MMAG 22 (1948/9) A110 A56857 A259691 A11971 A259697

&

ROOKS AND RHYMES

H. W. Becker

Kaplansky and Riordan have "unified and generalized" various results in statistics, in terms of the Problem of the Rooks, on a trapezoidal chessboard. A key theorem is: "the number of ways of putting c non-attacking rooks on a right-angled isosceles triangle of side $r-1$ is the Stirling number $\Delta^{r-c}\,0^r/(r-c)!$" = ${}_{r-c}\mathfrak{S}_r$. In other words, this is the number of selections of c points on such a board, such that none have any row or column index in common. [1]

We will exhibit well-ordered tables of these point sets thru $r = 5$, in 1 to 1 correspondence with the sequations (rhyme schemes) also enumerated by Stirling numbers [2,3]. We will then formulate other classifications, with respect to: 2) row location of the topmost rook; 3) number of rooks on the principal diagonal; 4) column vacancies; 5) column location of the bottom rook. Each of these except 3), of course has a dual interpretation, under interchange of row and column.

In Table I, the row index precedes the column index, and a vacant board is indicated by 0. The index 11 is at the top of the board, away from the observer, with rr at the bottom diagonal corner nearby on the right.

The sequations have further isomorphs in the substitution cycles [4]. All the rhyme repetitions of the kth letter correspond to all the letters in the kth substitution cycle. Thus the sequation *aaaaa* corresponds to the distribution or substitution cycle (*abcde*), while the sequation *abcde* corresponds to the distribution *a*/*b*/*c*/*d*/*e*, etc. These correspondences are too easily read off by inspection, to need printing here.

a	*a*	*a*	*a*	*a*	*a*	*a*	*a*
	a	*b*	*a*	*a*	*b*	*b*	*b*
			a	*b*	*a*	*b*	*c*
0	11	0	11	11	21	22	0
			22				

a	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*
a	*a*	*a*	*a*	*a*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
a	*a*	*b*	*b*	*b*	*a*	*a*	*a*	*b*	*b*	*b*	*c*	*c*	*c*	*c*
a	*b*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*d*
11	11	11	11	11	21	21	21	22	22	22	31	32	33	0
22	22	32	33		32	33		31	33					
33														

a	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*
a	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
a	*a*	*a*	*a*	*a*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*b*
a	*a*	*b*	*b*	*b*	*a*	*a*	*a*	*b*	*b*	*b*	*c*	*c*	*c*	*c*	*a*	*a*	*a*	*b*	*b*	*b*	*c*	*c*	*c*	*c*	*a*
a	*b*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*d*	*a*
11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	21	21	21	21	21	21	21	21	21	21	22
22	22	22	22	22	32	32	32	33	33	33	42	43	44		32	32	32	33	33	33	42	43	44		31
33	33	43	44		43	44		42	44						43	44		42	44						43
44																									

a	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*a*
b	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
b	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*	*c*
a	*a*	*b*	*b*	*b*	*c*	*c*	*c*	*c*	*a*	*a*	*a*	*a*	*b*	*b*	*b*	*b*	*c*	*c*	*c*	*c*	*d*	*d*	*d*	*d*	*d*
b	*c*	*a*	*b*	*c*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*a*	*b*	*c*	*d*	*c*	*b*	*c*	*d*	*e*
22	22	22	22	22	22	22	22	22	31	31	31	31	32	32	32	32	33	33	33	33	41	42	43	44	0
31	31	33	33	33	41	43	44		42	43	44		41	43	44		41	42	44						
44		41	44																						

Table I

The sequations are in lexical order, as they would appear in a dictionary, or directory. Each set of c rooks on the r-1 board is made to correspond with a sequation in ${}_{r-c}\mathcal{C}_r$, that is, one having r letters, r-c different. Whenever a sequation increases its range, goes from ${}_c\mathcal{C}_r$ to ${}_{1+c}\mathcal{C}_{r+1}$, the corresponding set of r-c rooks is unaltered on the larger board. Otherwise, when it goes only to ${}_c\mathcal{C}_{r+1}$ there will be an additional rook which may be located in any of c squares on the added bottom row. That is the significance of the recurrence

$$(1)\qquad {}_{r-c}\mathcal{C}_r = {}_{r-c-1}\mathcal{C}_{r-1} + (r-c)\cdot{}_{r-c}\mathcal{C}_{r-1} = {}_cR_{r-1} = {}_{c-1}R_{r-2}\cdot(r-c) + {}_cR_{r-2}$$

We observe that rook sets totaling ${}^{(q)}R_r$, the upper one of which is in the qth row of the r-board, occur together in the lexicon, and correspond to sequations whose first q letters are distinct, followed by a repetition. These last are known to be enumerated by

$$(2)\qquad {}^{(q)}R_r = q\cdot\mathcal{C}_{r-q}(q) = q\cdot(\mathcal{C}+q)^{r-q}$$

We adopt the convention in Table II that ${}^{(r+1)}R_r = 1$, the vacant board.

