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A FUNCTION RELATED TO THE SERIES FOR e^{e^x}

By Leo F. Epstein

I. Introduction and Definitions. In deriving a series expansion for e^{e^x}, an interesting group of integral constants is encountered, the coefficients in the expansion

$$(1) \qquad e^{e^x} = e\left[1 + K_1 x + \frac{K_2}{2!}x^2 + \frac{K_3}{3!}x^3 + \cdots + \frac{K_n}{n!}x^n + \cdots\right]$$

The numerical values of these constants can be obtained most directly by substituting the value $x = 0$ in the derivatives of e^{e^x}, the n'th derivative thus yielding K_n, following the rule for a Maclaurin[1] series. In this paper, we shall discuss a few of the simpler properties of these integers, and of a function derived from them.

As a study of the derivation from the derivatives of e^{e^x} will show, the K-numbers (as we shall call them) are given by the following iteration formula

$$(2) \qquad K_{n+1} = 1 + nK_1 + \frac{n(n-1)}{2!}K_2 + \frac{n(n-1)(n-2)}{3!}K_3 + \cdots + nK_{n-1} + K_n$$

$$K_1 = 1.$$

$$(3) \qquad K_{n+1} = 1 + \sum_{m=1}^{n}\binom{n}{m}K_m$$

where $\binom{n}{m}$ is the ordinary binomial coefficient. Thus K_n can be calculated at once if the values of K_1 to K_{n-1} are known following the above equations. In this respect, the K-numbers resemble Euler's numbers, which are also integers derivable from similar equations involving binomial coefficients providing all the values from E_1 to E_{n-1} are known. Since the binomial coefficients, which are the multiples of the K's,

[1] Whittaker and Watson *Modern Analysis* (Cambridge, 1935).

are all whole numbers, and since $K_1 = 1$, K_n *must always be an integer.* It should be observed that symbolically, we may write[2]

$$K_{n+1} = (1 + k)^n \tag{4}$$

where, in the expanded series, we substitute K_m for k^m in each term. It has further been demonstrated by various writers[3] that we may express K_n in the infinite series form

$$K_n = \frac{1}{e}\sum_{t=0}^{\infty}\frac{t^n}{t!} \qquad (n = 1, 2, 3, \cdots) \tag{5}$$

This equation is restricted to positive integral values of n; we shall now demonstrate that this restriction is unnecessary. In fact, we shall define the K-function

$$K_z = \frac{1}{e}\sum_{t=0}^{\infty}\frac{t^z}{t!} \tag{6}$$

where z is any number, real or complex. This definition is quite valid, for (6) converges absolutely for all values of z: applying the well-known ratio test, the ratio of the $(t + 1)$st term to the t'th term is

$$\frac{(t+1)^z}{(t+1)!}\cdot\frac{t!}{t^z}$$

and as t becomes infinite, the limit of this expression is identically zero for all finite values of z, real or complex.

By simple algebraic manipulation of the definition of K_z (Equation 6), we obtain, when x is a real positive integer,

$$eK_x = \frac{x^x}{x!} + (x-1)^x\sum_{n=1}^{\infty}\left[\frac{1}{(x-n)!}\left(\frac{x-n}{x-1}\right)^x + \frac{1}{(x+n)!}\left(\frac{x+n}{x-1}\right)^x\right]$$

the first term of the sum vanishing for $n \geqq x$. This equation is valid for all values of $x \geqq 0$, and is especially useful for the calculation of K_x for very large values of the argument.

[2] Equation (4) was known to d'Ocagne: Am. Jour. Math., 9, 370 (1887).

[3] This equation (5) for positive integers, the coefficients in the series expansion for e^{e^x} was given by Dobiński: Grunert's Archiv., 61, 333 (1877), and is said to have been known by Euler (cf. Bell, footnote 24c). Another early reference to these numbers, which seem to have been rediscovered many times is Cesaro: Nouvelles Annales de Math., 4, 39 (1885).

For negative values of z, equation (6) assumes the Dirichlet series form

$$K_{-s} = \frac{1}{e}\sum_{t=0}^{\infty}\frac{1}{t^s\cdot t!}. \tag{7}$$

Although equations (6) and (7) both converge absolutely for all values of z, it will be observed that while (7) converges very rapidly, the

TABLE I

K-Numbers for Positive and Negative Integers

n	K_n	K_{-n}
0	$1 - 1/e$	$1 - 1/e = 0.63212\ 05588$
1	1	.48482 91072
2	2	.42177 34383
3	5	.39340 93945
4	15	.38019 78350
5	52	.37389 58961
6	203	.37084 15557
7	877	.36934 54823
8	4,140	.36860 75427
9	21,147	.36824 18738
10	115,975	.36806 01230
11	678,570	.36796 96382
12	4,213,597	.36792 44638
13	27,644,437	.36790 19333
14	190,899,322	.36789 06810
15	1,382,958,545	.36788 50591
16	10,480,142,147	.36788 22496
17	82,864,869,804	.36788 08447
18	682,076,806,159	.36788 01431
19	5,832,742,205,057	.36787 97922
20	51,724,158,235,372	.36787 96167
⋮	⋮	⋮
∞	∞	$1/e =$ .36787 94412

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successive terms continually becoming smaller for all positive values of z, and approaching a limiting value most rapidly when z is large, the lower terms (small t) of equation (6) actually diverge at first since each term is greater than the preceding one, but eventually, when $t! > t^z$, the terms become successively smaller and the series converges. This initial divergence of the terms of (6) becomes increasingly apparent as we attempt to calculate K_z for large positive values of x from it. An asymptotic expansion, valid for large values of z is consequently highly desirable, and we shall show later how such an expansion can be obtained. For the exact computation of K_n equation (3) is most convenient; Table I has been calculated[4] using these principles.

