ON SOME CONJECTURES ON THE MONOTONICITY OF SOME ARITHMETICAL SEQUENCES *

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Abstract

Here, we prove some conjectures on the monotony of combinatorial and number–theoretical sequences from a recent paper of Zhi–Wei Sun.

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1. Introduction

In [Sun], it was conjectured that the sequences of general term $a_{n+1}^{1/(n+1)}/a_n^{1/n}$, or $a_n^{1/n}$ are monotonically increasing (or decreasing) for all $n \geq n_0$, for a large class of sequences $\mathbf{a} = (a_n)_{n \geq 1}$ appearing in combinatorics. Three of these conjectures were confirmed recently in [HSW]. Here, we confirm eight more of these conjectures (some partially, up to an explicit starting index n_0).

For a sequence $\mathbf{u} = (u_n)_{n \geq 1}$ and a positive integer k we write

$$\Delta^{(k)}u_n = u_{n+k} - \binom{k}{1}u_{n+k-1} + \cdots + (-1)^j \binom{k}{j}u_{n+k-j} + \cdots + (-1)^k u_n,$$

for the kth iterated difference of $(u_{n+k}, \ldots, u_n)$.

2. Bernoulli, Tangent and Euler numbers

The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are rational numbers given by

$$B_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad \text{for all} \quad n \geq 1,$$

whose exponential generating function is

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!}.$$

It is well-known that $B_{2n+1} = 0$ for all $n \geq 1$. Closely connected to the Bernoulli numbers are the Tangent numbers T_n and the Euler numbers E_n, defined by their exponential generating functions

$$\tan z = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{T_{2k+1} z^{2k+1}}{(2k+1)!}, \tag{2.1}$$

$$\sec z = \sum_{k=0}^{\infty} (-1)^k \frac{E_{2k} z^{2k}}{(2k)!}. \tag{2.2}$$

Thus, $E_{2k-1} = 0, T_{2k} = 0, k \geq 1$. We recall Stirling's formula

$$n! = (n/e)^n \sqrt{2\pi n}\, e^{\theta_n}, \quad \text{where} \quad \frac{1}{12n+1} < \theta_n < \frac{1}{12n}, \quad \text{for all} \quad n \geq 1. \tag{2.3}$$

Our first result gives an affirmative answer to Conjecture 2.15 and 3.5 in [Sun].

Theorem 2.1. *The sequences* $(|B_{2n}|^{1/n})_{n \geq 1}$, $(|T_{2n-1}|^{1/n})_{n \geq 1}$, $(|E_{2n}|^{1/n})_{n \geq 1}$ *are all increasing. Furthermore, the sequences* $(|B_{2n+2}|^{1/(n+1)}/|B_{2n}|^{1/n})_{n \geq 2}$, $(|T_{2n+1}|^{1/(n+1)}/|T_{2n-1}|^{1/n})_{n \geq 1}$, $(|E_{2n+2}|^{1/(n+1)}/|E_{2n}|^{1/n})_{n \geq 1}$ *are decreasing.*

Proof. We start with the Bernoulli numbers. We use the formula

$$|B_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n).$$

Clearly,

$$\zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots = 1 + \eta_n,$$

where

$$\eta_n = \frac{1}{2^{2n}} \left(1 + \frac{1}{1.5^{2n}} + \frac{1}{2^{2n}} + \cdots \right)$$
$$\leq \frac{1}{2^{2n}} \left(1 + \frac{1}{1.5^{2n}} + 2(\zeta(2n) - 1) \right) \leq \frac{3}{2^{2n}} \tag{2.4}$$

