets all permutations of $\{1, 2, \dots, n\}$. The problem of enumerating permutations of order n that avoid a set of permutation patterns of length 3 has been fully solved (see, e.g., [1, 17]) and leads to simple enumeration formulæ. In particular, the number of permutations of order n that avoid a single permutation pattern of length three is the same for all of the 6 possible patterns and is given by the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. It is now quite natural to ask, whether the problem of enumerating k-Stirling permutations of order n, which avoid a certain permutation pattern (or a set of patterns) of length three, also leads to "nice" enumeration formulæ. That this is indeed the case will be shown in this paper, where the avoidance of all possible sets of permutation patterns of length three is treated. A further aspect is that these formulæ allow to observe the influence of the multiplicities k of the labels to the growth of the number of restricted k-Stirling permutations subject to the specific patterns. E.g., we could show that for k-Stirling permutations there are two classes of single permutation patterns of length three, namely the class $\overline{312} = (312, 213)$ and the class $\overline{231} = (231, 132, 123, 321)$, leading (for general k) even to asymptotically different enumeration formulæ. Whereas avoiding a pattern of the former class leads to generalized Catalan numbers as enumeration formulæ, yield patterns of the latter class formulæ that appeared, for the special case $k = 2$, i.e., Stirling permutations, already in the context of enumerating certain labelled trees [9]; of course, for $k = 1$ both formulæ will specialize to the Catalan numbers leading to only one so-called Wilf-equivalent class (see, e.g., [16]). By extending this standard notion used in that context, we call two patterns *k-Wilf-equivalent* if the cardinalities of the k-Stirling permutations of order n that avoid the one or the other pattern, respectively, are, for all n, always equal; if two patterns are k-Wilf-equivalent, for all $k \ge 1$, we call them *$\mathbb{N}$-Wilf-equivalent*. Thus all single permutation patterns of length three are 1-Wilf-equivalent, but there are two different $\mathbb{N}$-Wilf-equivalent classes.

Besides methods that have been applied previously in the study of pattern avoiding permutations (as treating recurrences with the kernel method and generalizing a classical bijection due to Simion and Schmidt) to show our findings we use results that rely more specifically on the structure of k-Stirling permutations as close relations to increasing tree families and the so-called block decomposition of generalized Stirling permutations. Thus in Section 2 we collect basic facts about generalized Stirling permutations. A summary of our results is given in Section 3. Proofs concerning the avoidance of single patterns of length three are given in Section 4, whereas Section 5-6 are dedicated to the avoidance of multiple patterns of length three.

2. GENERALIZED STIRLING PERMUTATIONS

It is not difficult to see and already stated in [12] that there are exactly $\prod_{j=1}^{n}\big(1 + \sum_{i=1}^{j-1} k_i\big)$ different generalized Stirling permutations of the multiset $\{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$ (note that it is here convenient to allow also $n = 0$, i.e., the empty sequence). This result can be shown by induction, since the k_n copies of n have to form a substring and thus each such generalized Stirling permutation can be obtained uniquely by inserting the string n^{k_n} into a generalized Stirling permutation of the multiset $\{1^{k_1}, 2^{k_2}, \dots, (n-1)^{k_{n-1}}\}$ at one of the $1 + \sum_{i=1}^{n-1} k_i$ possible positions (viz., anywhere in the string, including first or last). Of course, this also gives a simple recursive algorithm to generate all generalized Stirling permutations, or to generate a random generalized Stirling permutation, of an arbitrary multiset. We remark that this enumeration result has also been obtained in [10]. When specializing to the family of k-Stirling permutations $\mathcal{Q}_{n,k}$ we obtain that the numbers $Q_{n,k} := |\mathcal{Q}_{n,k}|$ of different k-Stirling permutations of order n are given by

$$Q_{n,k} = \prod_{i=1}^{n-1} \big(ki+1\big) = n! k^n \binom{n-1+\frac{1}{k}}{n}. \tag{1}$$

As mentioned before there exist close connections between generalized Stirling permutations and various increasing trees families; see, e.g., [13]. Increasing trees are rooted labelled trees, where the nodes of a tree with n nodes (i.e., of order n) are labelled with distinct integers from a given label set L in such a way that each child node has a larger label than its parent node. In this work we only describe and later apply the connection to so-called d-ary increasing trees. A *d-ary tree* is an ordered tree (i.e., the left-to-right order of the children is important), where each node has exactly d positions, where a child might be attached or not (thus there are exactly $\binom{d}{l}$ different possibilities that the sequence of $0 \le l \le d$ nodes $v_1, v_2, \dots, v_l$ is attached to a node v in this left-to-right order). A *d-ary increasing tree* is then an increasingly labelled d-ary tree, where we always assume that the label set $L = \{1, 2, \dots, n\}$ will be used to label a tree of order n. Often it is appropriate to add at each position in a d-ary increasing tree, where no child has been attached, a so-called "external node" (which does not get any label), whereas the original nodes are now called "internal nodes"; then it holds that at each (internal) node there are attached exactly d nodes (internal or external).

Let us denote by $\mathcal{T}_{n,d}$ the family of d-ary increasing trees of order n and by $T_{n,d} := |\mathcal{T}_{n,d}|$ its cardinality. It has been shown in [12, 14] that there exists a bijection between the family $\mathcal{Q}_{n,k}$ of k-Stirling permutations of order n and the family $\mathcal{T}_{n,k+1}$ of $(k+1)$-ary increasing trees of order n, and thus that $Q_{n,k} = T_{n,k+1}$. To keep the paper self-contained we briefly describe this bijection here. Let $\sigma = \sigma_1\sigma_2 \dots \sigma_{kn}$ be a k-Stirling permutation of order n. We construct then the corresponding $(k+1)$-ary increasing tree T via the following recursive procedure. If σ is the empty sequence then we obtain the tree T containing only an external node as the root node; otherwise we decompose the sequence σ into substrings according to the k occurrences of the smallest label ℓ (of course, $\ell = 1$ in the first step) in the sequence, i.e., $\sigma = S_1\, \ell\, S_2\, \ell\, \dots\, \ell\, S_k\, \ell\, S_{k+1}$, where each S_i could be possibly empty; it holds that (after order-preserving relabellings) each S_i is itself a k-Stirling permutation. Thus we can apply recursively this procedure to each of the substrings $S_1, \dots, S_{k+1}$ leading to $k+1$ increasingly labelled trees $T_1, \dots, T_{k+1}$. We construct then the $(k+1)$-ary increasing tree T by attaching the root nodes of $T_1, \dots, T_{k+1}$ in this left-to-right order as subtrees to the node ℓ, which becomes the root node of T. It is not difficult to show that this procedure indeed gives a bijection; the inverse bijection, which we omit to state here, can be described nicely by using a so-called depth-first walk of the tree, see [12]. The bijection is illustrated in Figure 1. To end this paragraph on trees we state the well-known fact that the number of (unlabelled) d-ary trees of order n are given by the generalized Catalan numbers $\frac{1}{(d-1)n+1}\binom{dn}{n}$, see, e.g., [7].

In our studies we will use the block structure of k-Stirling permutations. A *block* in a generalized Stirling permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_s$ is a substring $\sigma_p \dots \sigma_q$, with $\sigma_p = \sigma_q$, that is maximal, i.e., which is not contained in any larger such substring. There is obviously at most one block for every $j \in \{1, 2, \dots, n\}$, extending from the first occurrence of j to the last one; we say that j forms a block if this substring is indeed a block, i.e., when it is not contained in a string $j' \dots j'$, for some

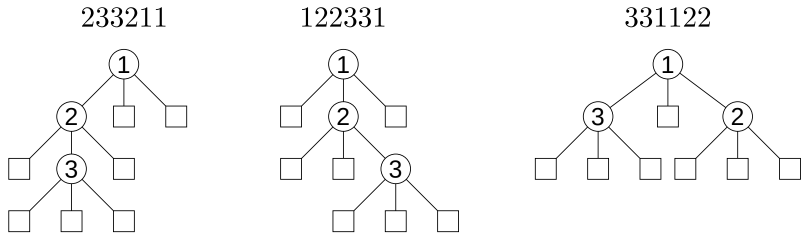

FIGURE 1. Three 2-Stirling permutations of order 3 and the corresponding ternary increasing trees.

$j' < j$. It can be shown easily by induction that any generalized Stirling permutation has a unique decomposition into a sequence of its blocks. As an example we consider the 3-Stirling permutation $\sigma = 355777534443112888221666$ of order 8, which can be decomposed into three blocks leading to the block decomposition $[355777534443][112888221][666]$.

3. RESULTS

We collect here our main results concerning the enumeration of pattern restricted k-Stirling permutations. To do this we first give some notation, which is used throughout this paper. Let α be a pattern; then we always denote by $\mathcal{Q}_{n,k}(\alpha)$ the combinatorial family of k-Stirling permutations of order n that avoid the pattern α and $Q_{n,k}(\alpha) := |\mathcal{Q}_{n,k}(\alpha)|$ its cardinality, i.e., the number of such restricted k-Stirling permutations of order n. The latter notion can be extended to k-Wilf-equivalent classes of patterns; thus $Q_{n,k}(\overline{\alpha})$ denotes the number of k-Stirling permutations of order n that avoid a member of the class $\overline{\alpha}$. We also use $\alpha \overset{(k)}{\equiv} \beta$ or $\alpha \overset{(\mathbb{N})}{\equiv} \beta$, for patterns α and β, to denote that they are k-Wilf-equivalent or $\mathbb{N}$-Wilf-equivalent, respectively. We also use the straightforward extensions of all these terms to a set Λ of patterns. To avoid ambiguity, we enclose sets of patterns into braces, whereas we use parentheses when collecting the members of a class of equivalent patterns.

When stating our results we use the obvious fact that avoiding a pattern $\alpha = \alpha_1 \dots \alpha_m$ and avoiding the reversal $\alpha' = \alpha_m\alpha_{m-1}\dots\alpha_1$ leads, for all generalized Stirling permutations of a multiset $\{1^{k_1}, \dots, n^{k_n}\}$, to the same enumeration formulæ. Thus we can always restrict ourselves to enumerate patterns, which are not obtained by applying the reversal to another pattern, i.e., we can consider classes of reversal-equivalent patterns.