We may derive several interesting relations from equation (7). Thus when $z = 1$

$$K_{-1} = \frac{1}{e}\sum_{t=1}^{\infty}\frac{1}{t\cdot t!}$$

Now since

$$\frac{e^x}{x} = \frac{1}{x} + \sum_{t=1}^{\infty}\frac{x^{t-1}}{t!},$$

$$\int_0^x \frac{e^x}{x}\,dx = \ln x + \sum_{t=1}^{\infty}\frac{x^t}{t\cdot t!}$$

If $x = 1$

$$\int_0^1 \frac{e^x}{x}\,dx = \ln 1 + \sum_{t=1}^{\infty}\frac{1}{t\cdot t!}.$$

or

$$K_{-1} = \frac{1}{e}\int_0^1 \frac{e^x}{x}\,dx \tag{8}$$

[4] For this table, and other numerical computations, free use was made of the following tables:
a. Fry *Probability and its Engineering Uses*. New York, 1928, Appendix III.
b. Hayashi *Sieben- und Mehrstellige Tafeln*. Berlin, 1926.
c. *Barlow's Tables* (ed. by Comrie). London, 1935.
d. Oakes *Tables of Reciprocals*. London, 1865.
e. Davis *Tables of the Higher Mathematical Functions*, v. I and II. Bloomington, Ind., 1933 and 1935.
f. Thompson *Logaritmetica Brittanica*. Cambridge U. Press, 1924–1937.
g. *Smithsonian Mathematical Formulae* (Adams, ed.), Washington, 1922.
A *Millionaire* computing machine with a ten place keyboard was employed. The values of K_{18}, K_{19}, and K_{20} were not computed independently, but were taken from the paper by Bell (cf. footnote 24c).

The exponential integral $\int_0^t \frac{e^x}{x}\,dx$ can be found tabulated in various places,[5] and can be used to calculate K_{-1} very accurately. Similarly we can readily prove

$$K_{-2} = \frac{1}{e}\int_0^1 \frac{1}{x_1}\int_0^{x_1}\frac{e^{x_2}}{x_2}\,dx_2\,dx_1$$

$$K_{-3} = \frac{1}{e}\int_0^1 \frac{1}{x_1}\int_0^{x_1}\frac{1}{x_2}\int_0^{x_2}\frac{e^{x_3}}{x_3}\,dx_3\,dx_2\,dx_1$$

or in general

$$K_{-n} = \frac{1}{e}\int_0^1 \frac{1}{x_1}\int_0^{x_1}\frac{1}{x_2}\int_0^{x_2}\frac{1}{x_3}\cdots\frac{1}{x_{n-1}}\int_0^{x_{n-1}}\frac{e^{x_n}}{x_n}\,dx_n\cdots dx_2\,dx_1 \tag{9}$$

a continued integral form, made up of n integrations, the final one with the limits zero to one. This equation applies, of course, only to integral values of n.

II. **Properties of the *K*-Function.** In this section we shall develop equations for some of the properties of the K-function. First, by Taylor's expansion

$$K_{x+\xi} = K_x + \xi K'_x + \frac{\xi^2}{2!}K''_x + \cdots + \frac{\xi^m}{m!}K_x^{(m)} + \cdots$$

$$K_x^{(m)} = \frac{1}{e}\sum_{t=1}^{\infty}\frac{t^x}{t!}(\ln t)^m \tag{10}$$

so that

$$K_{x+\xi} = \frac{1}{e}\sum_{m=0}^{\infty}\frac{\xi^m}{m!}\sum_{t=1}^{\infty}\frac{t^x}{t!}(\ln t)^m \tag{11}$$

This equation is most useful for computing K_y for large non-integral values of y, taking $y = x + \xi$, where ξ is less than one. The series converges for all values of x and ξ; for the coefficient of $\frac{\xi^m}{m!}$ in each case is

$$L_m(x) = \sum_{t=1}^{\infty}\frac{t^x}{t!}(\ln t)^m$$

[5] a) Jahnke-Emde *Funktionentafeln*. Leipzig, 1933.
b) British Assoc. Adv. Sci., Mathematical Tables I. London, 1931.