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for $n \geq 1$. Thus, putting $|B_{2n}| = \exp v_n$, we have that

$$
\begin{aligned}
\frac{v_n}{n} &= \frac{\log\left(2(2n)!(2\pi)^{-2n}(1+\eta_n)\right)}{n} \\
&= \frac{\log 2}{n} + \frac{\log(2n)!}{n} - 2\log(2\pi) + \frac{\log(1+\eta_n)}{n} \\
&= \frac{\log 2}{n} + 2\log(2n) - 2 + \frac{\log(4\pi n)}{2n} + \frac{\theta_{2n}}{n} - 2\log(2\pi) + \frac{\log(1+\eta_n)}{n} \\
&= 2\log n + c + \frac{\log n}{2n} + \frac{\log(16\pi)}{2n} + \frac{\theta_{2n}}{n} + \frac{\log(1+\eta_n)}{n}, \quad\quad (2.5)
\end{aligned}
$$

where $c = 2\log 2 - 2 - 2\log(2\pi) = -2 - 2\log \pi$. Taking the first iterated difference in (2.5) above, we get

$$
\begin{aligned}
\Delta^{(1)}\left(\frac{v_n}{n}\right) &= \frac{v_{n+1}}{n+1} - \frac{v_n}{n} \\
&= 2\log\left(1+\frac{1}{n}\right) + \left(\frac{\log(n+1)}{2(n+1)} - \frac{\log n}{2n}\right) - \frac{\log(16\pi)}{2n(n+1)} \\
&\quad + \left(\frac{\theta_{2n+2}}{n+1} - \frac{\theta_{2n}}{n}\right) + \left(\frac{\log(1+\eta_{n+1})}{n+1} - \frac{\log(1+\eta_n)}{n}\right).
\end{aligned}
$$

By the Intermediate Value Theorem

$$
\frac{\log(n+1)}{n+1} - \frac{\log n}{n} = \frac{d}{dx}\left(\frac{\log x}{x}\right)\Big|_{x=\zeta\in[n,n+1]},
$$

therefore

$$
\left|\frac{\log(n+1)}{n+1} - \frac{\log n}{n}\right| < \frac{\log(n+1)}{n^2}.
$$

Using the inequalities

$$
\log\left(1+\frac{1}{n}\right) \geq \frac{1}{2n} \quad \text{for} \quad n \geq 1,
$$

$$
\log(1+x) \leq x \quad \text{for all real numbers} \quad x, \quad\quad (2.6)
$$

with $x = \eta_n$ and $x = \eta_{n+1}$, inequality (2.4) together with Stirling's formula (2.3), we get

$$
\Delta^{(1)}\left(\frac{v_n}{n}\right) \geq \frac{1}{n} - \frac{\log(n+1)}{2n^2} - \frac{\log(16\pi)}{2n(n+1)} - \frac{1}{12n^2} - \frac{6}{2^{2n}n}, \quad\quad (2.7)
$$

and the above expression is positive for $n \geq 3$. This proves that $|B_{2n}|^{1/n}$ is increasing for $n \geq 3$, and by hand one checks that this is in fact true for all $n \geq 1$.

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Taking now the second iterated difference in (2.5), one gets

$$
\begin{aligned}
\Delta^{(2)}\left(\frac{v_n}{n}\right) &= \left(\frac{v_{n+2}}{n+2} - \frac{v_{n+1}}{n+1}\right) - \left(\frac{v_{n+1}}{n+1} - \frac{v_n}{n}\right) \\
&= 2\left(\log\left(1 + \frac{1}{n+1}\right) - \log\left(1 + \frac{1}{n}\right)\right) \\
&\quad + \frac{1}{2}\left(\frac{\log(n+2)}{(n+2)} - \frac{2\log(n+1)}{(n+1)} + \frac{\log n}{n}\right) \\
&\quad + \frac{\log(16\pi)}{n(n+1)(n+2)} \\
&\quad + \left(\frac{\theta_{2n+4}}{n+2} - \frac{2\theta_{2n+2}}{n+1} + \frac{\theta_{2n}}{n}\right) \\
&\quad + \left(\frac{\log(1 + \eta_{n+2})}{n+2} - \frac{2\log(1 + \eta_{n+1})}{n+1} + \frac{\log(1 + \eta_n)}{n}\right). \quad (2.8)
\end{aligned}
$$