3.1. Avoiding a single pattern of length three.

Theorem 1. *The numbers $Q_{n,k}(\alpha)$ of k-Stirling permutations of order n that avoid a single permutation pattern α of length three are given by the following enumeration formulæ, where we added for $k = 2$, i.e., Stirling permutations, the classification numbers of the sequences in the on-line encyclopedia of integer sequences* [18].

Class name	Representative α	Enumeration formula $Q_{n,k}(\alpha)$	$Q_{n,1}(\alpha)$	$Q_{n,2}(\alpha)$
$\overline{A}_1$	312	$\frac{1}{kn+1}\binom{(k+1)n}{n} \sim \gamma_1\, n^{-\frac{3}{2}}\kappa_1^n$	$\frac{1}{n+1}\binom{2n}{n}$	A001764
$\overline{A}_2$	231	$\sum_{j=0}^{n} \frac{\binom{n}{j}\binom{n+(k-1)j-1}{n-j}}{n+1-j} \sim \gamma_2\, n^{-\frac{3}{2}}\kappa_2^n$	$\frac{1}{n+1}\binom{2n}{n}$	A109081
$\overline{A}_3$	123	$\sum_{j=0}^{n} \frac{\binom{n}{j}\binom{n+(k-1)j-1}{n-j}}{n+1-j} \sim \gamma_2\, n^{-\frac{3}{2}}\kappa_2^n$	$\frac{1}{n+1}\binom{2n}{n}$	A109081

Class name	Representative Λ	Enumeration formula $Q_{n,k}(\Lambda)$	$Q_{n,1}(\Lambda)$	$Q_{n,2}(\Lambda)$
$\overline{B}_1$	$\{312, 213\}$	$(k+1)^{n-1}$	2^{n-1}	A000244
$\overline{B}_2$	$\{312, 231\}$	$\frac{1}{2}\big(\sqrt{k}+1\big)^n + \frac{1}{2}\big(1-\sqrt{k}\big)^n$	2^{n-1}	A001333
$\overline{B}_3$	$\{312, 132\}$	$\sum_{j=0}^{n} \binom{n-1+(k-1)j}{n-j}$	2^{n-1}	A001906
$\overline{B}_4$	$\{312, 123\}$	$1 + k\binom{n}{2}$	$1 + \binom{n}{2}$	A002061
$\overline{B}_5$	$\{312, 321\}$	$\sum_{j=0}^{n} \binom{n-1+(k-1)j}{n-j}$	2^{n-1}	A001906
$\overline{B}_6$	$\{231, 132\}$	$(k+1)2^{n-2}$, for $n \geq 2$	2^{n-1}	A003945
$\overline{B}_7$	$\{231, 123\}$	$1 + \binom{n}{2} - n + \binom{n-1+k}{k}$	$1 + \binom{n}{2}$	A002061
$\overline{B}_8$	$\{231, 321\}$	$\frac{1}{2}\big(\sqrt{k}+1\big)^n + \frac{1}{2}\big(1-\sqrt{k}\big)^n$	2^{n-1}	A001333
$\overline{B}_9$	$\{123, 321\}$	0, for $n > 4$	0, for $n > 4$	

TABLE 1. Avoiding a set of two permutation patterns. For $k = 2$, i.e., Stirling permutations, we added, if available, the classification number of the sequence in the on-line encyclopedia of integer sequences [18]. To compare the results easily we also stated explicitly the well-known formulæ for $k = 1$, i.e., ordinary permutations.

It follows that there are two $\mathbb{N}$-Wilf-equivalent classes of such patterns, namely $\overline{312} = (312, 231)$ *and* $\overline{231} = (231, 132, 123, 321)$*. The values of the constants appearing in the asymptotic expansions are given as follows:*

$$\gamma_1 = \sqrt{\frac{k+1}{2\pi k^3}}, \qquad \kappa_1 = \frac{(k+1)^{k+1}}{k^k},$$
$$\gamma_2 = \sqrt{\frac{(1+(k-1)\tau)^3(1-\tau)}{2\pi k^3\tau^3(2+(k-1)\tau)}}, \qquad \kappa_2 = \frac{1+(k-1)\tau}{k\tau^2},$$

where τ is the smallest positive real root of the equation $(1-\tau)^{k+1} = k\tau^2$.

For the particular instance $k = 2$, i.e., Stirling permutations, this leads to the following asymptotic formulæ:

$$Q_{n,2}(\overline{312}) \sim (0.24430125\ldots) \cdot \frac{(6.75)^n}{n^{\frac{3}{2}}}, \qquad Q_{n,2}(\overline{231}) \sim (0.53692389\ldots) \cdot \frac{(5.21913625\ldots)^n}{n^{\frac{3}{2}}}.$$

3.2. **Avoiding two patterns of length three.**

Theorem 2. *The numbers $Q_{n,k}(\Lambda)$ of k-Stirling permutations of order n that avoid a set $\Lambda = \{\alpha, \beta\}$ of two permutation patterns α and β of length three are, for $n \geq 1$, given in Table 1. As a consequence one obtains that there are seven $\mathbb{N}$-Wilf-equivalent classes of such patterns.*

3.3. **Avoiding three or more patterns of length three.**

Theorem 3. *The numbers $Q_{n,k}(\Lambda)$ of k-Stirling permutations of order n that avoid a set Λ of three permutation patterns of length three are, for $n \geq 1$, given in Table 2. If follows that there are five $\mathbb{N}$-Wilf-equivalent classes of such patterns.*

Theorem 4. *The numbers $Q_{n,k}(\Lambda)$ of k-Stirling permutations of order n that avoid a set Λ of more than three permutation patterns of length three are, for $n \geq 3$, given in Table 3.*

Class name	Representative Λ	Enumeration formula $Q_{n,k}(\Lambda)$	$Q_{n,1}(\Lambda)$
$\overline{C}_1$	$\{312, 231, 123\}$	$n+k-1$, for $n \geq 2$	n
$\overline{C}_2$	$\{312, 231, 321\}$	$\frac{1}{\sqrt{1+4k}}\left(\frac{\sqrt{1+4k}+1}{2}\right)^{n+1} - \frac{1}{\sqrt{1+4k}}\left(\frac{1-\sqrt{1+4k}}{2}\right)^{n+1}$	F_{n+1}
		$Q_{n,2}(\overline{C}_2) = \frac{1}{3}\left(2^{n+1} - (-1)^{n+1}\right)$; A001045	
$\overline{C}_3$	$\{312, 132, 123\}$	$1 + k(n-1)$	n
$\overline{C}_4$	$\{312, 132, 321\}$	$\binom{n-1+k}{k}$	n
$\overline{C}_5$	$\{312, 213, 231\}$	$1 + k(n-1)$	n
$\overline{C}_6$	$\{312, 213, 123\}$	$1 + k(n-1)$	n
$\overline{C}_7$	$\{231, 132, 312\}$	$n+k-1$, for $n \geq 2$	n
$\overline{C}_8$	$\{231, 132, 123\}$	$n+k-1$, for $n \geq 2$	n
$\overline{C}_9$	$\{123, 321, 312\}$	0, for $n > 4$	0, for $n > 4$
$\overline{C}_{10}$	$\{123, 321, 231\}$	0, for $n > 4$	0, for $n > 4$

TABLE 2. Avoiding a set of three permutation patterns. Here F_n denote the Fibonacci numbers, i.e., $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.

Class name	Representative Λ	Enumeration formula $Q_{n,k}(\Lambda)$
$\overline{D}_1$	$\{312, 213, 231, 132\}$	2
$\overline{D}_2$	$\{312, 213, 123, 321\}$	0, for $n \geq 4$; $Q_{3,k}(\overline{D}_2) = 2$
$\overline{D}_3$	$\{231, 132, 123, 321\}$	0, for $n \geq 4$; $Q_{3,k}(\overline{D}_3) = 2$
$\overline{D}_4$	$\{312, 213, 231, 123\}$	$k+1$
$\overline{D}_5$	$\{312, 213, 231, 321\}$	$k+1$
$\overline{D}_6$	$\{231, 132, 312, 123\}$	2
$\overline{D}_7$	$\{231, 132, 312, 321\}$	$k+1$
$\overline{D}_8$	$\{123, 321, 312, 231\}$	0, for $n > 4$; $Q_{3,k}(\overline{D}_8) = 2$; $Q_{4,k}(\overline{D}_8) = 1$
$\overline{D}_9$	$\{123, 321, 312, 132\}$	0, for $n \geq 4$; $Q_{3,k}(\overline{D}_9) = k+1$
$\overline{E}_1$	$\{312, 213, 231, 132, 123\}$	1
$\overline{E}_2$	$\{312, 213, 123, 321, 231\}$	0, for $n \geq 4$; $Q_{3,k}(\overline{E}_2) = 1$
$\overline{E}_3$	$\{231, 132, 123, 321, 312\}$	0, for $n \geq 4$; $Q_{3,k}(\overline{E}_3) = 1$

TABLE 3. Avoiding a set of four or five permutation patterns.

4. AVOIDING A SINGLE PATTERN OF LENGTH THREE

The aim of this section is to provide enumeration formulæ of $Q_{n,k}(\alpha)$, for all six permutation patterns α of length three. As pointed out before, due to the reversal operation, it actually suffices to study representatives of the three pattern classes $\overline{A}_1 = (312, 213)$, $\overline{A}_2 = (231, 132)$ and $\overline{A}_3 = (123, 321)$.

First we show the $\mathbb{N}$-Wilf-equivalence of the patterns 123 and 132 by generalizing the bijection of Simion and Schmidt given in [17] for permutations to k-Stirling permutations; this also implies that $\overline{A}_2 \overset{(\mathbb{N})}{\equiv} \overline{A}_3$.

