Cat ✓ Indexed 110 Sloane

TRANSACTIONS OF THE ROYAL SOCIETY OF CANADA 110
VOLUME XLIX : SERIES III : JUNE, 1955
SECTION THREE

# An Asymptotic Formula for the Bell Numbers

LEO MOSER and MAX WYMAN, F.R.S.C.

**1. Introduction.** Properties of the Bell numbers $G_n$, defined by

$$\sum_{n=0}^{\infty} \frac{G_n z^n}{n!} = e^{e^z-1} \tag{1.1}$$

have been studied by many authors. A recent thesis of Finlayson **(5)** lists over fifty references to these numbers. The problem of determining a formula for the asymptotic behaviour of $G_n$ for large $n$ has been suggested several times. However, as far as we are aware, only two such formulae have been derived. Knopp **(6)** gives the formula

$$\frac{G_n}{n!} = \left(\frac{1+\eta_n}{\log n}\right)^n \tag{1.2}$$

where $\eta_n \to 0$ as $n \to \infty$. Since the way in which $\eta_n$ approaches zero is not specified the formula is of no value for computation and does not constitute an asymptotic expansion for $G_n$. Epstein **(4)** gives the formula

$$G_n \sim \left(\frac{n\, e^{1/\log n}}{\log n}\right)^n \tag{1.3}$$

As we shall see, this result is in error. The method used to obtain it is rather long, but could be used to get a correct expression for the first term of an asymptotic expansion for $G_n$. In the present paper a complete asymptotic expansion for the Bell numbers will be obtained by an entirely different method.

**2. Asymptotic expansion.** Since the iterated exponential function occurs throughout the paper we shall use both $e^z$ and $\exp z$ to denote the exponential function.

From (1.1) and Cauchy's theorem

$$G_n = \frac{n!}{2\pi i}\int_C [\exp(e^z - 1)]/z^{n+1}\cdot dz, \tag{2.1}$$

where $C$ is the circle $z = Re^{i\theta}$. Hence,

$$G_n = \frac{n!}{2\pi e R^n}\int_{-\pi}^{\pi} \exp[\exp(Re^{i\theta}) - in\theta]\, d\theta. \tag{2.2}$$

This can be put in the form

$$G_n = A\int_{-\pi}^{\pi} \exp(F(\theta))\, d\theta, \tag{2.3}$$

where

(2.4) $$A = [n!\exp(e^R - 1)]/2\pi R^n$$

and

(2.5) $$F(\theta) = \exp(Re^{i\theta}) - in\theta - \exp R.$$

Let us define $\epsilon$ by

(2.6) $$\epsilon = e^{-3R/8}$$

and consider the integral $J$, defined by

(2.7) $$J = \int_{\epsilon}^{\pi} \exp(F(\theta))\, d\theta.$$

An easy computation proves the existence of a constant $k > 0$ such that

(2.8) $$|J| < \exp(-k\, Re^{R/4})$$

Since we shall show that the terms of this order may be neglected in our asymptotic expansion, we shall anticipate this result and use (2.3) to obtain

(2.9) $$G_n \sim A \int_{-\epsilon}^{\epsilon} \exp(F(\theta))\, d\theta.$$

Our next step is to expand $F(\theta)$, as given by (2.5) in a Maclaurin expansion about $\theta = 0$. If we introduce the operator $\Theta$ defined by

(2.10) $$\Theta = R\frac{d}{dR}$$

we may write

(2.11) $$F(\theta) = (Re^R - n)\, i\theta - \tfrac{1}{2}(R^2 + R)\, e^R\theta^2 + \sum_{k=3}^{\infty} (\Theta^k e^R)\frac{(i\theta)^k}{k!}.$$

At this stage we choose $R$ to be the unique real solution of the equation

(2.12) $$Re^R = n.$$

With this choice (2.11) becomes

(2.13) $$F(\theta) = -\tfrac{1}{2}(R^2 + R)\, e^R\theta^2 + \sum_{k=3}^{\infty} (\Theta^k e^R)\frac{(i\theta)^k}{k!}.$$

We now introduce the following notation:

(2.14) $$\phi = [\tfrac{1}{2}(R^2 + R)\, e^R]^{\frac{1}{2}}\theta$$

(2.15) $$B = A/[\tfrac{1}{2}(R^2 + R)\, e^R]^{\frac{1}{2}}$$

(2.16) $$h = \epsilon[\tfrac{1}{2}(R^2 + R)\, e^R]^{\frac{1}{2}} = \tfrac{1}{2}(R^2 + R)^{\frac{1}{2}} e^{R/8}$$

(2.17) $$a_k = e^{-R}\, \Theta^{k+2}(e^R)(i\phi)^{k+2}/(k+2)![\tfrac{1}{2}(R^2 + R)]^{k+2/2}$$

(2.18) $$z = e^{-R/2}$$

(2.19) $$f(z) = \sum_{k=1}^{\infty} a_k z^k.$$

We note in passing that the $a_k$ are polynomials in $\phi$.

Making the substitution (2.14) in (2.9) we find

$$G_n \sim B \int_{-h}^{h} e^{-\phi^2+f(z)} d\phi. \tag{2.20}$$

From (2.17) an easy calculation shows that there exists a fixed $R_0$ such that for all $R > R_0$,

$$|a_k| < |2\phi|^{k+2}. \tag{2.21}$$

We have defined $z$ as a function of $R$. However for the moment we consider $z$ to be an independent variable and expand $e^{f(z)}$ in a convergent Maclaurin expansion of the form

$$e^{f(z)} = \sum_{k=0}^{\infty} b_k z^k, \quad b_0 = 1. \tag{2.22}$$

Further, by (2.21), $z = e^{-R/2}$ is inside the domain of convergence. Hence at this stage we may again take $z = e^{-R/2}$. We note in passing that the $b_k$ are polynomials in $\phi$. Further, $b_{2k+1}$ contains only odd powers of $\phi$ and $b_{2k}$ only even powers of $\phi$.

Using (2.22) we may write (2.20) in the form

$$G_n = B\left\{ \sum_{k=0}^{s-1} \left( \int_{-h}^{h} e^{-\phi^2} b_k \, d\phi \right) z^k + R_s \right\} \tag{2.24}$$

where

$$R_s = \int_{-h}^{h} e^{-\phi^2} \left( \sum_{k=s}^{\infty} b_k z^k \right) d\phi. \tag{2.25}$$

From (2.12) we see that $R \to \infty$ as $n \to \infty$. Further, (2.16) implies that $h \to \infty$ as $R \to \infty$. From these facts and the known asymptotic expansion of functions of the form $\int_{-h}^{h} e^{-\phi^2}$ . (polynomial in $\phi$) $d\phi$, the replacement of $h$ by $\infty$ in (2.24) is easily justified. Hence

$$G_n \sim B\left\{ \sum_{k=0}^{s-1} \left( \int_{-\infty}^{\infty} e^{-\phi^2} b_k d\phi \right) z^k + R_s \right\}. \tag{2.25}$$

In order to complete our proof we must show that $R_s = 0(|z|^s)$. By a lemma of the authors **(7)**, (2.21) implies

$$|b_k| < |2\phi|^{k+2}(1 + |2\phi|^2)^{k-1}. \tag{2.25}$$

This yields

$$\left| \sum_{k=s}^{\infty} b_k z^k \right| < P_s(|\phi|) \, |z|^s/M, \tag{2.27}$$

where $P_s(|\phi|)$ is a polynomial in $|\phi|$ and $M$ is given by

$$M = 1 - |z| \, |2\phi| \, (1 + 2|\phi|^2). \tag{2.28}$$

Since $|\phi| \leqslant h$ and $z = e^{-R/2}$, $|\phi|^3|z| \leqslant [\frac{1}{2}(R^2 + R)]^{3/2}e^{-R/8} \to 0$ as $R \to \infty$. Hence for large $R$, $M \geqslant \frac{1}{2}$. Finally,

$$\int_{-\infty}^{\infty} e^{-\phi^2} P_s(|\phi|)\, d\phi$$

exists, which implies $|R_s| = O(|z|^s)$. This completes the proof that

$$G_n \sim B \sum_{k=0}^{\infty} \left( \int_{-\infty}^{\infty} e^{-\phi^2} b_k d\phi \right) e^{-kR/2}. \tag{2.29}$$