We have

$$
\log\left(1 + \frac{1}{n+1}\right) - \log\left(1 + \frac{1}{n}\right) = \log\left(1 - \frac{1}{(n+1)^2}\right) < -\frac{1}{(n+1)^2}. \quad (2.9)
$$

$$
\begin{aligned}
\frac{\log(n+2)}{n+2} &- \frac{2\log(n+1)}{n+1} + \frac{\log n}{n} = (\log n)\left(\frac{1}{n+2} - \frac{2}{n+1} + \frac{1}{n}\right) \\
&+ \frac{1}{n+2}\log\left(1 + \frac{2}{n}\right) - \frac{2}{n+1}\log\left(1 + \frac{1}{n}\right) \\
&\leq \frac{2\log n}{n(n+1)(n+2)} + \frac{2}{n(n+2)} - \frac{2}{n(n+1)} + \frac{2}{(n+1)n^2} \\
&\leq \frac{2\log n + 4}{n(n+1)(n+2)}, \quad (2.10)
\end{aligned}
$$

where we have used the fact that

$$
x - x^2 < \log(1 + x) < x \quad \text{holds for all} \quad x \in (0, 1/2).
$$

Next,

$$
\frac{\theta_{2n+4}}{n+2} - \frac{2\theta_{2n+2}}{n+1} + \frac{\theta_{2n}}{n} < \frac{1}{6n^2}, \quad (2.11)
$$

by (2.3). Further, by using inequality (2.6) with $x = \eta_n$, $x = \eta_{n+1}$, $x = \eta_{n+2}$ together with inequality (2.4), we get

$$
\left|\frac{\log(1 + \eta_{n+2})}{n+2} - \frac{2\log(1 + \eta_{n+1})}{n+1} + \frac{\log(1 + \eta_n)}{n}\right| < \frac{12}{2^{2n}n} \quad (2.12)
$$

for $n \geq 3$. Putting all these together, we have

$$
\Delta^{(2)}\left(\frac{v_n}{n}\right) \leq -\frac{2}{(n+1)^2} + \frac{\log n + 2 + \log(16\pi)}{n(n+1)(n+2)} + \frac{1}{6n^2} + \frac{12}{n2^{2n}}, \quad (2.13)
$$

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and this last expression is negative for $n \geq 3$. So, the sequence of general term $|B_{2n+2}|^{1/(n+1)}/|B_{2n}|^{1/n}$ is increasing for $n \geq 3$, and then one checks by hand that it is also increasing for $n = 2, 3$.

We next deal with the Tangent numbers. We have (see [BBD]),

$$|T_{2n-1}| = 2^{2n}(2^{2n}-1)\frac{|B_{2n}|}{2n} = 4^{2n}\left(\frac{2(2n)!}{(2\pi)^{2n}}\right)\left(\frac{1}{2n}\right)\left(\left(1-\frac{1}{2^{2n}}\right)\zeta(2n)\right). \quad (2.14)$$

Since

$$1 < \left(1-\frac{1}{2^{2n}}\right)\zeta(2n) < \zeta(2n),$$

it follows by (2.4) that

$$\left(1-\frac{1}{2^{2n}}\right)\zeta(2n) = 1 + \eta_n \qquad \text{for some} \qquad 0 < \eta_n < \frac{3}{2^{2n}}. \quad (2.15)$$

Writing $|T_{2n-1}| = \exp v_n$ and following along calculation (2.5), we get that

$$\begin{aligned}
\frac{v_n}{n} &= 4\log 2 + \frac{1}{n}\log\left(\frac{2(2n)!}{(2\pi)^{2n}}\right) - \frac{\log(2n)}{n} + \frac{\log(1+\eta_n)}{n} \\
&= 2\log n + c_1 - \frac{\log n}{2n} + \frac{\log(4\pi)}{2n} + \frac{\theta_{2n}}{n} + \frac{\log(1+\eta_n)}{n}, \quad (2.16)
\end{aligned}$$

where $c_1 = 4\log 2 + c = 4\log 2 - 2 - 2\log\pi$. Comparing the last row of (2.5) with the last row of (2.16), we see that the only differences are in the value of c, the fact that the term $(\log n)/(2n)$ has now changed sign and the positive constant $\log(16\pi)$ has been replaced by the smaller positive constant $\log(4\pi)$. Following along the arguments from (2.7) and (2.8), we note that such changes do not induce any significant change in the subsequent argument and so we get that the first iterated difference of v_n/n is positive for all $n \geq 3$ and the second iterated difference of v_n/n is negative for $n \geq 4$. The remaining small values of n are checked by hand.