Theorem 5. *There is a bijection between the family $\mathcal{Q}_{n,k}(123)$ of k-Stirling permutations of order n that avoid the pattern 123 and the family $\mathcal{Q}_{n,k}(132)$ of k-Stirling permutations of order n that avoid the pattern 132. Thus it holds $Q_{n,k}(\overline{A}_2) = Q_{n,k}(\overline{A}_3)$.*

Proof. Let us consider a 123-avoiding k-Stirling permutation $\sigma = \sigma_1\sigma_2\dots\sigma_{kn}$ of order n. We scan now the elements of σ from left to right and distinguish, whether an element σ_i is a (weak) left-to-right minimum or not:

- σ_i is a weak left-to-right minimum: this means that $\sigma_i \le \sigma_\ell$, for all $\ell < i$, or alternatively
$$\sigma_i = \min_{1\le \ell\le i} \sigma_\ell.$$
- σ_i is not a weak left-to-right minimum: then, due to the avoidance of the pattern 123, it must hold that there is no element larger than σ_i to the right of σ_i, i.e., σ_i is the largest element in the multiset $\{1^k, 2^k, \dots, n^k\} \setminus \bigcup_{1\le\ell<i} \sigma_\ell$. Since σ_i is not a left-to-right minimum, we can also characterize σ_i as follows (i.e., as the largest "available" element larger than the current left-to-right minimum):
$$\sigma_i = \max\big(\{(m_i+1)^k, (m_i+2)^k, \dots, n^k\} \setminus \bigcup_{1\le\ell<i} \sigma_\ell\big), \quad \text{with } m_i = \min_{1\le\ell<i} \sigma_\ell.$$
Please note that in the preceeding equation we consider the expression as a multiset, not as a set!

We obtain thus that each 123-avoiding k-Stirling permutation is determined completely by the locations and values of the weak left-to-right minima.

Now let us consider a 132-avoiding k-Stirling permutation $\sigma' = \sigma'_1\sigma'_2\dots\sigma'_{kn}$ of order n and distinguish again, whether an element σ'_i is a weak left-to-right minimum or not:

- σ'_i is a weak left-to-right minimum: so
$$\sigma'_i = \min_{1\le\ell\le i} \sigma'_\ell.$$
- σ'_i is not a weak left-to-right minimum: then, due to the avoidance of the pattern 132, it must hold that there is no element larger than the actual left-to-right minimum, but smaller than σ'_i, to the right of σ'_i. Thus σ'_i can be characterized as follows (i.e., as the smallest "available" element larger than the current left-to-right minimum):
$$\sigma'_i = \min\big(\{(m_i+1)^k, (m_i+2)^k, \dots, n^k\} \setminus \bigcup_{1\le\ell<i} \sigma'_\ell\big), \quad \text{with } m_i = \min_{1\le\ell<i} \sigma'_\ell.$$

Thus also each 132-avoiding k-Stirling permutation is determined completely by the locations and values of the weak left-to-right minima.

The bijection between the family of 123-avoiding k-Stirling permutations of order n and the family of 231-avoiding k-Stirling permutations of order n is then straightforward, i.e., keep all weak left-to-right minima and distribute all remaining elements in the unique possible way as described above not violating the pattern avoidance condition. However, we also state the bijection formally by the following algorithm, which, for each $\sigma \in \mathcal{Q}_{n,k}(123)$ gives a $\sigma' \in \mathcal{Q}_{n,k}(132)$ (we omit the inverse bijection).

Require: $\sigma = \sigma_1\sigma_2\dots\sigma_{kn} \in \mathcal{Q}_{n,k}(123)$
Ensure: Returns $\sigma' = \sigma'_1\sigma'_2\dots\sigma'_{kn} \in \mathcal{Q}_{n,k}(132)$
  for i from 1 to kn **do**
    if σ_i is a weak left-to-right minimum **then**
      $\sigma'_i := \sigma_i$
    else
      $\sigma'_i := \min\big(\{(m_i+1)^k, (m_i+2)^k, \dots, n^k\} \setminus \bigcup_{1\le\ell<i} \sigma'_\ell\big)$, with $m_i := \min_{1\le\ell<i} \sigma'_\ell$
    end if
  end for

It is immediate to see that, when carrying out this algorithm, one never gets stuck, i.e., that we get a sequence σ' (the multiset $\{(m_i+1)^k, (m_i+2)^k, \dots, n^k\} \setminus \bigcup_{1\le \ell<i} \sigma'_\ell$ will never be empty, since it has the same cardinality as the multiset $\{(m_i+1)^k, (m_i+2)^k, \dots, n^k\} \setminus \bigcup_{1\le \ell<i} \sigma_\ell$), and that σ' is a 132-avoiding permutation of the multiset $\{1^k, 2^k, \dots, n^k\}$. It remains to show that σ' is indeed a k-Stirling permutation, i.e., that it also avoids the pattern 212. To do this we consider the 123-avoiding k-Stirling permutation σ and assume that $\sigma_{i_1} = j_1$ and $\sigma_{i_2} = j_2 \le j_1$, with $i_1 < i_2$, are two consecutive weak left-to-right minima. Since σ is 212-avoiding it holds that if an element x appears between σ_{i_1} and σ_{i_2} then all k occurrences of x must appear between σ_{i_1} and σ_{i_2}. Due to the characterization given above this substring of σ will look as follows: $j_1\, x_p^k\, x_{p-1}^k\, \dots\, x_1^k\, j_2$, with $j_1 < x_1 < x_2 < \cdots < x_p$. In particular we obtain that the number of elements between two consecutive weak left-to-right minima is always a multiple of k. Thus when carrying out the above algorithm the corresponding substring of σ' will look as follows: $j_1\, y_1^k\, y_2^k\, \dots\, y_p^k\, j_2$, with $j_1 < y_1 < y_2 < \cdots < y_p$. Thus in σ' the 212-avoidance condition will never be violated, i.e., σ' is a k-Stirling permutation. $\square$

Due to this theorem we obtain the following $\mathbb{N}$-Wilf-equivalent pattern classes $\overline{312} = (312, 213)$ and $\overline{231} = (231, 132, 123, 321)$. Later, when providing enumeration formulæ, we show that these two classes are not k-Wilf-equivalent, for $k > 1$.

Before doing that we generalize the previous bijection further to show that even the number of generalized Stirling permutations of an arbitrary multiset $\{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$ that avoid the pattern 123 and the pattern 132, respectively, are always equal.

Theorem 6. *There is a bijection between the family $\mathcal{Q}_{n,(k_1,k_2,\dots,k_n)}(123)$ of generalized Stirling permutations of $\{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$ that avoid the pattern 123 and the family $\mathcal{Q}_{n,(k_1,k_2,\dots,k_n)}(132)$ of generalized Stirling permutations of $\{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$ that avoid the pattern 132. In particular this implies that $|\mathcal{Q}_{n,(k_1,k_2,\dots,k_n)}(123)| = |\mathcal{Q}_{n,(k_1,k_2,\dots,k_n)}(132)|$.*

Proof. Completely analogeous to the proof of Theorem 5 one obtains that each $\sigma \in \mathcal{Q}_{n,(k_1,k_2,\dots,k_n)}(123)$ and each $\sigma' \in \mathcal{Q}_{n,(k_1,k_2,\dots,k_n)}(132)$ is completely determined by the positions and values of its left-to-right minima. However, in order to provide a bijection between both families one cannot use the algorithm presented in Theorem 5, but one has to move the position of the left-to-right minima according to the number of occurrences k_q of any non-left-to-right minimum q. Such an algorithm, which, for each $\sigma \in \mathcal{Q}_{n,(k_1,k_2,\dots,k_n)}(123)$ gives a $\sigma' \in \mathcal{Q}_{n,(k_1,k_2,\dots,k_n)}(132)$, is presented in the following (again we omit the straightforward inverse bijection).

Require: $\sigma = \sigma_1\sigma_2\dots\sigma_{k_1+\cdots+k_n} \in \mathcal{Q}_{n,(k_1,\dots,k_n)}(123)$
Ensure: Returns $\sigma' = \sigma'_1\sigma'_2\dots\sigma'_{k_1+\cdots+k_n} \in \mathcal{Q}_{n,(k_1,\dots,k_n)}(132)$
  $i := 1; j := 1$
  while $i \le k_1 + \cdots + k_n$ **do**
    if σ_i is a weak left-to-right minimum **then**
      $\sigma'_j := \sigma_i$
      $i := i + 1; j := j + 1$
    else
      $p := \sigma_i; i := i + k_p$
      $q := \min\big(\{(m_j+1)^{k_{m_j+1}}, (m_j+2)^{k_{m_j+2}}, \dots, n^{k_n}\} \setminus \bigcup_{1\le\ell<j} \sigma'_\ell\big)$, with $m_j := \min_{1\le\ell<j} \sigma'_\ell$
      $\sigma'_j\sigma'_{j+1}\dots\sigma'_{j+k_q-1} := q^{k_q}; j := j + k_q$
    end if
  end while

We remark that if $k_r > 0$, for all $1 \le r \le n$, one can replace the corresponding part by the following simpler expression: $q := \min\big(\{(m_j+1), (m_j+2), \dots, n\} \setminus \bigcup_{1\le \ell<j} \sigma'_\ell\big)$ dealing with sets instead of multisets. Following the argumentations as given in the proof of Theorem 5 one easily shows that this algorithm is indeed correct. $\square$

Now we return to the problem of providing enumeration formulæ for $\mathbb{N}$-Wilf-equivalent pattern classes and start with the pattern class $\overline{A}_1 = \overline{312} = (312, 213)$. It is here and later convenient to write $A \prec B$, for strings A and B, if each label contained in A is smaller than any label contained in B.