The ratio of the $(t + 1)$st to the t'th term of this series is

$$\left[\frac{\ln (t+1)}{\ln t}\right]^m \left(1 + \frac{1}{t}\right)^x \frac{1}{t+1}$$

and as t approaches infinity, this approaches

$$\lim_{t \to \infty} \left\{ \frac{1}{t+1} \left[\frac{\ln t\left(1 + \frac{1}{t}\right)}{\ln t} \right]^m \right\} = 0, \text{ for all values of } m, \text{ finite or infinite.}$$

The series for $L_m(x)$ thus converges to a constant value. The ratio of the $(m + 1)$st to the m'th term of equation (14) is

$$\frac{\xi^{m+1}}{\xi^m} \cdot \frac{m!}{(m+1)!} \cdot \frac{L_{m+1}}{L_m} = \frac{\xi}{m+1} \cdot \frac{L_{m+1}}{L_m}$$

Then as $m \to \infty$, this equation also approaches zero as a limit, since the ratio of the two finite constants $\frac{L_{m+1}}{L_m}$ is always finite. Thus equation (14) converges for all values such that

$$-\infty < \xi < +\infty.$$

As a special case of (14), we have the Maclaurin series expansion for K_x

$$K_x = \frac{1}{e} \sum_{m=0}^{\infty} \frac{p_m}{m!} x^m \tag{12}$$

the coefficients of this series

$$p_m = \sum_{t=1}^{\infty} \frac{(\ln t)^m}{t!}$$

have been calculated, and are summarized in Table II. A check on this computation is obtained from the interesting relation

$$\sum_{m=1}^{\infty} \frac{p_m}{m!} = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{t=1}^{\infty} \frac{(\ln t)^m}{t!} = 1$$

which can be derived from (12). The power series formulation is limited in its usefulness for calculating K_x only by the wearily slow convergence of the series for large values of x.

TABLE II

Coefficients of the Power Series Expansion for K_x

m	p_m	$p_m/m!$	$\sum_{k=1}^{m} p_k/k!$
0	1.71828 18285	—	—
1	0.60378 28628	0.60378 28628	.60378 28628
2	.54837 82849	.27418 91425	.87797 20053
3	.54296 35131	.09049 39188 7	.96846 59241
4	.58570 49319	.02440 43721 6	.99287 02963
5	.68236 68995	.00568 63908 29	.99855 66871
6	.84815 49252	.00117 79929 52	.99973 46801
7	1.11172 0107	.00022 05793 851	.99995 52594
8	1.52259 0134	.00003 77626 5211	.99999 30221
9	2.16436 2125	.00000 59644 01799	.99999 89865
10	3.17790 0086	.00000 08757 44071	.99999 98622
11	4.80224 7807	.00000 01203 06433	.99999 99826
12	7.44736 1338	.00000 00155 47675	.99999 99981
13	11.82463 354	.00000 00018 98923	1.00000 00000
14	19.18356 156	.00000 00002 20050	1.00000 00002
15	31.74519 554	.00000 00000 24276	1.00000 00002

The last figure is uncertain in the second and fourth columns.

We have already considered the differentiation of K_x in equation (10). Integration formulae may be readily derived from our definition (6). Thus

$$\int K_x f(x)\, dx = \frac{1}{e} \sum_{t=0}^{\infty} \frac{1}{t!} \int f(x) e^{x \ln t}\, dx \tag{13}$$

A large number of indefinite and definite integrals involving K_x may be evaluated using this equation.

So far, we have discussed K_x only as a function of a real variable; if $x = iy$

$$K_{iy} = \frac{1}{e} \sum_{t=0}^{\infty} \frac{t^{iy}}{t!} = \frac{1}{e} \left[\sum_{t=0}^{\infty} \frac{\cos (\ln t)y}{t!} + i \sum_{t=0}^{\infty} \frac{\sin (\ln t)y}{t!} \right] \tag{14}$$

that is, K_{iy} is a complex quantity. Similarly, for a complex argument

$$K_{x+iy} = \frac{1}{e} \left[\sum_{t=0}^{\infty} \frac{t^x \cos (\ln t)y}{t!} + i \sum_{t=0}^{\infty} \frac{t^x \sin (\ln t)y}{t!} \right] \tag{15}$$

In this last equation, we have terms of the form

$$\frac{t^x \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (\ln t)y}{t!}$$

since neither sin x nor cos x is greater than one

$$\left| \frac{t^x \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (\ln t)y}{t!} \right| \leq \frac{t^x}{t!}$$

But $\frac{t^x}{t!}$ is the t'th term of the series for K_x which we have already proven absolutely convergent. Thus following the comparison test for convergence,[6] equation (15) is absolutely convergent for all values of x and y; and therefore, equation (14) which is a special case of (15) is also absolutely convergent. Now since $e^{x+iy} = e^{x+i(y+2\pi k)}$

$$K_{x+i(y+2\pi k)} = K_{x+iy} \tag{16}$$

where k is an integer, and it follows that the K-function is periodic with the imaginary period $2\pi i$.

We shall conclude this section with a number of simply derived summation formulas, which we write without proof:

$$\sum_{n=0}^{\infty} \frac{K_n}{n!} = \frac{1}{e}(e^e - 1) = 5.2056 \cdots \tag{17}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{K_n}{n!} = \frac{1}{e}(e^{1/e} - 1) = 0.16359 \cdots$$

$$\left\{ \begin{aligned} \sum_{n=0}^{\infty} \frac{K_{2n}}{(2n)!} &= \frac{1}{2e}(e^e + e^{1/e} - 2) \\ \sum_{n=0}^{\infty} \frac{K_{2n+1}}{(2n+1)!} &= \frac{1}{2e}(e^e - e^{1/e}) \end{aligned} \right. \tag{18}$$

$$\left\{ \begin{aligned} e^{e^x \cos y} \cos(\sin y) &= e + e \sum_{n=1}^{\infty} \frac{K_n}{n!} r^n \cos n\theta \\ e^{e^x \cos y} \sin(\sin y) &= e \sum_{n=1}^{\infty} \frac{K_n}{n!} r^n \sin n\theta \end{aligned} \right. \tag{19}$$

In these last interesting Fourier series, the symbols r and θ have their usual polar-coordinate significance.

