Ainley (see p.5) A110

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 Q A          QQ    A L     AA         R E        RR C O             H U          HH
 Q T          QQ    A G      AA        R A        RR C M             H A          HH
 Q I          QQ    A E       AA       R L       RRR C P             H T          HH
 Q O          QQ    A B        AA      R R     RRRR  C L             H E          HH
 Q N          QQ    AABRAAAAAAAAA      RRRRRRRRRR    C E             HHRHHHHHHHHHHH
 Q A   QQQQ   QQ    AAAAAAAAAAAAAA     RRRRRRRRRR    C X             HHHHHHHHHHHHHH
 Q L   QQQQ   QQ    A I          AA    R R     RRRR  C C             H I          HH
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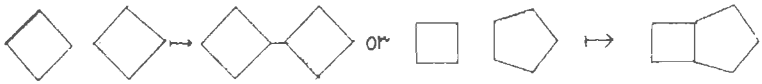
Solutions and Comments

Two more of our problems have received the John Rickard treatment. In no. 11 it is not possible to place less than 16 coins flat on a table such that each of the coins touches exactly three others; moreover the solution for 16 is unique. In no. 14 the safe can always be opened in k^n+n-1 moves, as was also proved by Michael Giles. The Rickardian comment on no. 12 was, "Does this mean, 'is this the only problem which takes a mathematician time proportional to his ability to solve?'?". Incidentally, I challenge any anglicist to criticise my use of punctuation here.

11. We use a graph-theory notation with a vertex representing each coin, and an edge joining touching coins. Then the solution will be a planar graph of girth at least seven, all of whose vertices have degree three.

Define a connected arrangement of at least three coins to be 'simple' iff
(i) every vertex has degree at most three, and every non-boundary vertex has degree exactly three and
(ii) the subgraph of boundary vertices is a cycle. That is, considering them in order, each occurs exactly once and touches only those boundary vertices adjacent to it in that order.
Also, define a simple arrangement to be 'inert' iff it is a solution, ie every vertex has degree three. It may quickly be seen that an inert graph has at least four non-boundary coins, and at least twelve on the boundary, and that this is unique.

Any non-simple solution may be expressed as a tree (defining now a graph at a higher level) whose vertices represent simple arrangements and whose edges represent the joining together of two simple arrangements exemplified below:



This may be proved by induction by considering the boundary of the figure and the two ways in which it can violate simplicity, breaking it in two.

Any simple arrangement capable of being an endvertex of such a tree must have at least 13 coins, hence any nonsimple solution must have at least 26 coins, and these bounds are attained. Thus there is no solution with fewer than 16 coins, and that with 16 is unique, up to graph isomorphism.

14. The problem can always be solved in the minimal number of moves, namely k^n+n-1. For consider a directed graph whose edges represent the k^n n-letter codes, and whose vertices represent the k^{n-1} (n-1)-letter codes. An edge goes from x to y iff x forms the first and y the last n-1 letters of the code. Then each vertex has exactly k edges going in to it, and k coming out. Hence there exists a directed Euler circuit, ie one which passes along each directed edge exactly once. This may be seen to solve the problem.

The proof of this well-known theorem runs as follows. Suppose a circuit has been found, which passes along each edge (in the correct direction) at most once. Then if it is not an Euler circuit, there exists an edge not in it. The terminal vertex of this must have at least one 'spare' outgoing edge, and it may be seen that another circuit can be constructed. If this has a vertex in common with the previous one, join it on in the obvious way by 'rerouting'. Otherwise, since the graph is connected (as may easily be seen to be so in our case), there exists a path from one circuit to the other which is disjoint from them. This may be extended to another circuit in the same way, and joined likewise to both of them. In either case a strictly bigger circuit has been found, and since the graph is finite the process terminates, giving an Euler circuit.

(14.2) Suppose now we restrict the sequence of digits (a_i) such that

$$a_{i-1} \leqslant a_i \geqslant a_{i+1} \qquad \text{for } i=2,4,6,\ldots$$

Is the solution the same?

Eureka

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Problems

15. Various people have interpreted QARCH I as meaning that one or all of the problems 1 to 6, or even the entire text, was due to Mark King, our President in 1979-80. In fact only the wording of no. 4 was his, but this problem is certainly due to him, and bears his name. Given a rectangular bar of chocolate-covered toffee, describe how to cut it with two plane cuts into three pieces each with the same amount of toffee and chocolate.

16. This one was posed by Dr. David Singmaster at a recent TMS meeting. Find the general rectangular prism all of whose edges, face diagonals and body diagonals are integers. He would also like to find Ramanujan's taxi no. 1729, and the venue. (see, for example, Martin Gardner's "Mathematical Puzzles and Diversions", p. 92)

17. This problem was posed by Professor Churchhouse at the first SMP lecture given to honour that year's British IMO team in September 1978, and two years later in his Presidential Address to the Maths & Physics section of the British Association. Find the smallest number, k, such that every natural number, n, may be written in the form $n = a^2 + b^3 + \ldots + r^k$, for natural numbers a, b, ..., r.

18. Dr. John Mason asks, Is it possible to partition a circle into congruent pieces such that the centre lies in the interior of a piece, or at least not on the boundary of every piece?

19. Stephen Ainley is looking for an explicit (non-recursive) formula for the sequence defined by $a_0 = 1$, $D^n a_0 = a_{n-1}$, where $Da_i = a_{i+1} - a_i$. That is, starting

```
 1
 2   1
 5   3   2
15  10   7   5
52  37  27  20  15
   114  87  67  52
       322 255 203
          1080 877
              4160
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20. Finally, Dr. Mason also asks, must a convex 3D solid, all of whose shadows (projections) have the same area, be a sphere?