Theorem 7. *There is a bijection between the family $\mathcal{Q}_{n,k}(312)$ of k-Stirling permutations of order n that avoid the pattern 312 and the family of $(k+1)$-ary trees of order n. Thus the numbers $Q_{n,k}(312) = Q_{n,k}(\overline{A}_1)$ are given by the generalized Catalan numbers:*

$$Q_{n,k}(312) = Q_{n,k}(\overline{A}_1) = \frac{1}{kn+1}\binom{(k+1)n}{n} \sim \sqrt{\tfrac{k+1}{2\pi k^3}} \cdot \frac{\big(\frac{(k+1)^{k+1}}{k^k}\big)^n}{n^{\frac{3}{2}}}.$$

Proof. We use the connection between k-Stirling permutations and $(k+1)$-ary increasing trees described in Section 2. Consider a k-Stirling permutation σ of order $n \ge 1$ that avoids the pattern 312. If we look at the decomposition according to the smallest element 1, i.e., $\sigma = S_1\,1\,S_2\,1\,\dots\,1\,S_k\,1\,S_{k+1}$, it must hold that $S_p \prec S_q$, for all $1 \le p < q \le k+1$; otherwise a subsequence $s_p 1 s_q$ would give the pattern 312. Thus we can write

$$\sigma = S_1\,1\,S_2\,1\,\dots\,1\,S_k\,1\,S_{k+1}, \quad \text{with } 1 \prec S_1 \prec S_2 \prec \cdots \prec S_{k+1}, \tag{2}$$

where each substring S_i, $1 \le i \le k+1$, is (after an order-preserving relabelling) itself a (possibly empty) 312-avoiding k-Stirling permutation. Therefore the same argument can be applied recursively to the substrings $S_1, \dots, S_{k+1}$. If we consider the corresponding $(k+1)$-ary increasing tree T it holds thus that one can remove all of the labels of the tree and could still regain the restricted k-Stirling permutation σ. In other words, for each $(k+1)$-ary tree of order n, there exists exactly one increasing labelling such that the corresponding k-Stirling permutation avoids the pattern 312; this labelling can be described recursively as follows: the root gets the smallest label and all remaining labels are distributed amongst the branches $T_1, T_2, \dots, T_{k+1}$ of the root, such that each label of the branch T_p is smaller than any label of the branch T_q, if $p < q$. This, together with the well-known enumeration formula of $(k+1)$-ary trees, shows the theorem. The given asymptotic expansion follows from a direct application of Stirling's formula for the factorials [7]. $\square$

Now we enumerate the pattern class $\overline{231} = (231, 132, 123, 321)$ by providing a combinatorial decomposition of the k-Stirling permutations avoiding the pattern $231 \in \overline{A}_2$.

Theorem 8. *The numbers $Q_{n,k}(231) = Q_{n,k}(\overline{A}_2)$ of k-Stirling permutations of order n that avoid the pattern 231 are given as follows:*

$$Q_{n,k}(231) = Q_{n,k}(\overline{A}_2) = \sum_{j=0}^{n} \frac{\binom{n}{j}\binom{n+(k-1)j-1}{n-j}}{n+1-j} \sim \sqrt{\frac{(1+(k-1)\tau)^3(1-\tau)}{2\pi k^3\tau^3(2+(k-1)\tau)}} \cdot \frac{\big(\frac{1+(k-1)\tau}{k\tau^2}\big)^n}{n^{\frac{3}{2}}},$$

where τ is the smallest positive real root of the equation $(1-\tau)^{k+1} = k\tau^2$.

Proof. Consider a 231-avoiding k-Stirling permutation σ of order $n \ge 1$. We consider now the decomposition of σ according to the substring n^k formed by the largest element, where we also take into account the block decomposition of k-Stirling permutations described in Section 2. We distinguish two cases.

- The substring n^k is forming a block. Thus when considering the decomposition $\sigma = A\, n^k\, B$ one obtains that the elements of A and B do not have common labels. Since σ is 231-avoiding it must hold that $A \prec B$ (otherwise a subsequence anb would violate this condition). Furthermore (after an order-preserving relabelling), the substrings A and B are itself (possibly empty) 231-avoiding k-Stirling permutations. We write

$$\sigma = A\, n^k\, B, \quad \text{with } A \prec B \prec n. \tag{3}$$

- The substring n^k does not form a block. Then n^k is contained in a block; let us assume this block is formed by element j, with $1 \le j < n$. Let us now consider the decomposition of σ according to the first and last occurrence of j (i.e., the j-block): $\sigma = A\, j\, R\, j\, B$. R contains n^k and, since j is forming a block, R contains only elements $\ge j$. Furthermore, it must hold that A contains only elements $< j$, since otherwise a subsequence anj would give a 231 pattern. On the other hand it must hold that B contains only elements $> j$; otherwise a subsequence jnb would give a 231 pattern. Therefore, the substring A contains all elements $< j$ and is itself a (possibly empty) 231-avoiding k-Stirling permutation.

  Consider now the decomposition of R according to n^k, i.e., $R = P\, n^k\, R'$. It must hold that P does not contain elements $\neq j$; otherwise a subsequence pnj would violate the 231-avoidance condition. In other words, P can be only a (possibly empty) substring formed by consecutive j's. Using all this information we obtain the following refinement of the decomposition of σ according to the k occurrences of j, with $1 \le \ell \le k-1$:

$$\sigma = A\, j^\ell\, n^k\, R_{k-\ell}\, j\, R_{k-\ell-1}\, j\, \dots\, j\, R_1\, j\, B.$$

  We consider now the substrings R_i. For each R_i, $1 \le i \le k-\ell$, it must hold that the elements contained in R_i are forming a non-increasing sequence, since otherwise a subsequence $r_i r_i' j$ would give a 231 pattern. Moreover, it must hold that $R_p \prec R_q$, for all $1 \le p < q \le k-\ell$; otherwise a subsequence $r_q r_p j$ would violate the 231-avoidance condition. Therefore the substrings $R_1, R_2, \dots, R_{k-\ell}$ are given as follows:

$$\begin{aligned} &R_1 = r_{t_1}^k r_{t_1-1}^k \dots r_1^k, \quad R_2 = r_{t_1+t_2}^k r_{t_1+t_2-1}^k \dots r_{t_1+1}^k, \quad \dots, \\ &R_{k-\ell} = r_{t_1+\dots+t_{k-\ell}}^k r_{t_1+\dots+t_{k-\ell}-1}^k \dots r_{t_1+\dots+t_{k-\ell-1}+1}^k, \end{aligned} \tag{4}$$

  with $j < r_1 < r_2 < \dots < r_{t_1+\dots+t_{k-\ell}} < n$ and $t_1, \dots, t_{k-\ell} \ge 0$.

  Finally we consider the substring B. Take an arbitrary element r_i, with $1 \le i \le t_1 + \dots + t_{k-\ell}$, as defined before. Then it must hold that each element contained in B with a label smaller than r_i must preceed any element contained in B with a label larger than r_i; otherwise a substring $r_i b b'$ would occur that violates the 231-avoidance condition. This implies the following decomposition of B into substrings: $B = C_0 C_1 \dots C_{t_1+\dots+t_{k-\ell}}$, where the substring C_i, $0 \le i \le t_1 + \dots + t_{k-\ell}$, contains all elements with a label x satisfying $r_i < x < r_{i+1}$; we set here $r_0 := j$ and $r_{t_1+\dots+t_{k-\ell}+1} := n$. Furthermore, each of the substrings C_i, $0 \le i \le t_1 + \dots + t_{k-\ell}$, is, after an order-preserving relabelling, itself a 231-avoiding k-Stirling permutation. Thus in this case the decomposition of σ can be illustrated as follows:

$$\sigma = A\, j^\ell\, n^k\, R_{k-\ell}\, j\, R_{k-\ell-1}\, j\, \dots\, j\, R_1\, j\, C_0\, C_1\, \dots\, C_{t_1+\dots+t_{k-\ell}}, \tag{5}$$

  $1 \le \ell \le k-1$, with $A \prec j \prec C_0 \prec r_1 \prec C_1 \prec r_2 \prec C_2 \prec \dots \prec r_{t_1+\dots+t_{k-\ell}} \prec C_{t_1+\dots+t_{k-\ell}} \prec n$ and where $R_1, \dots, R_{k-\ell}$ are specified in (4).

We denote now by $\mathcal{Q}$ the combinatorial family of (possibly empty) 231-avoiding k-Stirling permutations (for notational convenience we suppress here the occurrence of k). The decompositions described before can then be translated easily into the following formal equation for $\mathcal{Q}$:

$$\mathcal{Q} = \{\epsilon\} \,\dot\cup\, \mathcal{Z} \times \mathcal{Q} \times \mathcal{Q} \,\dot\cup\, \dot{\bigcup}_{\ell=1}^{k-1} \Big(\mathcal{Q} \times \mathcal{Z} \times \mathcal{Z} \times \big((\mathcal{Z} \times \mathcal{Q})^{*} \big)^{k-\ell} \times \mathcal{Q} \Big), \tag{6}$$

where ϵ denotes the empty string, $\mathcal{Z}$ the family containing a generic string of k copies of a label, $\dot\cup$ the disjoint union and $\times$ the cartesian product of combinatorial families, $\mathcal{A}^r$ the family of sequences of length r of objects from a family $\mathcal{A}$, and $\mathcal{A}^*$ the family of (possibly empty) sequences of objects from $\mathcal{A}$; see, e.g., [7].

Now we introduce the generating function

$$Q(z) := \sum_{n \geq 0} Q_{n,k}(231) z^n$$

of the number $Q_{n,k}(231)$ of 231-avoiding k-Stirling permutations of order n. Using the so-called symbolic method, again see [7], the formal equation (6) can be translated directly into the following equation for $Q = Q(z)$:

$$Q = 1 + zQ^2 + \sum_{\ell=1}^{k-1} \frac{z^2 Q^2}{(1 - zQ)^{k-\ell}},$$

which, after simple manipulations, gives

$$Q = 1 + zQ(Q-1) + \frac{zQ}{(1 - zQ)^{k-1}},$$

and eventually

$$Q = 1 + \frac{zQ}{(1 - zQ)^{k}}. \tag{7}$$

Introducing $\tilde{Q}(z) := zQ(z)$ equation (7) leads to the following equation for $\tilde{Q} = \tilde{Q}(z)$, which will be advantageous when extracting coefficients:

$$\tilde{Q} = z \left(1 + \frac{\tilde{Q}}{(1 - \tilde{Q})^{k}} \right). \tag{8}$$

To extract coefficients from $\tilde{Q}$ we apply the Lagrange inversion formula (see, e.g., [7, 19]) to (8) and obtain:

$$\begin{aligned} Q_{n,k}(231) &= [z^n]Q(z) = [z^{n+1}]\tilde{Q}(z) = \frac{1}{n+1} [\tilde{Q}^n] \Big(1 + \frac{\tilde{Q}}{(1-\tilde{Q})^k} \Big)^{n+1} \\ &= \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} [\tilde{Q}^{n-j}] \frac{1}{(1-\tilde{Q})^{kj}} = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \binom{n-j+kj-1}{n-j} \\ &= \sum_{j=0}^{n} \frac{\binom{n}{j}\binom{n+(k-1)j-1}{n-j}}{n+1-j}. \end{aligned} \tag{9}$$

