Counting Forests with Stirling and Bell Numbers

Dennis Walsh
Middle Tennessee State University

We will show how unsigned Stirling numbers of the first kind and Stirling numbers of the second kind provide counts for forests of certain types. The forests in which we are interested have monotonic-labeled vertices and a fixed number of rooted trees. In particular we will show

(i) the number of monotonic-labeled forests on n vertices with exactly k rooted trees is given by $|s(n,k)|$, an unsigned Stirling number of the first kind;

(ii) the number of monotonic-labeled forests on n vertices with exactly k rooted trees, each of height one or less, is given by $S(n,k)$, a Stirling number of the second kind.

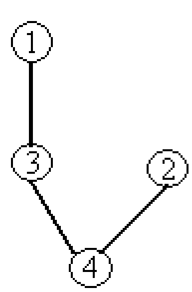
I. Monotonic-Labeled Forests, No-Less-Than Functions, and Permutations

In this section we will provide definitions of various terms (monotonic-labeled forests, no-less-than functions, etc.), along with examples, that are needed in deriving our results for counting forests. We also introduce a bijection between a set of forests and a set of no-less-than functions.

■ Monotonic-Labeled Forests

Definition. A labeled, rooted tree is *monotonic-labeled* if the label y of any parent vertex is greater than the label x of any offspring vertex.

Example 1. The following rooted tree is labeled monotonically.

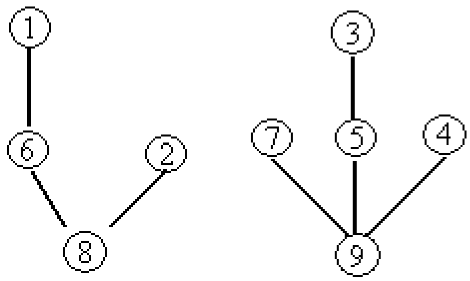


◇

Notation. Let $[n]$ denote the set $\{1, 2, ..., n\}$.

Definition. A monotonic-labeled, rooted forest on vertex set $[n]$ is a disjoint union of monotonic-labeled rooted trees.

Example 2. The following graph is a monotonic-labeled, rooted forest on vertex set $\{1, ..., 9\}$.



◇

▪ No-Less-Than Functions

Definition. A function $f : [n] \rightarrow [n]$ is called a *no-less-than function* on $[n]$ if $f(x) \geq x$ for all $x \in [n]$.

Example 3. The function $f = \{(1, 4),\ (2, 2),\ (3, 5),\ (4, 5),\ (5, 5)\}$ is a no-less-than function on $\{1, 2, 3, 4, 5\}$. ◇

Theorem 1. There is a one-to-one correspondence between the set of monotonic-labeled, rooted forests on vertex set $[n]$ and the set of no-less-than functions on $[n]$.

Proof. We construct the bijection that maps a monotonic-labeled, rooted forest F to a no-less-than function f. Given a monotonic-labeled, rooted forest F on set $[n]$, let $f : [n] \rightarrow [n]$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is a root of a tree in the forest} \\ \\ f(x) = y & \text{if } x \text{ is a child of } y. \end{cases}$$

A vertex of F is either a root or a child. Hence f is a no-less-than function since $f(x) = x$ (when x is a root) or $f(x) = y > x$ (when x is a child whose label is less than the label of its parent). This mapping of F to f is one-to-one since a forest different from F has either a different set of roots or a difference in parent-child labelings.

On the other hand, given a no-less-than function f on $[n]$, define a labeled forest F such that x is a root of F if $f(x) = x$ and x is a child of y if $f(x) = y$. Note that F is monotonically labeled since $x < f(x) = y$ when x is a child of y. This inverse mapping of f to F is one-to-one since a different no-less-than-function will have either different fixed elements or a different function value for some non-fixed element. □

Example 4. The forest in the example 2 above corresponds to the no-less-function

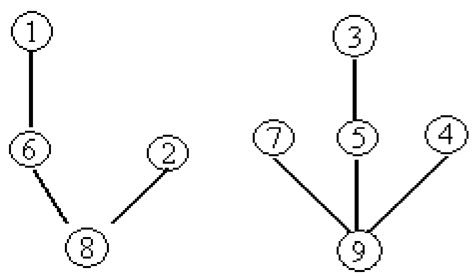
$$f = \{(1,6),\ (2,8),\ (3,5),\ (4,9),\ (5,9),\ (6,8),\ (7,9),\ (8,8),\ (9,9)\}. \qquad \diamond$$

The cardinality of the set of no-less-than functions from $[n]$ to $[n]$ is readily seen to be $n!$ since there are n choices for $f(1)$, $(n-1)$ choices for $f(2)$, ..., and 1 choice for $f(n)$. Since $n!$ is also the number of permutations on n elements, there exists a one-to-one correspondence between the symmetric group S_n, the set of permutations on $[n]$, and the set of no-less-than functions on $[n]$. We provide a correspondence below.