Regarding the Euler numbers, we use the inequality

$$\frac{4^{2n+1}(2n)!}{\pi^{2n+1}} > |E_{2n}| > \frac{4^{2n+1}(2n)!}{\pi^{2n+1}}\left(\frac{1}{1+3^{-2n-1}}\right). \quad (2.17)$$

Since

$$1 > \frac{1}{1+3^{-2n-1}} > 1 - \frac{1}{3^{2n+1}},$$

we can write

$$|E_{2n}| = 16^n\left(\frac{2(2n)!}{(2\pi)^{2n}}\right)\left(\frac{2}{\pi}\right)(1+\eta_n),$$

where

$$0 < |\eta_n| < \frac{1}{3^{2n+1}} < \frac{3}{2^{2n}}. \quad (2.18)$$

Writing $|E_{2n}| = \exp v_n$ and following along calculation (2.5), we get that

$$\begin{aligned}
\frac{v_n}{n} &= 4\log 2 + \frac{1}{n}\log\left(\frac{2(2n)!}{(2\pi)^{2n}}\right) + \frac{\log(2/\pi)}{n} + \frac{\log(1+\eta_n)}{n} \\
&= 2\log n + c_1 + \frac{\log n}{2n} + \frac{\log(64/\pi)}{2n} + \frac{\theta_{2n}}{n} + \frac{\log(1+\eta_n)}{n}. \quad (2.19)
\end{aligned}$$

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Since now η_n is negative, instead of (2.6) we need to use

$$|\log(1 - x)| \le 2|x| \quad \text{for} \quad x \in [0, 1/2]$$

with $x = -\eta_n$ and $n \ge 2$. Comparing the last row of (2.5) with the last row of (2.19), we see that the only differences are in the value of c and the positive constant $\log(16\pi)$ has been replaced by the smaller positive constant $\log(64/\pi)$. So, in (2.7) and (2.13), aside from replacing $\log(16\pi)$ by $\log(64/\pi)$, also the terms $6/2^{2n}n$ and $12/2^{2n}n$ need to be replaced by their doubles $12/2^{2n}n$ and $24/2^{2n}n$, respectively. As in the case of the Tangent numbers, such changes do not induce any significant change and so we get that the first iterated difference of v_n/n is positive for all $n \ge 3$ and the second iterated difference of v_n/n is negative for $n \ge 4$, and the remaining values are checked by hand. $\qquad\square$

3. Apéry, Delannoy and Franel numbers

Let $\mathbf{r} = (r_0, r_1, \ldots, r_m)$ be fixed nonnegative integers and put

$$S^{(\mathbf{r})}(n) = \sum_{k=0}^{n} \binom{n}{k}^{r_0} \binom{n+k}{k}^{r_1} \cdots \binom{n+km}{k}^{r_m} \quad \text{for} \quad n \ge 0. \qquad (3.20)$$

In what follows, we put $r = r_0 + \cdots + r_m$. We assume that $r_0 > 0$. When $\mathbf{r} = (r)$ for some positive integer r, we get that

$$S^{(r)}(n) = \sum_{k=0}^{n} \binom{n}{k}^{r} = b_n^{(r)} \quad \text{for all} \quad n \ge 0, \qquad (3.21)$$

where $b_n^{(1)} = 2^n$, $b_n^{(2)} = \binom{2n}{n}$ is the middle binomial coefficient, and $b_n^{(3)}$ is the Franel number. When $\mathbf{r} = (1, 1)$, we get that

$$S^{(1,1)}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = d_n \quad \text{for all} \quad n \ge 0, \qquad (3.22)$$

is the central Delannoy number. When $\mathbf{r} = (2, 2)$, we get that

$$S^{(2,2)}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = A_n \quad \text{for all} \quad n \ge 0, \qquad (3.23)$$

where A_n is the nth Apéry number. The next result answers in the affirmative the three Conjectures 3.8–3.10 from [Sun].