To get an asymptotic expansion of these numbers it is advantagous not to deal with the exact formula but applying singularity analysis [7] to the corresponding generating function. By using singularity analysis in [7] a formula for the asymptotic expansion of the coefficients of generating functions of the form $y(z) = z\Phi\big(y(z)\big)$ has already been given. It holds (under certain conditions, see [7, p. 452–454]):

$$[z^n]y(z) = \sqrt{\frac{\phi(\tau)}{2\phi''(\tau)}}\frac{\rho^{-n}}{\sqrt{\pi n^3}}\Big(1 + \mathcal{O}\big(\frac{1}{n}\big)\Big),$$

where τ is the smallest positive real root of the equation $\phi(t) - t\phi'(t) = 0$ and $\rho = \frac{\tau}{\phi(\tau)}$. Due to equation (8) for $\tilde{Q} = zQ$ a direct application of this formula with $\phi(t) = 1 + \frac{t}{(1-t)^k}$ leads, after easy computations, which are omitted here, to the asymptotic expansion given in Theorem 8. This completes the proof of the theorem. □

Although we have shown in Theorem 5 via a bijection that $Q_{n,k}(123) = Q_{n,k}(132)$ we give in the following also a direct proof of the enumeration formulæ for $Q_{n,k}(123) = Q_{n,k}(\overline{A}_3)$. To do this we establish a recurrence, which will be treated by an application of the so-called kernel method (see, e.g., [2, 15]).

Theorem 9. *The numbers $Q_{n,k}(123)$ of 123-avoiding k-Stirling permutations of order n satisfy*

$$Q_{n,k}(123) = Q_{n,k}(\overline{A}_3) = \sum_{j=0}^{n} \frac{\binom{n}{j}\binom{n+(k-1)j-1}{n-j}}{n+1-j}.$$

Proof. Let σ be a k-Stirling permutation of order $n = n(\sigma)$: $\sigma = \sigma_1\sigma_2\dots\sigma_{kn}$. We denote by $m = m(\sigma)$ the number of elements σ_i, such that, when inserting the string $(n(\sigma)+1)^k$ directly after σ_i, the resulting k-Stirling permutation $\sigma' = \sigma_1\dots\sigma_i(n+1)^k\sigma_{i+1}\dots\sigma_{kn}$ is still 123-avoiding. Of course, $m = m(\sigma)$ is the first index, where an ascent occurs, i.e., $\sigma_m < \sigma_{m+1}$ (if there is such one; otherwise, $m = kn$).

We consider now all possible ways of inserting the string $(n+1)^k$ into σ leading to a 123-avoiding k-Stirling permutation σ' and determine $m(\sigma')$.

- Insert $(n+1)^k$ into σ before σ_1, i.e., $\sigma' = (n+1)^k\sigma_1\dots\sigma_{kn}$. Then it holds:
$$m(\sigma') = m(\sigma) + k.$$
- Insert $(n+1)^k$ into σ after the ℓ-th of the $m(\sigma)$ possible elements, i.e., after σ_ℓ: $\sigma' = \sigma_1\dots\sigma_\ell(n+1)^k\sigma_{\ell+1}\dots\sigma_{kn}$. It holds then:
$$m(\sigma') = \ell, \quad \text{with } 1 \le \ell \le m(\sigma).$$

Let us denote now by $F_{n,m}$ the number of 123-avoiding k-Stirling permutations σ of order $n = n(\sigma)$ with $m = m(\sigma)$. Using the previous considerations we obtain the following recursive description of $F_{n,m}$:

$$F_{n,m} = F_{n-1,m-k} + \sum_{\ell=m}^{k(n-1)} F_{n-1,\ell}, \quad \text{for } n \ge 1 \text{ and } 1 \le m \le kn, \qquad F_{0,0} = 0, \tag{10}$$

where we assume that $F_{n,m} = 0$, otherwise. We introduce now the generating function

$$Q(z,u) := \sum_{n\ge0}\sum_{0\le m\le n} F_{n,m}z^nu^m.$$

Please note that evaluating $Q(z,u)$ at $u = 1$ leads to the generating function of the required numbers $Q_{n,k}(123)$, i.e., $Q(z,1) = \sum_{n\ge0} Q_{n,k}(123)z^n$. The recurrence (10) leads then, after straightforward computations, to the following equation involving $Q(z,u)$ and $Q(z,1)$:

$$\left(1 - zu^k + \frac{zu}{1-u}\right)Q(z,u) = 1 + \frac{zu}{1-u}Q(z,1). \tag{11}$$

Now let $u(z)$ be the suitable root of the kernel (see [2]), i.e.,

$$1 - zu(z)^k + \frac{zu(z)}{1-u(z)} = 0. \tag{12}$$

Plugging $u(z)$ as given by (12) into (11) leads to

$$Q := Q(z,1) = \frac{u(z)-1}{zu(z)}$$

or $u(z) = \frac{1}{1-zQ}$. Plugging this into equation (12) shows, after simple manipulations, that $Q = Q(z,1)$ satisfies the following equation:

$$Q = 1 + \frac{zQ}{(1-zQ)^k}. \tag{13}$$

But (13) is exactly equation (7), which is satisfied also by the generating function of the number $Q_{n,k}(231)$ of 231-avoiding k-Stirling permutations of order n. Extracting coefficients as carried out in Theorem 8 shows then the enumeration formula. □

We remark that a similar approach using the kernel method could be used also to obtain the enumeration results for $Q_{n,k}(\overline{A}_1)$ and $Q_{n,k}(\overline{A}_2)$. However, it seems that the proofs presented in Theorem 5-8 have the advantage of revealing more information on the structure of these restricted k-Stirling permutations and furthermore they are also useful in the next sections when considering sets of patterns.

5. Avoiding a set of two patterns of length three

In this section we prove the enumeration formulæ of $Q_{n,k}(\Lambda)$, where Λ consists of two permutation patterns of length three, given in Theorem 2. Due to the reversal operation it actually suffices to study representatives of the nine pattern classes $\overline{B}_1, \dots, \overline{B}_9$ as given in Theorem 2. To show the results we heavily use equations (2), (3) and (5), i.e., the decomposition of 312-avoiding k-Stirling permutations according to the smallest element and of 231-avoiding k-Stirling permutations according to the largest element as given in the proof of Theorem 7 and Theorem 8, respectively. Furthermore, we use here and later the notation $S\nearrow$ and $S\searrow$ to express that the substring S of a k-Stirling permutation is forming a sequence of non-decreasing labels or non-increasing labels, respectively, i.e., $S = r_1^k\, r_2^k\, \dots\, r_p^k$ or $S = r_p^k\, r_{p-1}^k\, \dots\, r_1^k$, with $r_1 < r_2 < \dots < r_p$.

Theorem 10. *The numbers $Q_{n,k}(\{312, 213\}) = Q_{n,k}(\overline{B}_1)$ are given as follows:*

$$Q_{n,k}(\overline{B}_1) = (k+1)^{n-1}, \quad \text{for } n \ge 1, \qquad Q_{0,k}(\overline{B}_1) = 1.$$

Proof. We start with the decomposition of a non-empty 312-avoiding k-Stirling permutation σ into $\sigma = S_1\,1\,S_2\,1\,\dots\,1\,S_k\,1\,S_{k+1}$, with $1 \prec S_1 \prec S_2 \prec \dots \prec S_{k+1}$, as given by (2). Since σ is also 213-avoiding it follows that at most one of the substrings S_i, $1 \le i \le k+1$, is not the empty string ϵ (otherwise a subsequence $s_p 1 s_q$ would violate this condition). Thus only the following two cases can occur:

- $S_1 = S_2 = \dots = S_{k+1} = \epsilon$, i.e., $\sigma = 1^k$.
- There exists an ℓ, $1 \le \ell \le k+1$, such that $S_\ell \ne \epsilon$, but $S_i = \epsilon$, for all $i \ne \ell$. Then S_ℓ is (after an order-preserving relabelling) a non-empty $\{312, 213\}$-avoiding k-Stirling permutation, i.e., it holds $\sigma = 1^{\ell-1}\, S_\ell\, 1^{k+1-\ell}$, $1 \le \ell \le k+1$, with $1 \prec (S_\ell \ne \epsilon)$.

Using this characterization one obtains that the generating function $Q := Q(z) = \sum_{n\ge 0} Q_{n,k}(\{312, 213\}) z^n$ satisfies the equation

$$Q = 1 + z + (k+1)z(Q-1),$$

which implies

$$Q = \frac{1-kz}{1-(k+1)z}. \tag{14}$$

Extracting coefficients from (14) immediately shows Theorem 10. ☐

Theorem 11. *The numbers $Q_{n,k}(\{312, 231\}) = Q_{n,k}(\overline{B}_2)$ are given as follows:*

$$Q_{n,k}(\overline{B}_2) = \frac{1}{2}\big(\sqrt{k}+1\big)^n + \frac{1}{2}\big(1-\sqrt{k}\big)^n, \quad \text{for } n \geq 1, \qquad Q_{0,k}(\overline{B}_2) = 1.$$

Proof. We consider the decomposition (2) of a non-empty 312-avoiding k-Stirling permutation σ. Since σ is also 231-avoiding it follows that at most one of the substrings S_i, $1 \leq i \leq k$, is not the empty string ϵ (otherwise a subsequence $s_p 1 s_q$ would violate this condition). Thus only the following two cases can occur:

- $S_1 = S_2 = \cdots = S_k = \epsilon$. Then S_{k+1} is a possibly empty $\{312, 231\}$-avoiding k-Stirling permutation, i.e., it holds $\sigma = 1^k\, S_{k+1}$, with $1 \prec S_{k+1}$.
- There exists an ℓ, $1 \leq \ell \leq k$, such that $S_\ell \neq \epsilon$, but $S_i = \epsilon$, for $1 \leq i \leq k$ and $i \neq \ell$. In this case it must hold that S_ℓ is forming a (non-empty) non-increasing sequence of labels (otherwise a subsequence $s_\ell s'_\ell 1$ would give a 231 pattern), whereas S_{k+1} is a possibly empty $\{312, 231\}$-avoiding k-Stirling permutation, i.e., $\sigma = 1^{\ell-1}\, S_\ell\, 1^{k+1-\ell}\, S_{k+1}$, $1 \leq \ell \leq k$, with $1 \prec (S_\ell \neq \epsilon) \searrow \prec S_{k+1}$.