- **Permutations**

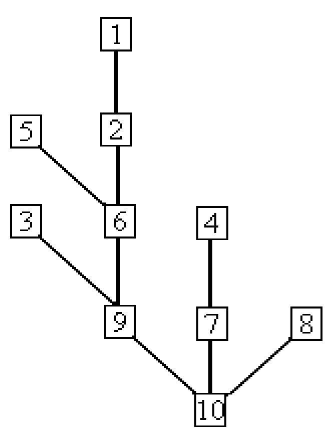
Given a permutation π, write π in cyclic form. The unique no-less-than function corresponding to π is defined as follows. For any x that is not the largest element of a cycle, let $f(x) = \pi^k(x)$ where k is the smallest positive integer such that $\pi^k(x) > x$. If x is the largest element in a cycle, then $f(x) = x$.

Example 5. Let $\pi = (1\ 6\ 2\ 8)\,(7\ 3\ 5\ 4\ 9)$. Then $f(1) = 6,\ f(6) = 8,\ f(2) = 8,$ $f(8) = 8,\ f(7) = 9,\ f(3) = 5,\ f(5) = 9,\ f(4) = 9,$ and $f(9) = 9$. Note that f is exactly the no-less-than function in example 4 above, which itself corresponds to the forest in example 2



$\diamond$

Example 6. Given $\pi = (5\ \ 1\ \ 2\ \ \ 6\ \ 3\ \ 9\ \ \ 8\ \ 4\ \ 7\ \ 10)$, the corresponding no-less-function is $f = \{(1,2),\ (2,6),\ (3,9),\ (4,\ 7),\ (5,6),\ (6,9),\ (7,10),\ (8,\ 10),\ (9,\ 10),\ (10,10)\}$.The corresponding monotonic-labeled, rooted forest consists of the one tree given below.



$\diamond$

II. The Count of Monotonically-Labeled Rooted Forests

From the results in Section I, we see that each permutation in S_n can be uniquely associated with a monotonically-labeled rooted forest on n vertices. If a permutation has k cycles, its corresponding forest has k trees. Furthermore, since the number of permutations in S_n with exactly k cycles is $|s(n,k)|$, where $s(n,k)$ denotes a Stirling number of the first kind, the number of monotonically-labeled rooted forests with k trees and n vertices is also an unsigned Stirling number of the first kind.

Theorem 2. Let $F_{n,k}$ denote the set of monotonically-labeled rooted forests on n vertices and with exactly k trees. The cardinality of $F_{n,k}$ is $|s(n,k)|$, a unsigned Stirling number of the first kind, given by

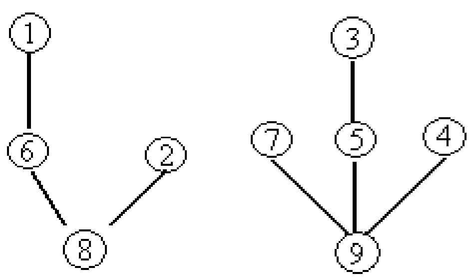
$$|s(n,k)| = \sum_A a_1 a_2 \cdot \cdot \cdot a_{n-k} \tag{1}$$

where the sum is over all $\binom{n-1}{n-k}$ subsets A of $\{1, 2, ..., n-1\}$.

***Proof*.** As shown in Section I above, the cardinality of $F_{n,k}$ equals the cardinality of the set of permutations on $[n]$ with exactly k cycles, that is, $|s(n,k)|$. A proof that identity 1 holds is given in my article *A Short Note on Unsigned Stirling Numbers* (available online at *http://frank.mtsu.edu/~dwalsh/STIRLIN1.pdf*) $\square$

Example 7. Below is a monotonically-labeled rooted forests on 9 vertices and with exactly 2 trees, one element of $F_{9,2}$

.



The cardinality of $F_{9,2}$ is given by

$$|s(9,2)| = \sum_A a_1 a_2 \cdot \cdot \cdot a_{n-k}$$

where the sum is over all $\binom{8}{7}$ subsets A of $\{1, 2, ..., 8\}$.

Hence

$$\begin{aligned} |s(9,2)| &= 8!(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) \\ &= 8!(\tfrac{761}{280}) = 109,584. \end{aligned}$$ $\diamond$

Table of Unsigned Stirling Numbers

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	2	3	1							
4	6	11	6	1						
5	24	50	35	10	1					
6	120	274	225	85	15	1				
7	720	1764	1624	735	175	21	1			
8	5040	13068	13132	6769	1960	322	28	1		
9	40320	109584	118124	67284	22449	4536	546	36	1	
10	362880	1026756	1172700	723680	269325	63273	9450	870	45	1

III. The Count of Monotonically-Labeled Rooted Forests with a Height Restriction

Suppose we place a height restriction on our labeled-rooted forests, namely that the height of all trees be one or less. When we do, we obtain the following result.