[6] Cf. Whittaker and Watson (1), ¶2.34.

III. **An Asymptotic Expansion**[7] **for** K_n. In this section, we shall attempt to develop a more convenient asymptotic formula for the function considered above. Briefly, the procedure to be followed consists of two steps: (A) the convergent infinite series formulation of K_n (equation 6) is converted into an asymptotically equivalent infinite integral by the use of the Euler-Maclaurin sum formula; and (B) this integral is then approximated asymptotically using a method developed by Laplace.

(A) We start then with the Euler-Maclaurin equation[8]

$$\begin{aligned} \sum_{x=0}^{\infty} f(x) = \int_0^{\infty} f(x)\,dx &- \frac{1}{2}[f(\infty) - f(0)] + \frac{B_1}{2!}[f'(\infty) - f'(0)] \\ &- \frac{B_2}{4!}[f'''(\infty) - f'''(0)] + \cdots \\ &+ (-1)^m \frac{B_m}{(2m)!}[f^{(2m-1)}(\infty) - f^{(2m-1)}(0)] \\ &+ \cdots \end{aligned} \tag{20}$$

In the case of the K-function

$$f(x) = \frac{x^n}{\Gamma(x+1)} \tag{21}$$

where the Gamma function now replaces the factorial in the denominator in order to obtain a continuous analytic function. Then we may write

$$\begin{aligned} eK_n = \int_0^{\infty} \frac{x^n\,dn}{\Gamma(x+1)} &- \tfrac{1}{2}[f(\infty) - f(0)] + \tfrac{1}{12}[f'(\infty) - f'(0)] \\ &- \frac{1}{720}[f'''(\infty) - f'''(0)] + \frac{1}{30{,}240}[f^{v}(\infty) - f^{v}(0)] \\ &- \cdots \\ &\cdots - (-1)^m \frac{B_m}{(2m)!}[f^{(2m-1)}(\infty) - f^{(2m-1)}(0)] \\ &+ \cdots \end{aligned} \tag{22}$$

[7] The restriction to integral values of the argument is made here only for convenience in formulation. Many of the results finally obtained, it will subsequently appear, are also valid for K_z, where z is non-integral.

[8] Cf. Ford *Studies on Divergent Series*, Macmillan (1916); also Ford *Asymptotic Developments* (U. of Mich., 1936).

or since $f(\infty)$ and $f(0)$ are both zero, we may write

(23a) $$eK_n = \int_0^\infty \frac{x^n\,dx}{\Gamma(x+1)} - s_n$$

where

(23b) $$s_n = \sum_{m=1} (-1)^m \frac{B_m}{(2m)!} [f^{(2m-1)}(\infty) - f^{(2m-1)}(0)]$$

We shall now evaluate $f^{(k)}(\infty)$ and $f^{(k)}(0)$ using Leibnitz's theorem[9] in the form

(24) $$f^{(k)}(x) = \sum_{i=0}^{k} \binom{k}{i} \frac{d^i}{dx^i}(x^n) \frac{d^{k-i}}{dx^{k-i}}\left[\frac{1}{\Gamma(x+1)}\right]$$

But since

$$\frac{d^i}{dx^i}(x^n) = n(n-1)(n-2)\cdots(n-i+1)x^{n-i}$$

(25) $$f^{(k)}(x) = \sum_{i=0}^{k} \binom{k}{i} \{n(n-1)(n-2)\cdots(n-i+1)x^{n-i}\} \cdot \frac{d^{k-i}}{dx^{k-i}}\left[\frac{1}{\Gamma(x+1)}\right]$$

We shall first use (25) to prove $f^{(k)}(\infty) = 0$; since $1/\Gamma(x)$ resembles e^{-x} in its behavior at $x = \infty$, in that the function and all of its finite derivatives are zeros of infinite order, it follows that

(26) $$f^{(k)}(\infty) = \lim_{x\to\infty} \frac{d^m}{dx^m}\left[\frac{x^n}{\Gamma(x+1)}\right] = 0$$

Again from (25), we observe that $f^{(k)}(0)$ is zero unless $i = n$. When this is true

$$f^{(k)}(0) = \binom{k}{n} \cdot n! \frac{d^{k-n}}{dx^{k-n}}\left[\frac{1}{\Gamma(x+1)}\right]_{x=0}$$

Such a term will occur only when $k \geqq n$; so that

(27) $$\begin{aligned} f^{(k)}(0) &= 0 && \text{when } 0 < k < n \\ &= \frac{k!}{(k-n)!} \frac{d^{k-n}}{dx^{k-n}}\left[\frac{1}{\Gamma(x+1)}\right]_{x=0} && \text{when } k \geqq n \end{aligned}$$

[9] Cf. e.g. Hardy *Pure Mathematics*, Cambridge (1933).