This implies that the generating function $Q := Q(z) = \sum_{n\geq 0} Q_{n,k}(\{312, 231\}) z^n$ satisfies the equation

$$Q = 1 + zQ + \frac{kz^2 Q}{1-z},$$

and is thus given by

$$Q = \frac{1-z}{1-2z-(k-1)z^2}. \tag{15}$$

Since partial fraction expansion of (15) leads to

$$Q = \frac{1}{2\big(1-\frac{(k-1)z}{\sqrt{k}-1}\big)} + \frac{1}{2\big(1+\frac{(k-1)z}{\sqrt{k}+1}\big)}, \tag{16}$$

extracting coefficients from (16) easily shows Theorem 11. ☐

Of course, from (15) follows further that the numbers $Q_{n,k}(\overline{B}_2)$ satisfy, for $n \geq 2$, the recurrence $Q_n = 2Q_{n-1} + (k-1)Q_{n-2}$.

Theorem 12. *The numbers $Q_{n,k}(\{312, 132\}) = Q_{n,k}(\overline{B}_3)$ are given as follows:*

$$Q_{n,k}(\overline{B}_3) = \sum_{j=0}^{n} \binom{n-1+(k-1)j}{n-j}.$$

For the special case $k = 2$, i.e., Stirling permutations, one obtains that $Q_{n,2}(\overline{B}_3) = F_{2n}$, for $n \geq 1$, where F_n denote the Fibonacci-numbers, i.e., $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.

Proof. The decomposition (2) of a non-empty 312-avoiding k-Stirling permutation σ implies that each of the substrings $S_2, \dots, S_{k+1}$ is forming a (possibly empty) non-decreasing sequence of labels (otherwise the 132-avoidance condition would be violated by a sequence $1 s_i s'_i$), whereas the substring S_1 is an arbitrary possibly empty $\{312, 132\}$-avoiding k-Stirling permutation, i.e., $\sigma = S_1\, 1\, S_2\, 1\, \dots\, 1\, S_{k+1}$, with $1 \prec S_1 \prec S_2 \nearrow \prec S_3 \nearrow \prec \cdots \prec S_{k+1} \nearrow$.

Thus the generating function $Q := Q(z) = \sum_{n\geq 0} Q_{n,k}(\{312, 132\})z^n$ satisfies the equation

$$Q = 1 + \frac{zQ}{(1-z)^k}$$

and is therefore given by

$$Q = \frac{1}{1 - \frac{z}{(1-z)^k}}. \tag{17}$$

Extracting coefficients from (17) easily shows the first part of Theorem 12.

For $k = 2$ one obtains $Q = 1 + \frac{z}{1-3z+z^2}$. The generating function $F(z) = \sum_{n\geq 0} F_n z^n$ of the Fibonacci-numbers is given by $F(z) = \frac{z}{1-z-z^2}$, which implies that the generating function $\tilde{F}(z) = \sum_{n\geq 0} F_{2n} z^n$ of the even-indexed Fibonacci numbers can be obtained by $\tilde{F}(z) = \frac{F(\sqrt{z})+F(-\sqrt{z})}{2}$; after simple manipulations one obtains $\tilde{F}(z) = \frac{z}{1-3z+z^2}$. Since $\tilde{F}(z) = Q(z) - 1$ this completes the proof of the theorem. □

Theorem 13. *The numbers $Q_{n,k}(\{312, 123\}) = Q_{n,k}(\overline{B}_4)$ are given as follows:*

$$Q_{n,k}(\overline{B}_4) = 1 + k\binom{n}{2}.$$

Proof. When considering the decomposition (2) of a non-empty 312-avoiding k-Stirling permutation σ one obtains that at most one of the substrings S_i, $2 \leq i \leq k+1$, is not the empty string ϵ, since otherwise a substring $1s_p s_q$ would violate the 123-avoidance condition. Thus only the following two cases can occur:

- $S_2 = S_3 = \cdots = S_{k+1} = \epsilon$. Then S_1 is a possibly empty $\{312, 123\}$-avoiding k-Stirling permutation, i.e., it holds $\sigma = S_1\, 1^k$, with $1 \prec S_1$.
- There exists an ℓ, $2 \leq \ell \leq k+1$, such that $S_\ell \neq \epsilon$, but $S_i = \epsilon$, for $2 \leq i \leq k+1$ and $i \neq \ell$. In this case it must hold that the substring S_1 is forming a possibly empty – and the substring S_ℓ is forming a non-empty – non-increasing sequence of labels (otherwise a subsequence $s_1 s_1' s_\ell$ or a subsequence $1 s_\ell s_\ell'$ would give a 123 pattern), i.e., $\sigma = S_1\, 1^{\ell-1}\, S_\ell\, 1^{k+1-\ell}$, $2 \leq \ell \leq k+1$, with $1 \prec S_1 \searrow \prec (S_\ell \neq \epsilon) \searrow$.

Thus the generating function $Q := Q(z) = \sum_{n\geq 0} Q_{n,k}(\{312, 123\})z^n$ satisfies the equation

$$Q = 1 + zQ + \frac{kz^2}{(1-z)^2},$$

which implies

$$Q = \frac{1}{1-z} + \frac{kz^2}{(1-z)^3}. \tag{18}$$

Extracting coefficients from (18) immediately proves Theorem 13. □

Theorem 14. *The numbers $Q_{n,k}(\{312, 321\}) = Q_{n,k}(\overline{B}_5)$ are given as follows:*

$$Q_{n,k}(\overline{B}_5) = \sum_{j=0}^{n} \binom{n-1+(k-1)j}{n-j}.$$

Proof. The decomposition (2) of a non-empty 312-avoiding k-Stirling permutation σ implies that each of the substrings $S_1, \dots, S_k$ is forming a (possibly empty) non-decreasing sequence of labels (otherwise the 321-avoidance condition would be violated by a sequence $1s_i s_i'$), whereas the substring S_{k+1} is an arbitrary possibly empty $\{312, 321\}$-avoiding k-Stirling permutation, i.e., $\sigma = S_1\, 1\, S_2\, 1\, \dots\, S_{k+1}$, with $1 \prec S_1 \nearrow \prec S_2 \nearrow \prec \cdots \prec S_k \nearrow \prec S_{k+1}$.

Therefore the generating function $Q := Q(z) = \sum_{n\ge 0} Q_{n,k}(\{312, 321\})z^n$ satisfies the equation

$$Q = 1 + \frac{zQ}{(1-z)^k},$$

which gives

$$Q = \frac{1}{1 - \frac{z}{(1-z)^k}}. \tag{19}$$

Thus the generating functions (19) and (17) coincide, which proves Theorem 14. ☐

Theorem 15. *The numbers* $Q_{n,k}(\{231, 132\}) = Q_{n,k}(\overline{B}_6)$ *are given as follows:*

$$Q_{n,k}(\overline{B}_6) = (k+1)2^{n-2}, \quad \textit{for } n \ge 2, \qquad Q_{0,k}(\overline{B}_6) = Q_{1,k}(\overline{B}_6) = 1.$$

Proof. We start with the decomposition of a non-empty 231-avoiding k-Stirling permutation σ according to the occurrence of the largest element $n = n(\sigma)$ as given by (3) and (5), respectively.

- If n^k is forming a block then (3) gives $\sigma = A\, n^k\, B$, with $A \prec B \prec n$. Since σ is also 132-avoiding it must hold that $A = \epsilon$ or $B = \epsilon$ (otherwise a substring anb would violate this condition). This leads to the following three possibilities: $(i) : \sigma = 1^k$; $(ii) : \sigma = A\, n^k$, with $(A \ne \epsilon) \prec n$; $(iii) : \sigma = n^k\, B$, with $(B \ne \epsilon) \prec n$, where A and B, respectively, are non-empty $\{231, 132\}$-avoiding k-Stirling permutations.
- If n^k is not forming a block then (5) and (4) lead to $\sigma = A\, j^\ell\, n^k\, R_{k-\ell}\, j\, R_{k-\ell-1}\, j\, \dots\, j\, R_1\, j\, C_0\, C_1\, \dots\, C_{t_1+\dots+t_{k-\ell}}$, $1 \le \ell \le k-1$, with $A \prec j \prec C_0 \prec r_1 \prec C_1 \prec r_2 \prec C_2 \prec \cdots \prec r_{t_1+\dots+t_{k-\ell}} \prec C_{t_1+\dots+t_{k-\ell}} \prec n$. Since σ is also 132-avoiding it must hold that $R_1 = R_2 = \cdots = R_{k-\ell} = \epsilon$ (which implies that $C_1 = C_2 = \cdots = C_{t_1+\dots+t_{k-\ell}} = \epsilon$), since otherwise a subsequence jnr would violate this condition. Moreover, one gets $C_0 = \epsilon$ (otherwise a subsequence jnc gives a 132-pattern) and $A = \epsilon$ (otherwise a subsequence anj gives a 132-pattern). Therefore in this case one obtains that $\sigma = 1^\ell\, 2^k\, 1^{k-\ell}$, with $1 \le \ell \le k-1$.

Using this characterization one obtains that the generating function $Q := Q(z) = \sum_{n\ge 0} Q_{n,k}(\{231, 132\})z^n$ satisfies the equation

$$Q = 1 + z\big(1 + 2(Q-1)\big) + (k-1)z^2,$$

which implies

$$Q = \frac{1 - z + (k-1)z^2}{1-2z}. \tag{20}$$

Extracting coefficients from (20) immediately shows Theorem 15. ☐

Theorem 16. *The numbers* $Q_{n,k}(\{231, 123\}) = Q_{n,k}(\overline{B}_7)$ *are given as follows:*

$$Q_{n,k}(\overline{B}_7) = 1 + \binom{n}{2} - n + \binom{n-1+k}{k}.$$

Proof. We consider the decompositions (3) and (5), respectively, of a non-empty 231-avoiding k-Stirling permutation σ.