Theorem 3. Consider monotonic-labeled forests on n vertices with exactly k rooted trees, each of height one or less. The number of such forests is equal to

$$S_2(n,k) = \tfrac{1}{k!}\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}j^n,$$

a Stirling number of the second kind.

Proof. $S_2(n,k)$ denotes the number of ways to assign n numbered balls into k indistinguishable boxes, leaving none of the boxes empty. Such a ball assignment has a one-to-one correspondence to a permutation with exactly k cycles, each cycle having entries in decreasing (or increasing) order. Furthermore, using results from Section 1, such a permutation has a one-to-one correspondence to a monotonically-labeled forest with exactly k rooted trees of height 1 or less. $\square$

Example 8. Consider monotonic-labeled forests on 7 vertices with exactly 3 rooted trees, each of height one or less. The number of such forests is equal to

$$\begin{aligned} S_2(7,3) &= \tfrac{1}{3!}(3-3(2^7)+3^7) \\ &= 301 \end{aligned}$$

$\diamond$

Table of $S_2(n,k)$, Stirling Numbers of the Second Kind

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

An Identity for Stirling Numbers of the Second Kind

For positive integers j, k and n, the following identity holds for Stirling numbers of the second kind :

$$\boxed{S_2(n,j+k)=\frac{1}{\binom{j+k}{j}}\sum_{i=k}^{n-j}\binom{n}{i}S_2(i,k)S_2(n-i,j)}.$$

Example 9.

$$S_2(6,4)=S_2(6,3+1)=\frac{1}{\binom{4}{1}}\sum_{i=3}^{5}\binom{6}{i}S_2(i,3)S_2(6-i,1)$$

$$=\tfrac{1}{4}[20(1)(1)+15(6)(1)+6(25)(1)]=65$$

and, also,

$$S_2(6,4)=S_2(6,2+2)=\frac{1}{\binom{4}{2}}\sum_{i=2}^{4}\binom{6}{i}S_2(i,2)S_2(6-i,2)$$

$$=\tfrac{1}{6}[15(1)(7)+20(3)(3)+15(7)(1)]=65$$

Example 10. Using Maple software, we calculate $S_2(n,2+1)$ for $n=1..10$.

```
> s2:=(n,k,j)->sum(binomial(n,i)*stirling2(i,k)*stirling2(n-i,j),i=k..n-j)/binomial(k+j,k);

> seq(s2(n,2,1),n=1..10);
            0, 0, 1, 6, 25, 90, 301, 966, 3025, 9330
```

Theorem 4. Consider monotonic-labeled forests on n vertices with rooted trees all of height one or less. The number of such forests is equal to the Bell number B_n. (See sequence A000110, OEIS, at http://oeis.org/A000110).

Proof. By Theorem 3, the number of such forests is

$$\sum_{k=1}^{n} S_2(n,k) \ = B_n. \qquad \square$$

Bell Numbers

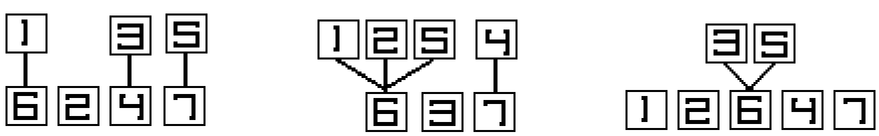
n	1	2	3	4	5	6	7	8	9	10	11	12
B_n	1	2	5	15	52	203	877	4140	21147	115975	678570	4213597

We note that

$$B_n \ = \sum_{k=1}^{n} \frac{1}{k!} \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} j^n = \sum_{j=0}^{\infty} \frac{j^n}{j!} \sum_{k=j}^{\infty} \frac{(-1)^{j-k}}{(k-j)!}$$

$$= \ e^{-1} \sum_{i=0}^{\infty} \frac{i^n}{i!} \, .$$

Example 9. Consider monotonic-labeled forests on 7 vertices with rooted trees all of height one or less. Several examples of such forests are shown below.



The number of such forests is equal to the Bell number $B_7 = 877$. $\diamond$

Example 10. A group of 12 people are occupying a downtown public space. If the police attempt to disperse the group, a "dispersement" occurs. A null dsipersement means the group of 12 remain intact, that is, the police failed. A complete dispersement means all 12 people are individually isolated. A partial dispersement means the set of 12 people has been broken up into smaller groups but not all 12 are individually isolated. In essence, a dipsersement is a partition of the set of 12 people. How many possible dispersements are there? The answer is the twelfth Bell number: $4,213,597$.