Now Weirstrass long ago demonstrated that the reciprocal of the Gamma function can be developed in a convergent power series

$$\frac{1}{\Gamma(x)} = \sum_{k=1}^{\infty} a_k x^k$$

or

$$\frac{1}{\Gamma(x+1)} = \sum_{k=1}^{\infty} a_k x^{k-1}$$

From Taylor's theorem, we observe that the expression required for (27) can be written most simply as

$$\frac{d^{k-n}}{dx^{k-n}}\left[\frac{1}{\Gamma(x+1)}\right]_{x=0} = a_{k-n+1}\cdot(k-n)!$$

Whence

(28) $$\begin{aligned} f^{(k)}(0) &= 0 && (0 < k < n) \\ &= k!\cdot a_{k-n+1} && (k \geqq n) \end{aligned}$$

Substituting the values of the derivatives at infinity and at the origin (equations 26 and 28) in (23b), we obtain

(29) $$s_n = -\sum_{m=m_0} (-1)^m \frac{B_m}{2m} \cdot a_{2m-n}$$

where

(30) $$m_0 = \begin{cases} (n+1)/2 & \text{when } n \text{ is odd} \\ (n+2)/2 & \text{when } n \text{ is even} \end{cases}$$

A table of the coefficients a_1 to a_{23} to sixteen places (the last doubtful according to Jensen[10]) is given by Bourguet.[11] Schlömilch[12] also studied the properties of these constants and obtained a recursion formula and an infinite integral form. Nevertheless, the behavior of a_k in the limit of large values of k seems to be only poorly known; and correspondingly, an exact study of the properties of our series (29) is not possible on the basis of present knowledge. Applying the ratio test

[10] Ann. of Math., 17, 123 (1915). Translated by Gronwall.
[11] Acta Mathematica, 2, 261 (1883). Cf. also Bull. des Sci. Math., 16, 43 (1883), Davis (cf. 4e), v. I, p. 185.
[12] Z. für Math. und Physik, XXV, 103 (1880).

for absolute convergence, and using the familiar expression for the Bernoulli numbers[1,4b,4e,5a]

$$B_m = \frac{2}{(2\pi)^{2m}}\cdot\zeta(2m)\cdot(2m)! \tag{31}$$

where $\zeta(x)$ is the Riemann Zeta function, we obtain

$$\frac{1}{2\pi^2}\left|\lim_{m\to\infty}\frac{\zeta(2m+2)}{\zeta(2m)}\cdot m(2m+1)\cdot\frac{a_{2m-n+2}}{a_{2m-n}}\right| \tag{32}$$

Thus, in order to obtain absolute convergence

$$\lim_{m\to\infty}\left|\frac{a_{2m-n+2}}{a_{2m-n}}\right| \leq \lim_{m\to\infty}\frac{2\pi^2}{m(2m+1)}$$

an extremely rigorous requirement, probably not fulfilled by a_k. We can however prove that the series is at least asymptotic, so that it is suitable for calculation, using the following theorem,[13] (using the following theorem,[13]) given by Knopp (Equation 300): "If $f(x)$ is of constant sign for $x > 0$, and, together with all its derivatives, tends monotonely to zero as $x \to \infty$, Euler's summation formula may be stated in the simplified form

$$\begin{aligned}\sum_{r=0}^{n} f(r) = \int_0^n f(x)\,dx - \tfrac{1}{2}[f(n)+f(0)] + \frac{B_1}{2!}[f'(n)-f'(0)]\\ + \cdots - (-1)^m\frac{B_m}{(2m)!}[f^{(2m-1)}(n)-f^{(2m-1)}(0)]\\ - (-1)^{m+1}\theta\frac{B_{m+1}}{(2m+2)!}[f^{(2m+1)}(n)-f^{(2m+1)}(0)]\end{aligned} \tag{33}$$

where $0 < \theta < 1$." Since, by equation (26) the required condition is fulfilled, we may terminate series (29) after any number of terms, with the assurance that the error thus committed will be less in absolute magnitude than the next term of the series.

(B) We now consider the infinite integral which we have developed above

$$I_n = \int_0^\infty y(x)\,dx; \qquad y(x) = \frac{x^n}{\Gamma(x+1)} \tag{34}$$

[13] *Theory and Application of Infinite Series*, translated by Young. Blackie (London, 1928).

Laplace[14] demonstrated that a definite integral of this type, taken between the limits of x which cause $y(x)$ to vanish, can be developed in an asymptotic series (subject to restrictions that are rather readily fulfilled),

$$\begin{aligned}\int_{x_1}^{x_2} y\,dx = Y\sqrt{\pi}\Big(U + \frac{1}{2}\cdot\frac{1}{2!}\frac{d^2U^3}{dx^2} + \frac{1\cdot3}{2^2}\cdot\frac{1}{4!}\frac{d^4}{dx^4}U^5\\ + \cdots + \frac{1\cdot3\cdot5\cdot\cdots\cdot(2m-1)}{2^m\cdot(2m)!}\frac{d^{2m}}{dx^{2m}}U^{2m+1} + \cdots\Big)\end{aligned} \tag{35}$$

where Y is the maximum value of y between the assigned limits and

$$U = \left[-\frac{1}{2y}\cdot\frac{d^2y}{dx^2}\right]^{-\frac{1}{2}}_{y=Y} \tag{36}$$

In the limit then, we may use only the first term of this series,

$$\int_{x_1}^{x_2} y\,dx \sim \frac{\sqrt{2\pi}\,Y^{3/2}}{\sqrt{-\dfrac{d^2Y}{dx^2}}} \tag{37}$$

where $\dfrac{d^2Y}{dx^2}$ is written for $\left(\dfrac{d^2y}{dx^2}\right)_{y=Y}$. We shall not attempt to justify this formula of Laplace's further at this time, but shall proceed to employ it for the evaluation of K_n.