We have noted that the $b_{2k+1}$, as polynomials in $\phi$, contain only odd powers. Hence

$$G_n \sim B \sum_{k=0}^{\infty} \left( \int_{-\infty}^{\infty} e^{-\phi^2} b_{2k} d\phi \right) e^{-kR}. \tag{2.30}$$

**3. The first terms of the expansion.** By calculation we obtain

$$b_0 = 1 \quad b_2 = \frac{R^3 + 6R^2 + 7R + 1}{6R(R+1)^2}\phi^4 - \frac{R^2 + 3R + 1}{9R(R+1)^3}\phi^6. \tag{3.1}$$

Hence from (2.29) we find

$$G_n \sim \pi^{\frac{1}{2}} B \left[ 1 - \frac{2R^4 + 9R^3 + 16R^2 + 6R + 2}{24Re^R(R+1)^3} \right]. \tag{3.2}$$

Using (2.12) we may write

$$G_n \sim \pi^{\frac{1}{2}} B \left[ 1 - \frac{2R^4 + 9R^3 + 16R^2 + 6R + 2}{24n(R+1)^3} \right]. \tag{3.3}$$

From (2.4) and (2.15) we have

$$\pi^{\frac{1}{2}} B = n!\exp(e^R - 1)/R^n[2\pi(R+1)\, Re^R]^{\frac{1}{2}}. \tag{3.4}$$

Since $Re^R = n$ we have $e^R = nR^{-1}$, $R^n = n^n e^{-nR}$.

This implies

$$\pi^{\frac{1}{2}} B = \{n!\exp[n(R + R^{-1}) - 1]\}/[2\pi(R+1)]^{\frac{1}{2}} n^{n+\frac{1}{2}}. \tag{3.5}$$

We now use Stirling's expansion for $n!$, namely

$$n! \sim (2\pi)^{\frac{1}{2}} e^{-n} n^{n+\frac{1}{2}} \cdot \left( 1 + \frac{1}{12n} \right), \tag{3.6}$$

to obtain from (3.5),

$$\pi^{\frac{1}{2}} B \sim (\exp[n(R + R^{-1} - 1) - 1])(1 + 1/12n)/(R+1)^{\frac{1}{2}}. \tag{3.7}$$

Hence from (3.3) and (3.7) we obtain

$$G_n \sim (R+1)^{-\frac{1}{2}} \exp[n(R + R^{-1} - 1) - 1] \left( 1 - \frac{R^2(2R^2 + 7R + 10)}{24n(R+1)^3} \right). \tag{3.8}$$

ie $\log G_n \sim n \log n$
and $R \sim \log n$

It would be natural to express $R$ asymptotically in terms of $n$ by means of (2.12) and thus to obtain an asymptotic expression for $G_n$ entirely in terms of $n$. However, this procedure is not satisfactory as we shall now show. We may rewrite (2.12) in the form

$$R = \log n - \log R. \tag{3.9}$$

Starting with the approximation $R = \log n$ and iterating we find

$$R = \log n - \log(\log n) + \frac{\log(\log n)}{\log n} + \cdots. \tag{3.10}$$

Further terms contain higher powers of $\log n$ in the denominators. However, in (3.8) we have a term of the type $nR$. It is clear that none of the terms in $nR$ will approach zero as $n \to \infty$. Hence none of these terms can be dropped. For this reason it is better to retain (3.8) as our final result.

This point is overlooked by Epstein **(4)** thus leading to the inaccuracy of (1.3). For $n = 50$, (2.12) gives $R = 2.8608902\ldots$ and (3.8) then yields $G_{50} = 1.85730\ldots \times 10^{47}$. This is an excellent agreement with exact value of $G_{50}$ given in the next section.