- n^k is forming a block. Since σ is also 123-avoiding one obtains from (3) that only the following two cases can occur: $(i) : A = \epsilon$, then B is an arbitrary possibly empty $\{231, 123\}$-avoiding k-Stirling permutation, i.e., $\sigma = n^k B$, with $B \prec n$; $(ii) : A \ne \epsilon$, then it must hold that the substring A is forming a non-empty – and the substring B is forming a possibly empty – non-increasing sequence of labels (otherwise a subsequence $aa'n$ or a subsequence abb' would give a 123 pattern), i.e., $\sigma = A\, n^k\, B$, with $(A \ne \epsilon) \searrow \prec B \searrow \prec n$.

- n^k is not forming a block. Since σ is also 123-avoiding we get from (5) and (4) that $A = \epsilon$ (otherwise a subsequence ajn would violate this condition) and $C_1 = C_2 = \cdots = C_{t_1+\cdots+t_{k-\ell}} = \epsilon$ (otherwise a subsequence jrc would violate this condition). Furthermore, C_0 has to form a (possibly empty) non-increasing sequence of labels; otherwise a subsequence jcc' gives the pattern 123. Thus this case implies $\sigma = 1^\ell\, n^k\, R_{k-\ell}\, 1\, R_{k-\ell-1}\, 1\, \dots\, 1\, R_1\, 1\, C_0$, $1 \le \ell \le k-1$, with $1 \prec C_0 \searrow \prec R_1 \searrow \prec \cdots \prec R_{k-\ell} \searrow$.

Therefore the generating function $Q := Q(z) = \sum_{n\ge 0} Q_{n,k}(\{231, 123\})z^n$ satisfies the equation

$$Q = 1 + zQ + \frac{z^2}{(1-z)^2} + \frac{z^2}{1-z}\sum_{\ell=1}^{k-1}\frac{1}{(1-z)^{k-\ell}},$$

which implies

$$Q = \frac{1}{1-z} + \frac{z^2}{(1-z)^3} - \frac{z}{(1-z)^2} + \frac{z}{(1-z)^{k+1}}. \tag{21}$$

Extracting coefficients from (21) immediately shows Theorem 16. $\square$

We remark that, since $Q_{n,2}(\overline{B}_7) = Q_{n,2}(\overline{B}_4) = 1 + 2\binom{n}{2}$, it holds $\overline{B}_7 \overset{(2)}{\equiv} \overline{B}_4$, but $\overline{B}_7 \overset{(\mathbb{N})}{\not\equiv} \overline{B}_4$.

Theorem 17. *The numbers $Q_{n,k}(\{231, 321\}) = Q_{n,k}(\overline{B}_8)$ are given as follows:*

$$Q_{n,k}(\overline{B}_8) = \frac{1}{2}\big(\sqrt{k}+1\big)^n + \frac{1}{2}\big(1-\sqrt{k}\big)^n, \quad \text{for } n \ge 1, \qquad Q_{0,k}(\overline{B}_8) = 1.$$

Proof. Again we consider the decompositions (3) and (5), respectively, of a non-empty 231-avoiding k-Stirling permutation σ.

- n^k is forming a block. Since σ is also 321-avoiding we obtain from (3) that B has to form a (possibly empty) non-decreasing sequence of labels (otherwise a subsequence nbb' would violate this condition), whereas A might be an arbitrary possibly empty $\{231, 321\}$-avoiding k-Stirling permutation, i.e., $\sigma = A\, n^k\, B$, with $A \prec B \nearrow \prec n$.
- n^k is not forming a block. Since σ is also 321-avoiding one gets from (5) and (4) that $R_1 = R_2 = \cdots = R_{k-\ell} = \epsilon$ (which also implies $C_1 = C_2 = \cdots = C_{t_1+\cdots+t_{k-\ell}} = \epsilon$); otherwise a subsequence nrj would give a 321 pattern. Furthermore, C_0 has to form a (possibly empty) non-decreasing sequence of labels (otherwise a subsequence ncc' would give a 321 pattern), whereas A is a possibly empty $\{231, 321\}$-avoiding k-Stirling permutation, i.e., $\sigma = A\, j^\ell\, n^k\, j^{n-\ell}\, C_0$, with $A \prec j \prec C_0 \nearrow \prec n$.

Thus the generating function $Q := Q(z) = \sum_{n\ge 0} Q_{n,k}(\{231, 321\})z^n$ satisfies the equation

$$Q = 1 + \frac{zQ}{1-z} + \frac{(k-1)z^2 Q}{1-z},$$

which gives

$$Q = \frac{1-z}{1-2z-(k-1)z^2}. \tag{22}$$

Thus the generating functions (22) and (15) coincide, which proves Theorem 17. $\square$

Theorem 18. *The numbers $Q_{n,k}(\{123, 321\}) = Q_{n,k}(\overline{B}_9)$ are given as follows:*

$$Q_{0,k}(\overline{B}_9) = Q_{1,k}(\overline{B}_9) = 1, \qquad Q_{2,k}(\overline{B}_9) = k+1, \qquad Q_{3,k}(\overline{B}_9) = Q_{4,k}(\overline{B}_9) = 2k+2,$$
$$Q_{n,k}(\overline{B}_9) = 0, \quad \text{for } n > 4.$$

Proof. A special instance of the Erdős-Szekeres theorem (see, e.g, [20]) shows that any sequence containing more than four distinct labels contains the pattern 123 or the pattern 321. Thus $Q_{n,k}(\{123, 321\}) = 0$, for $n > 4$.

The results for the remaining cases follow easily by inspection: for $n \le 2$ the theorem is trivially true; for $n = 3$ one obtains that either $\sigma = 2^k\, 1^{\ell-1}\, 3^k\, 1^{k+1-\ell}$, with $1 \le \ell \le k+1$, or $\sigma = 1^{\ell-1}\, 3^k\, 1^{k+1-\ell}\, 2^k$, $1 \le \ell \le k+1$; and for $n = 4$ one gets that either $\sigma = 2^k\, 1^{\ell-1}\, 4^k\, 1^{k+1-\ell}\, 3^k$, with $1 \le \ell \le k+1$, or $\sigma = 3^k\, 1^{\ell-1}\, 4^k\, 1^{k+1-\ell}\, 2^k$, with $1 \le \ell \le k+1$; thus the theorem also holds for the two latter cases. □

6. AVOIDING A SET OF AT LEAST THREE PATTERNS OF LENGTH THREE

This section is devoted to a study of k-Stirling permutations avoiding a set of three or more patterns of length three. The results for more than three patterns stated in Theorem 4 are just given for the sake of completeness; since they can be obtained easily from the proof of Theorem (3) by further inspection we omit here these straightforward considerations. But also when proving the enumeration formulæ of $Q_{n,k}(\Lambda)$, where Λ consists of three permutation patterns of length three, which are given in Theorem 3, we will be more brief than in the preceeding sections, since the results follow easily by starting with the characterizations appearing in the proofs of Theorem 10-18 when avoiding a set of two patterns and adapting them to the additional restrictions.

6.1. **Proof of Theorem 3.** Due to the reversal operation it fully suffices to study representatives of the ten pattern classes $\overline{C}_1, \dots, \overline{C}_{10}$ as given in Theorem 3. During the proofs we use the generic generating function $Q := Q(z) = \sum_{n\ge0} Q_{n,k}(\overline{C}) z^n$, where $\overline{C}$ is the pattern class considered.

Pattern $\{312, 231, 123\} \in \overline{C}_1$*:* Adapting the characterization given in the proof of Theorem 11 to satisfy also the additional 123-avoidance shows that each non-empty restricted k-Stirling permutation σ is described by one of the following three cases:

- $\sigma = 1^k\, S_{k+1}$, with $1 \prec S_{k+1} \searrow$.
- $\sigma = S_1\, 1^k\, S_{k+1}$, with $1 \prec (S_1 \neq \epsilon) \searrow \prec S_{k+1} \searrow$.
- $\sigma = 1^{\ell-1}\, S_\ell\, 1^{k+1-\ell}$, $2 \le \ell \le k$, with $1 \prec (S_\ell \neq \epsilon) \searrow$.

This implies that

$$Q = 1 + \frac{z}{1-z} + \frac{z^2}{(1-z)^2} + \frac{(k-1)z^2}{1-z},$$

and extracting coefficients gives

$$Q_{n,k}(\overline{C}_1) = n + k - 1, \quad \text{for } n \ge 2, \qquad Q_{0,k}(\overline{C}_1) = Q_{1,k}(\overline{C}_1) = 1.$$

Pattern $\{312, 231, 321\} \in \overline{C}_2$*:* The proof of Theorem 11 together with the additional 321-avoidance shows that each non-empty restricted k-Stirling permutation σ is described by one of the following two cases:

- $\sigma = 1^k\, S_{k+1}$, with $1 \prec S_{k+1}$.
- $\sigma = 1^{\ell-1}\, 2^k\, 1^{k+1-\ell}\, S_{k+1}$, $1 \le \ell \le k$, with $1 \prec 2 \prec S_{k+1}$.

This leads to the equation $Q = 1 + zQ + kz^2 Q$, which gives

$$Q = \frac{1}{1 - z - kz^2}.$$

Applying the partial fraction expansion and extracting coefficients easily shows that

$$Q_{n,k}(\overline{C}_2) = \frac{1}{\sqrt{1+4k}} \Big(\frac{\sqrt{1+4k}+1}{2}\Big)^{n+1} - \frac{1}{\sqrt{1+4k}} \Big(\frac{1-\sqrt{1+4k}}{2}\Big)^{n+1}.$$

Of course, from the generating function Q follows immediately that the numbers $Q_{n,k}(\overline{C}_2)$ satisfy, for $n \geq 2$, the recurrence $Q_n = Q_{n-1} + kQ_{n-2}$. For the special instance $k = 2$, i.e., Stirling permutations, we obtain $Q_{n,2}(\overline{C}_2) = \frac{1}{3}\big(2^{n+1} - (-1)^{n+1}\big)$, the so-called Jacobsthal numbers.