In this problem, y goes through a maximum when $\dfrac{dy}{dx} = 0$; but

$$y' = \frac{x^n}{\Gamma(x+1)}\left[\frac{n}{x} - \Psi(x+1)\right]$$

where

$$\Psi(x) = \frac{d}{dx}\ln\Gamma(x)$$

so that y is a maximum when $x = \alpha_n$, where α_n is the solution of the transcendental equation

$$\alpha_n = \frac{n}{\Psi(\alpha_n+1)} \tag{38}$$

[14] Oeuvres, Tome 7, p. 103.

This equation can quite readily be solved numerically by successive approximations using the Newton-Raphson method. Thus if x_k is the k'th approximation to the root of

$$x\Psi(x+1) = n,$$

$$x_{k+1} = x_k + \frac{n - x_k\Psi(x_k+1)}{x_k\Psi'(x_k+1) + \Psi(x_k+1)}$$

$$= x_k - \frac{x_k\Psi^2(x_k+1) - n\Psi(x_k+1)}{\Psi^2(x_k+1) + n\Psi'(x_k+1)}$$

These equations lead to very rapid and convenient convergence in solving for α_n with tables of Ψ and Ψ'. Then

$$Y = \frac{\alpha_n^n}{\Gamma(\alpha_n+1)}; \qquad -\frac{d^2y}{dx^2} = \left\{\frac{\alpha_n^n}{\Gamma(\alpha_n+1)}\right\}\left[\frac{n}{\alpha_n^2} + \Psi'(\alpha_n+1)\right]$$

Substituting these values in (37) and (23a)

$$(39) \qquad \int_0^\infty \frac{x^n\,dx}{\Gamma(x+1)} \sim \sqrt{2\pi}\,\frac{\alpha_n^{n+1}}{\Gamma(\alpha_n+1)}\cdot\frac{1}{\sqrt{n+\alpha_n^2\Psi'(\alpha_n+1)}}$$

and

$$(40) \qquad K_n \sim \frac{\sqrt{2\pi}}{e}\cdot\frac{\alpha_n^{n+1}}{\Gamma(\alpha_n+1)}\cdot\frac{1}{\sqrt{n+\alpha_n^2\Psi'(\alpha_n+1)}} + \frac{1}{e}\sum_{m=m_0}(-1)^m\cdot\frac{B_m}{2m}\cdot a_{2m-n}$$

Generally speaking, the first term of equation 40, omitting the summation term entirely, is quite a fair approximation for the higher values of K_n. Thus while at $n = 10$, the result is about eighteen percent too high, at $n = 16$ where

$$\alpha_{16} = 7.63200\ 18618$$

we calculate

$$K_{16} \cong 10{,}481{,}629{,}910$$

which is greater than the true value of K_{16} (cf. Table I) by only about 0.014%. The first term of the series, $m = m_0 = 9$, is -1.76279; we note that, in this case at least, the summation term amounts to only a negligible fraction of the net error; so that the higher terms of the Laplace expansion (35) are much more important than the higher terms of (40).

For very large values of n, leading to large values of α_n, we may apply further approximations to (40). Introducing the approximations, valid for large n,

$$(41) \qquad \begin{cases} \Gamma(\alpha_n+1) \to \sqrt{2\pi\alpha_n}\left(\dfrac{\alpha_n}{e}\right)^{\alpha_n} \\ \Psi(\alpha_n+1) \to \ln(\alpha_n+1) \\ \Psi'(\alpha_n+1) \to 0 \end{cases}$$

we obtain

$$(42) \qquad K_n \sim \frac{\alpha_n^{n-\alpha_n}e^{\alpha_n-1}}{\sqrt{\ln\alpha_n}}$$

If the restrictions of (41) are valid, α_n is the solution of the transcendental equation,

$$(43) \qquad \alpha_n = \frac{n}{\ln(\alpha_n+1)}$$

and the solution of this equation can be obtained by iteration in the curious convergent form

$$(44) \qquad \alpha_n = \cfrac{n}{\ln\left(1 + \cfrac{n}{\ln\left(1 + \cfrac{n}{\ln\left(1 + \cfrac{n}{\ln(1+\cdots}\right.}\right.}\right.}$$

If n is taken quite large, this is equivalent to

$$\alpha_n = \frac{n}{\ln n - \ln(\ln\alpha_n)}$$

or as a very crude approximation, valid only in the limit of extremely large n,

$$(45) \qquad \alpha_n \sim \frac{n}{\ln n}$$

With this value of α_n, equation (42) becomes

$$(46) \qquad K_n \sim \left[\frac{ne^{1/\ln n}}{\ln n}\right]^n$$

By taking logarithms of this equation, we obtain

$$\ln K_n \sim n\left[\ln\left(\frac{n}{e \ln n}\right) + \left\{1 + \frac{1}{\ln n}\right\}\right] \tag{47}$$

which is nearly identical with the asymptotic expansion for $\ln K_n$ given by Knopp[15]

$$\ln K_n \sim n\left[\ln\left(\frac{n}{e \ln n}\right) + \epsilon_n\right]$$

where, the author says $\epsilon_n \to 0$ as $n \to \infty$. We observe that from our calculations above ϵ_n approaches one rather than zero, a difference which is negligible, since the two expressions for $\ln K_n$ rapidly approach each other for large values of n.