Theorem 3.2. *For each $\mathbf{r}$ such that $r > 1$, there exists n_r such that the sequence $(S_{n+1}^{(\mathbf{r})})^{1/(n+1)}/(S_n^{(\mathbf{r})})^{1/n}$ is strictly decreasing for $n \ge n_{\mathbf{r}}$.*

Proof. We start with McIntosh's asymptotic formula for $S^{(\mathbf{r})}(n)$ (see [Mc]).

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Lemma 3.1. *For each nonnegative integer p,*

$$S^{(\mathbf{r})}(n) = \frac{\mu^{n+1/2}}{\sqrt{\nu(2\pi\lambda n)^{r-1}}} \left(1 + \sum_{k=1}^{p} \frac{R_k}{n^k} + O\left(\frac{1}{n^{p+1}}\right) \right), \tag{3.24}$$

where $0 < \lambda < 1$ is defined by

$$1 = \prod_{j=0}^{m} \left(\frac{(1+j\lambda)^j}{\lambda(1+(j-1)\lambda)^{j-1}} \right)^{r_j},$$

$$\mu = \prod_{j=0}^{m} \left(\frac{1+j\lambda}{1+(j-1)\lambda} \right)^{r_j},$$

$$\nu = \sum_{j=0}^{m} \frac{r_j}{(1+(j-1)\lambda)(1+j\lambda)},$$

and each R_k is a rational function of the exponents $r_0, r_1, \ldots, r_m$ and λ.

Put $f(n)$ for the function such that

$$S_n^{(\mathbf{r})} = \frac{\mu^{n+1/2}}{\sqrt{\nu(2\pi\lambda n)^{r-1}}} f(n).$$

Put $S_n^{(\mathbf{r})} = \exp v_n$. Then

$$\frac{v_n}{n} = \log\mu + \frac{c}{n} - \frac{(r-1)\log n}{2n} + \frac{\log(f(n))}{n},$$

where $c = \log(\mu^{1/2}\nu^{-1/2}(2\pi\lambda)^{(1-r)/2})$. Thus,

$$\begin{aligned}
\Delta^{(2)}\left(\frac{v_n}{n}\right) &= \frac{2c}{n(n+1)(n+2)} \\
&\quad - \frac{(r-1)}{2}\left(\frac{\log(n+2)}{n+2} - \frac{2\log(n+1)}{n+1} + \frac{\log n}{n} \right) \\
&\quad + \left(\frac{\log(f(n+2))}{n+2} - \frac{2\log(f(n+1))}{n+1} + \frac{\log(f(n))}{n} \right).
\end{aligned}$$

The argument from the proof of Theorem 2.1 shows that

$$\begin{aligned}
\frac{\log(n+2)}{n+2} - \frac{2\log(n+1)}{n+1} + \frac{\log n}{n} &= \frac{2\log n}{n(n+1)(n+2)} + O\left(\frac{1}{n^3}\right) \\
&= \frac{2\log n}{n^3} + O\left(\frac{1}{n^3}\right).
\end{aligned}$$

Next, write

$$f(x) = 1 + \frac{R}{x} + O\left(\frac{1}{x^2}\right) \tag{3.25}$$

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for some rational R as in Lemma 3.1. For simplicity, put

$$g(x) = 1 + \frac{R}{x}.$$

Thus,

$$\frac{\log f(x)}{x} = \frac{\log(g(x))}{x} + O\left(\frac{1}{x^3}\right). \tag{3.26}$$

Furthermore, by the Intermediate Value Theorem, we have

$$\left|\Delta^{(1)}\left(\frac{\log g(m))}{m}\right)\right| = \left|\frac{d}{dx}\left(\frac{\log g(x)}{x}\right)\Big|_{x=\zeta\in[m,m+1]}\right|$$

$$= \left|\frac{\zeta g'(\zeta)/g(\zeta) + \log(g(\zeta))}{\zeta^2}\right|$$

$$= O\left(\frac{1}{m^3}\right)$$

for large enough positive integers m, simply by differentiating the form (3.26), and using the interval for ζ.