Pattern $\{312, 132, 123\} \in \overline{C}_3$*:* Adapting the characterization given in the proof of Theorem 12 to satisfy also the additional 123-avoidance shows that each non-empty restricted k-Stirling permutation σ is described by one of the following two cases:

- $\sigma = S_1\, 1^k$, with $1 \prec S_1$.
- $\sigma = S_1\, 1^{\ell-1}\, n^k\, 1^{k+1-\ell}$, $2 \leq \ell \leq k+1$, with $1 \prec S_1 \searrow \prec n$.

Thus we obtain the equation $Q = 1 + zQ + \frac{kz^2}{1-z}$, and further

$$Q = \frac{1}{1-z} + \frac{kz^2}{(1-z)^2}.$$

Extracting coefficients immediately gives

$$Q_{n,k}(\overline{C}_3) = 1 + k(n-1), \quad \text{for } n \geq 1, \qquad Q_{0,k}(\overline{C}_3) = 1.$$

Pattern $\{312, 132, 321\} \in \overline{C}_4$*:* The proof of Theorem 12 together with the additional 321-avoidance shows that each non-empty restricted k-Stirling permutation σ can be described as follows:

- $\sigma = S_1\, 1\, S_2\, 1\, \dots\, 1\, S_{k+1}$, with $1 \prec S_1 \nearrow \prec S_2 \nearrow \prec \cdots \prec S_{k+1} \nearrow$.

This implies

$$Q = 1 + \frac{z}{(1-z)^{k+1}},$$

and extracting coefficients leads to

$$Q_{n,k}(\overline{C}_4) = \binom{n-1+k}{k}, \quad \text{for } n \geq 1, \qquad Q_{0,k}(\overline{C}_4) = 1.$$

Pattern $\{312, 213, 231\} \in \overline{C}_5$*:* Adapting the characterization given in the proof of Theorem 10 to satisfy also the additional 231-avoidance shows that each non-empty restricted k-Stirling permutation σ is described by one of the following two cases:

- $\sigma = 1^k\, S_{k+1}$, with $1 \prec S_{k+1}$.
- $\sigma = 1^{\ell-1}\, S_\ell\, 1^{k+1-\ell}$, $1 \leq \ell \leq k$, with $1 \prec (S_\ell \neq \epsilon) \searrow$.

This leads to the equation $Q = 1 + zQ + \frac{kz^2}{1-z}$, and further

$$Q = \frac{1}{1-z} + \frac{kz^2}{(1-z)^2}.$$

Thus we obtain the same result as for the pattern class $\overline{C}_3$.

Pattern $\{312, 213, 123\} \in \overline{C}_6$*:* The proof of Theorem 10 together with the additional 123-avoidance shows that each non-empty restricted k-Stirling permutation σ is described by one of the following two cases:

- $\sigma = S_1\, 1^k$, with $1 \prec S_1$.
- $\sigma = 1^{\ell-1}\, S_\ell\, 1^{k+1-\ell}$, $2 \leq \ell \leq k+1$, with $1 \prec (S_\ell \neq \epsilon) \searrow$.

This implies the equation $Q = 1 + zQ + \frac{kz^2}{1-z}$, and thus

$$Q = \frac{1}{1-z} + \frac{kz^2}{(1-z)^2}.$$

Therefore we also obtain the same result as for the pattern class $\overline{C}_3$.

Pattern $\{231, 132, 312\} \in \overline{C}_7$*:* Adapting the characterization given in the proof of Theorem 15 to satisfy also the additional 312-avoidance shows that each non-empty restricted k-Stirling permutation σ is described by one of the following three cases:

- $\sigma = A\, n^k$, with $A \prec n$.
- $\sigma = n^k\, B$, with $(B \neq \epsilon) \searrow \prec n$.
- $\sigma = 1^\ell\, 2^k\, 1^{k-\ell}$, $1 \le \ell \le k-1$.

Thus we obtain the equation $Q = 1 + zQ + \frac{z^2}{1-z} + (k-1)z^2$, and further

$$Q = \frac{1}{1-z} + \frac{z^2}{(1-z)^2} + \frac{(k-1)z^2}{1-z}.$$

Therefore we obtain the same result as for the pattern class $\overline{C}_1$.

Pattern $\{231, 132, 123\} \in \overline{C}_8$*:* The proof of Theorem 15 together with the additional 123-avoidance shows that each non-empty restricted k-Stirling permutation σ is described by one of the following three cases:

- $\sigma = n^k\, B$, with $B \prec n$.
- $\sigma = A\, n^k$, with $(A \neq \epsilon) \searrow \prec n$.
- $\sigma = 1^\ell\, 2^k\, 1^{k-\ell}$, $1 \le \ell \le k-1$.

This gives the equation $Q = 1 + zQ + \frac{z^2}{1-z} + (k-1)z^2$, and further

$$Q = \frac{1}{1-z} + \frac{z^2}{(1-z)^2} + \frac{(k-1)z^2}{1-z}.$$

Thus we also obtain the same result as for the pattern class $\overline{C}_1$.

Pattern $\{123, 321, 312\} \in \overline{C}_9$*:* From Theorem 18 it already follows that $Q_{n,k}(\overline{C}_9) = 0$, for $n > 4$. For $n \le 4$ one gets the following:

$$Q_{0,k}(\overline{C}_9) = Q_{1,k}(\overline{C}_9) = 1, \quad Q_{2,k}(\overline{C}_9) = k+1, \quad Q_{3,k}(\overline{C}_9) = k+2, \quad Q_{4,k}(\overline{C}_9) = 1.$$

This can be shown easily by inspection. For $n = 3$ one obtains that σ is given by one of the following two cases:

$$(i) : \sigma = 2^k\, 1^{\ell-1}\, 3^k\, 1^{k+1-\ell}, \;\; 1 \le \ell \le k+1; \qquad (ii) : \sigma = 1^k\, 3^k\, 2^k,$$

whereas for $n = 4$ the only possible case is $\sigma = 2^k\, 1^k\, 4^k\, 3^k$.

Pattern $\{123, 321, 231\} \in \overline{C}_{10}$*:* Again it follows from Theorem 18 that $Q_{n,k}(\overline{C}_{10}) = 0$, for $n > 4$. But also for $n \le 4$ one obtains the same enumeration results as for the pattern class $\overline{C}_9$, i.e.,

$$Q_{0,k}(\overline{C}_{10}) = Q_{1,k}(\overline{C}_{10}) = 1, \quad Q_{2,k}(\overline{C}_{10}) = k+1, \quad Q_{3,k}(\overline{C}_{10}) = k+2, \quad Q_{4,k}(\overline{C}_{10}) = 1.$$

Here one obtains for $n = 3$ that σ is given by one of the following two cases:

$$(i) : \sigma = 1^{\ell-1}\, 3^k\, 1^{k+1-\ell}\, 2^k, \;\; 1 \le \ell \le k+1; \qquad (ii) : \sigma = 2^k\, 1^k\, 3^k,$$

whereas for $n = 4$ the only possible case is $\sigma = 2^k\, 1^k\, 4^k\, 3^k$.

REFERENCES

[1] M. Albert, R. Aldred, M. Atkinson, C. Handley and D. Holton, Permutations of a multiset avoiding permutations of length 3, *European Journal of Combinatorics* 22, 1021–1031, 2001.

[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy and D. Gouyou-Beauchamps, Generating functions for generating trees, *Discrete Mathematics* 246, 29–55, 2002.

[3] M. Bona, *Combinatorics of permutations*, Chapman & Hall, Boca Raton, 2004.

[4] M. Bona, Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley, *SIAM Journal on Discrete Mathematics* 23, 401–406, 2008.

[5] F. Brenti, Unimodal, log-concave, and Pólya frequency sequences in combinatorics, *Memoirs of the American Mathematical Society* 81, no. 413, 1989.

[6] F. Brenti, Hilbert polynomials in combinatorics, *Journal of Algebraic Combinatorics* 7, 127–156, 1998.

[7] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.

[8] I. Gessel and R. P. Stanley, Stirling polynomials, *Journal of Combinatorial Theory, Series A* 24, no. 1, 24–33, 1978.

[9] N. Gu, N. Li and T. Mansour, 2-binary trees: bijections and related issues, *Discrete Mathematics* 308, 1209–1221, 2008.

[10] S. Heubach and T. Mansour, Avoiding patterns of length three in compositions and multiset permutations, *Advances in Applied Mathematics* 36, 156–174, 2006.

[11] S. Janson, Plane recursive trees, Stirling permutations and an urn model. *Discrete Mathematics and Theoretical Computer Science, Proceedings AI*, 541–547, 2008.

[12] S. Janson, M. Kuba and A. Panholzer, Generalized Stirling permutations, families of increasing trees and urn models, *Journal of Combinatorial Theory, Series A*, to appear, 2010.

[13] M. Kuba and A. Panholzer, Analysis of statistics for generalized Stirling permutations, submitted, 2010.

[14] S. K. Park, The r-multipermutations, *Journal of Combinatorial Theory, Series A* 67, no. 1, 44–71, 1994.

[15] H. Prodinger, The kernel method: a collection of examples, *Séminaire Lotharingien de Combinatoire* 50, Article B50f, 2004.

[16] A. Robertson, Restricted permutations from Catalan to Fine and back, *Séminaire Lotharingien de Combinatoire* 50, Article B50g, 2004.

[17] R. Simion and F. Schmidt, Restricted permutations, *European Journal of Combinatorics* 6, 383–406, 1985.

[18] N. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://www.research.att.com/~njas/sequences/`.

[19] R. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, Cambridge, 1999.

[20] J. M. Steele, Variations on the monotone subsequence theme of Erdős and Szekeres, in: *Discrete probability and algorithms*, 111-131, The IMA Volumes in Mathematics and its Applications, 72, Springer, New York, 1995.

MARKUS KUBA, INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, TECHNISCHE UNIVERSITÄT WIEN, WIEDNER HAUPTSTR. 8-10/104, 1040 WIEN, AUSTRIA

E-mail address: `kuba@dmg.tuwien.ac.at`

ALOIS PANHOLZER, INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE, TECHNISCHE UNIVERSITÄT WIEN, WIEDNER HAUPTSTR. 8-10/104, 1040 WIEN, AUSTRIA

E-mail address: `Alois.Panholzer@tuwien.ac.at`