By a direct comparison of the higher terms of the infinite series for K_x, $x^m e^x$, and e^{e^x}, we can show that in the limit of large x

$$x^m e^x < K_x < e^{e^x} \tag{48a}$$

for all finite values of m. A much tighter inequality follows from (46):

$$K_n \sim e^{n \ln n + n/\ln n - n \ln(\ln n)}$$

so that

$$e^{(n \ln n)(1-\eta_n)} < K_n < e^{(n \ln n)(1+\xi_n)}$$

where

$$\eta_n = \frac{1}{\ln n}\left[\frac{1}{\ln n}\right], \qquad \xi_n = \frac{1}{\ln n}[\ln(\ln n)]$$

Defining $g(n)$ as the difference between η_n and ξ_n, so that

$$g(n) = \frac{1}{\ln n}\left[\frac{1}{\ln n} - \ln(\ln n)\right]$$

we note that

$$\operatorname*{Lim}_{n\to\infty} g(n) = -\operatorname*{Lim}_{n\to\infty} \frac{\ln(\ln n)}{\ln n} = 0$$

and hence

$$\operatorname*{Lim}_{n\to\infty} \eta_n/\xi_n = 1$$

[15] Cf. Knopp (footnote 13), example 236.

Then letting the limiting value of η_n and ξ_n as n approaches infinity be ζ_n, we have

$$e^{(n \ln n)(1-\zeta_n)} < K_n < e^{(n \ln n)(1+\zeta_n)}$$

or

$$n^{n(1-\zeta_n)} < K_n < n^{n(1+\zeta_n)} \tag{48b}$$

where

$$\zeta_n = O\left(\frac{1}{\ln n}\right)^2$$

In conclusion, the writer wishes to sincerely thank Professor Philip Franklin for his inspiration and many helpful suggestions in the preparation of this paper, and Julius E. Epstein for his aid with the lengthy numerical calculations.

Appendix I

Subsequent to the preparation of the larger part of this paper, several references with bearing on the problems herein considered came to the attention of the writer. These results are here presented in brief.

1. Schwatt,[16] in studying the properties of the differential operator $\left(x\frac{d}{dx}\right)^n$, finds

$$\begin{aligned} K_n &= \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \alpha^n \\ &= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}\left[\binom{k}{1}1^n - \binom{k}{2}2^n + \binom{k}{3}3^n - \cdots + (-1)^{k-1}\binom{k}{k}k^n\right] \\ &= \sum_{k=1}^{n} (-1)^{k+1}\left[\frac{1^n}{1!}\cdot\frac{1}{(k-1)!} - \frac{2^n}{2!}\cdot\frac{1}{(k-2)!} + \frac{3^n}{3!}\cdot\frac{1}{(k-3)!} - \cdots \right. \\ &\qquad \left. \cdots (-1)^{k-1}\frac{k^n}{k!}\right] \end{aligned} \tag{I-1}$$

2. Chiellini[17] studies $\sum_{n=0}^{\infty} \frac{n^r}{n!}$ for integral r, by expanding in the inverse factorial series

$$\frac{n^r}{n!} = \sum_{k=1}^{r} \frac{b_{rk}}{(n-k)!} \tag{I-2}$$

[16] *An Introduction to the Operations with Series.* U. of Penn. Press, Philadelphia (1924).

[17] Boll. Un. Mat. It., X, 134 (1931).

where

I-3 $$b_{rk} = \frac{(-1)^k}{k!} \sum_{\alpha=1}^{k} (-1)^\alpha \binom{k}{\alpha} \alpha^r$$

I-4 $$b_{rk} = k b_{r-1,k} + b_{r-1,k-1}$$

so that

$$b_{r1} = 1^{r-1}$$

$$b_{r2} + b_{r1} = 2^{r-1}$$

$$b_{r3} + b_{r2} + \frac{b_{r1}}{2!} = \frac{3^{r-1}}{2!}$$

$$\vdots$$

I-5 $$b_{rr} + b_{r,r-1} + \frac{b_{r,r-2}}{2!} + \frac{b_{r,r-3}}{3!} + \cdots + \frac{b_{r1}}{(r-1)!} = \frac{r^{r-1}}{(r-1)!},$$

finds

I-6 $$\sum_{n=1}^{\infty} \frac{n^r}{n!} = e \sum_{k=1}^{r} b_{rk} = b_r e$$

So that his b_r is identical with our K_r. He includes a table of b_r for integral r's from 1 to 10, with the misprints

$$4{,}140 = b_8 \neq 4{,}138$$

$$21{,}147 = b_9 \neq 22{,}147$$

3. Broggi[18] also studies $\sum_{n=1}^{\infty} \frac{n^h}{n!}$, h being integral and positive, by using the classical Stirling expansion