A259691

r \ q	1	2	3	4	5	6	7
0	1						
1	1	1					
2	2	2	1				
3	5	6	3	1			
4	15	20	12	4	1		
5	52	74	51	20	5	1	
6	203	302	231	104	30	6	1

A110 Table II, ${}^{(q)}R_r$

A56857

0	1	2	3	4	5	6
1						
1	1					
2	2	1				
5	6	3	1			
15	20	12	4	1		
52	75	50	20	5	1	
203	312	225	100	30	6	1

A110 Table III, ${}^qR_r$

Tables II and III are identical thru $r = 4$, then deviate more and more. It is inherent in the mode of ordering the rook and rhyme schemes, that whenever we append a letter just equal to the range or previous high letter to a sequation, we add a rook in the outer or principal diagonal of the corresponding rook pattern. Then if ${}^qR_r$ = the number of rook sets on an r-board, with q rooks on the principal diagonal, we have, (r,q) being a binomial coefficient $(r)_q/q!$

$$(3)\qquad {}^qR_r = (r,q)\cdot R_{r-q} = {}^{q+1}\mathcal{C}_{r+1} = (r,q)\cdot\mathcal{C}_{r-q}$$

The latter expression denotes the number of sequations of r+1 letters having q+1 letters just equal to the previous high letter of the sequation. This table has two other rhyme interpretations: with q+1 as the number of a's (also the number of substitution cycles of r+1 letters with q+1 letters in the a-cycle); and—the rows written in reverse order—q+1 is the number of changes, or non-repetitive pairs of consecutive letters.

The evolution of (3) is: add q outer diagonals to the $(r$-$q)$-board, and permute the q rooks on the principal diagonal and intersecting column-rows, in all $(r$-$q)$ ways amongst the patterns of R_{r-q}.

Let $_{c'}R_r$ = the number of patterns on the r-board in which the cth column is empty. If we adopt the convention that $_{r+1'}R_r = R_r$, Table IV is the same as the difference table of $R_r = @_{r+1}$ with the rows and columns interchanged. Or, in terms of the operators ∇ and E such that $\nabla U_n = U_n - U_{n-1}$, and $E^{-1} U_n = U_{n-1}$, so $\nabla = 1 - E^{-1}$,

(4.1) $\quad {}_{c'}R_r = \nabla^{r-c-1} R_r = (1 - E^{-1})^{r-c-1} R_r = (1 + E^{-1})^{c-1} R_{r-1}$

These follow by induction, working either way from the obvious

(4.2) $\quad {}_{r'}R_r = \nabla R_r = R_r - R_{r-1}$,

(4.3) $\quad {}_{1'}R_r = R_{r-1}$

The process is the same as in the Problem of the Incompatible Mechanics [5], whence the correspondence: $_{c'}R_r$ = the number of non-attacking rook patterns on a right triangular chessboard of side r and cth column vacant = the number of organizations of $r+1$ men into crews under the restriction that one man is incompatible with and must be segregated from $r-c$ other men.

A11971

c'	1	2	3	4	5	6	7
r							
0	1						
1	1	2					
2	2	3	5				
3	5	7	10	15			
4	15	20	27	37	52		
5	52	67	87	114	151	203	
6	203	255	322	409	523	674	877

Table IV, $_{c'}R_r$

A259697

(c) 0	1	2	3	4	5	6
1						
1	1					
1	2	2				
1	4	5	5			
1	9	12	15	15		
1	24	32	42	52	52	
1	76	99	129	166	203	203

Table V, $_{(c)}R_r$

Let $_{(c)}R_r$ = the number of patterns on the r-board such that the bottom rook is in the cth column. Then it is plain that

(5.1) $\quad {}_{(0)}R_r = 1, \quad {}_{(c)}R_r = {}_{(c)}R_{r-1} + {}_{c'}R_{r-1} = {}_{r-1}\Sigma_{s=c-1}\ {}_{c'}R_s$,

the summation being between $c-1$ and $r-1$, a notation which facilitates printing. The $_{(c)}R_r$ may be given absolute evaluations as polynomials in R or @. Abbreviating $_r\Sigma_{c=1}\ @_c = @_{r)}$, we have

(5.2) $\quad {}_{(2)}R_{r+2} = @_{r)} + 1$,

$$(5.3)\qquad {}_{(3)}R_{r+2} = @_{r)} + @_{r-1)} + 1$$

$$(5.4)\qquad {}_{(4)}R_{r+2} = @_{r)} + 2@_{r-1)} + @_{r-2)} - 1$$

$$(5.5)\qquad {}_{(r)}R_r = {}_{(r-1)}R_r = @_r$$

$$(5.6)\qquad {}_{(r-2)}R_r = @_r - @_{r-1} + @_{r-2}$$

$$(5.7)\qquad {}_{(r-3)}R_r = @_r - 2@_{r-1} + 2@_{r-2}$$

BIBLIOGRAPHY

1. I. Kaplansky and J. Riordan, *The Problem of the Rooks and its Applications*, Duke Math Jr., 13 (1946) 259-268.
2. H. W. Becker, Am. Math. Monthly, 48 (1941) 701.
3. H. W. Becker, Bull. Am. Math. Society, 52 (1946) 415.
4. J. Touchard, *Sur les Cycles des Substitutions*, Acta Math. 70 (1939) 249.
5. D. H. Browne, Am. Math. Monthly, 51 (1944) 47.
6. M. Cotlar, *A Generalization of the Factorials*, Math. Reviews, 7 (1946) 242.
7. W. W. R. Ball, *Mathematical Recreations and Essays*.
8. Whitworth, *Choice and Chance*.

Omaha, Nebraska