Further, this shows that

$$\frac{\log(f(n+2))}{n+2} - \frac{2\log(f(n+1))}{n+1} + \frac{\log(f(n))}{n} = O\left(\frac{1}{n^3}\right).$$

Hence,

$$\Delta^{(2)}\left(\frac{v_n}{n}\right) = (r-1)\frac{\log n}{n^3} + O\left(\frac{1}{n^3}\right),$$

and the above expression is positive when $r > 1$ for $n > n_{\mathbf{r}}$, which is what we wanted to prove. $\qquad\square$

4. Motzkin numbers, Schröder numbers and Trinomial coefficients

The nth Motzkin number is

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}\frac{1}{k+1}$$

and counts the number of lattice paths from $(0,0)$ to $(n,0)$ which never dip below the line $y = 0$ and which are made up only of steps $(1,0)$, $(1,1)$ and $(1,-1)$.

The nth Schröder number is

$$S_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}$$

and counts the number of lattice paths form $(0,0)$ to (n,n) with steps $(1,0)$, $(0,1)$ and $(1,1)$ that never rise above the line $y = x$.

The nth trinomial coefficient Tr_n is the coefficient of x^n in the expansion of $(x^2 + x + 1)^n$. Its formula is

$$\mathrm{Tr}_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k}.$$

The following result gives a partial affirmative answer (up to the values of n_0) to Conjectures 3.6, 3.7 and 3.11 from [Sun].

Theorem 4.3. *Each of the sequences $(M_n^{1/n})_{n \geq n_0}$, $(S_n^{1/n})_{n \geq n_0}$ and $(\mathrm{Tr}_n^{1/n})_{n \geq n_0}$ is strictly increasing while each of $(M_{n+1}^{1/(n+1)}/M_n^{1/n})_{n \geq n_0}$, $(S_{n+1}^{1/(n+1)}/S_n^{1/n})_{n \geq n_0}$ and $(\mathrm{Tr}_{n+1}^{1/(n+1)}/\mathrm{Tr}_n^{1/n})_{n \geq n_0}$ is strictly decreasing.*

Proof. It is similar to the proof of Theorem 3.2 and it is based on the existence of analogues of asymptotic expansions for M_n, S_n and Tr_n to the one of Lemma 3.1 (in fact, as we have seen in the proof of Theorem 3.2, the existence of an expansion with the first two terms, as in (3.25), suffices). For example,

$$M_n = \sqrt{\frac{3}{4\pi n^3}} 3^n \left(1 - \frac{15}{16n} + \frac{505}{512n^2} - \frac{8085}{8192n^3} + \frac{505659}{524288n^4} + O\left(\frac{1}{n^5}\right)\right) \quad (4.27)$$

(see Example VI.3 on page 396 in [FlS]),

$$S_n = \sqrt{\frac{4 + 3\sqrt{2}}{4\pi n^3}}(3 + 2\sqrt{2})^n \left(1 - \frac{24 + 9\sqrt{2}}{32n} + \frac{665 + 360\sqrt{2}}{1024n^2} + O\left(\frac{1}{n^3}\right)\right) \quad (4.28)$$

(see [W1]), and

$$\mathrm{Tr}_n = \sqrt{\frac{1 + \sqrt{2}}{4\pi n}}(1 + \sqrt{2})^n \left(1 - \frac{3}{16n} + O\left(\frac{1}{n^2}\right)\right) \quad (4.29)$$

(see [W2]). We give no further details. $\square$

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