**4. Table of values of G .** Using the recursion formula

$$G_0 = 1,\ G_{n+1} = \sum_{r=0}^{n} \binom{n}{r} G_r \tag{4.1}$$

Bell **(3)** constructed a table of $G_n$ for $n < 20$. By means of an algorithm due to Aitkin **(1)**, H. Finlayson recalculated these values and extended the table up to $n = 25$. At the request of the authors, F. L. Miksa checked these values and calculated the $G_n$ up to $n = 51$. $G_{30}$ was independently calculated by H. W. Becker **(2)**. The Table will be found on page 54.

## References


1. A. C Aitkin, *A problem on combinations*, Math. Notes Edinburgh, *21* (1933), 18–23.
2. H. W Becker, *Solution to problem*, Amer. Math. Monthly, *48* (1941), 701–702.
3. E. T. Bell, *Exponential polynomials*, Ann. Math., *35* (1934), 258–277.
4. L. F. Epstein, *A function related to the series for* $e^{e^x}$, J. Math. & Phys., *18* (1939), 166–182.
5. H. Finlayson, *Numbers generated by* $e^{e^x-1}$. Master's thesis, University of Alberta (1954), unpublished.
6. K. Knopp, *Theory and applications of infinite series* (Blackie and Sons, 1928), p. 563.
7. L. Moser and M. Wyman, *On solutions of* $x^d = 1$ *in symmetric groups*. Can. J. Math., 7 (1955), 159–168.

TABLE OF $G_n$ $(0 \leqslant n \leqslant 51)$

| $n$ | |
|---|---|
| 0 | 1 |
| 1 | 1 |
| 2 | 2 |
| 3 | 5 |
| 4 | 15 |
| 5 | 52 |
| 6 | 203 |
| 7 | 877 |
| 8 | 4140 |
| 9 | 21147 |
| 10 | 1 15975 |
| 11 | 6 78570 |
| 12 | 42 13597 |
| 13 | 276 44437 |
| 14 | 1908 99322 |
| 15 | 13829 58545 |
| 16 | 1 04801 42147 |
| 17 | 8 28648 69804 |
| 18 | 68 20768 06159 |
| 19 | 583 27422 05057 |
| 20 | 5172 41582 35372 |
| 21 | 47486 98161 56751 |
| 22 | 4 50671 57384 47323 |
| 23 | 44 15200 58550 84346 |
| 24 | 445 95886 92948 05289 |
| 25 | 4638 59033 22299 99353 |
| 26 | 49631 24652 36187 56274 |
| 27 | 5 45717 04793 60599 89389 |
| 28 | 61 60539 40459 99346 52455 |
| 29 | 713 39801 93886 02751 91172 |
| 30 | 8467 49014 51180 93324 50147 |
| 31 | 1 02933 58946 22637 64850 95653 |
| 32 | 12 80646 70049 90871 38189 25644 |
| 33 | 162 95958 92846 00760 67647 28147 |
| 34 | 2119 50393 88640 36046 23886 56799 |
| 35 | 28160 02030 19560 26656 33404 26570 |
| 36 | 3 81971 47298 94818 33997 55256 81317 |
| 37 | 52 86836 62085 50447 90194 55756 24941 |
| 38 | 746 28989 20956 25330 52309 95406 39146 |
| 39 | 10738 82333 07746 92832 76885 79864 25209 |
| 40 | 1 57450 58839 12049 31289 32434 47025 31067 |
| 41 | 23 51152 50774 06176 28200 69407 72437 88988 |
| 42 | 357 42549 19887 26172 91353 50865 66266 42567 |
| 43 | 5529 50118 79716 54843 21714 69328 07377 67385 |
| 44 | 87019 63427 38705 50890 23600 53185 57971 48876 |
| 45 | 13 92585 05266 26366 96023 47053 99365 40796 93415 |
| 46 | 226 54182 19334 49400 29284 84444 70539 22761 58355 |
| 47 | 3745 00595 02461 51119 65053 42096 43151 01201 74682 |
| 48 | 62891 97963 03118 41542 02104 54071 84953 77460 15761 |
| 49 | 10 72613 71545 73358 40034 22155 18590 00263 39172 47281 |
| 50 | 185 72426 87710 78270 43825 77671 81908 91749 92218 52770 |
| 51 | 3263 98387 00041 11524 85695 18301 91582 52441 92558 19477 |