I-7 $$\prod_{p=0}^{n} \frac{1}{(x+p)} = \sum_{s=0}^{\infty} (-1)^s \frac{C_n^s}{x^{n+s}}$$

where

$$C_n^s = \frac{1}{n!} \sum_{r=0}^{n-1} (-1)^r (n-r)^{n+s} \binom{n}{r},$$

[18] Ist. Lombardo Rend. LXVI (IIa), 196 (1933).

so that the C_n^s's are what Nielsen[19] calls the Stirling numbers of the second kind, and designates $\mathfrak{C}_{n+1}^s$. Then from the known properties of these numbers, he demonstrates

I-8 $$K_h = 1 + \frac{1 \cdot (h-1)^h}{(h-1)!} + \frac{\left(1 - \frac{1}{1!}\right)(h-2)^h}{(h-2)!} + \cdots$$
$$\cdots + \frac{1 - 1/1! + 1/2! - \cdots + (-1)^h 1/(h-2)!}{1!}$$

I-9 $$K_h = C_h^0 + C_h^1 + C_{h-2}^2 + \cdots + C_1^{h-1}$$

and he proves that C_r^{r-k} is identical with Chiellini's b_{rk}. Finally he derives the asymptotic series

I-10 $$\frac{1}{e} \int_0^1 t^{x-1} e^t \, dt = \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n \frac{K_n}{x^{n+1}}$$

4. Whitworth[20] shows that "the total number of ways in which n different things can be distributed into 1, 2, 3, 4, $\cdots$ or n indifferent parcels is $n!$ times the coefficient of x^n in the expansion of e^{e^x}/e," i.e. K_n. He further proves

I-11 $$K_n = n! \sum_{t=1}^{n} N_t$$

where N_t is the number of t-partitions of n different things, and shows that N_t is $n!$ times the coefficient of x^n in the expansion of

$$\frac{(e^x - 1)^t}{t!}$$

Glover[21] used equation I-11 above to compute the first six values of $K_n/n!$

[19] *Handbuch der Theorie der Gamma funktion.* Leipzig, 1906. 26, 27, 109.
[20] *Choice and Chance*, p. 96. Stechert, New York, 1925.
[21] *Tables of Applied Mathematics*, Ann Arbor, 1923. The writer is indebted to Dr. Glover for the reference to Whitworth above, and for his method of computing $K_n/n!$ (private communication).

5. Anderegg[22] proves

$$\text{I-12} \qquad K_n = \begin{vmatrix} 1 & -\binom{n-1}{0} & -\binom{n-1}{1} & \cdots & \binom{n-1}{n-2} \\ 1 & 1 & -\binom{n-2}{0} & \cdots & \binom{n-2}{n-3} \\ 1 & 0 & 1 & \cdots & \binom{n-3}{n-4} \\ \vdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 \quad 1 & & \binom{1}{0} \\ 1 & 0 & 0 \cdots 0 & & 1 \end{vmatrix}$$

and also gives the values of K_1 to K_{10}.

6. P. Epstein,[23] Krug, Tauchard, and Aitken have studied the arithmetical properties of K_n. Tauchard gives K_1 to K_{11}.

7. Bell[24] has contributed several papers on the numbers we have considered here. He shows[24a] that

$$\text{I-13} \qquad \xi_n^{(1)}(\text{Bell}) \equiv K_n = \sum_{s=1}^{n} \frac{1}{(s-1)!} \left[\sum_{r=0}^{s-1} (-1)^r \binom{s-1}{r} (s-r)^{n-1} \right] \qquad (n > 0)$$

and discusses some of the arithmetic properties of these numbers. Further applications to the theory of numbers are also treated in the second[24b] and third[24c] papers. In the second paper he demonstrates

$$\text{I-14} \qquad K_n = \sum_{p=1}^{n} \frac{\Delta^p O^n}{p!}$$

where $\Delta^p O^n$ are the "differences of zero" discussed by various writers.[25] Equation I-14 is used, with tables of differences of zero, to find K_1 to K_{20}. The last three values in our Table I are taken from this paper—the rest were computed independently as indicated. The latest paper contains some interesting generalizations of the K-numbers, and discusses applications to computation; also an interesting interpretation of the significance of K_n in combinatory analysis: K_n "is the total number of possible rhyme schemes for a stanza of n verses."

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

[22] Am. Math. Monthly, 8, 54 (1901) and 9, 11 (1901).

[23] a) P. Epstein: Archiv. der Math. und Physik., 8, 329 (1904-5).
b) Krug: Ibid., 9, 189 (1905).
c) Aitken: Math. Notes (Edinburgh), 28, 18 (1933).
d) Tauchard: Ann. Soc. Sci. Bruxelles A, 53, 21 (1933).

[24] a) Am. Math. Monthly, 41, 411 (1934).
b) Ann. of Math., 35, 264 (1934).
c) Ibid., 39, 539 (1938).

[25] Cf. Steffensen *Interpolation*, Williams & Wilkins, Baltimore (1927). See also Davis, v. II (footnote 4e), p. 210.