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Occurrences of the integer $(2n-2)!/n!(n-1)!$

1. Introduction. In a multiplicative set G lacking both the associative and the commutative properties, an element x may have two distinct third powers; namely, $x(xx)$ and $(xx)x$. Let $N(n)$ denote the maximum number of possibly distinct n-th powers of an element in a non-associative and non-commutative groupoid. In [3], [5], and [2] appear proofs of the equality

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$$\text{(I)} \qquad N(n) = (2n-2)!/n!(n-1)!,$$

where n is a positive integer. All three of these proofs rely upon the obvious recursive relation

$$\text{(II)} \qquad N(n) = \sum_{k=1}^{n-1} N(k)N(n-k),$$

and the first two of the proofs, which use generating functions, are analytic in character.

In Section 2 we offer an elementary combinatorial proof, of (I), in which we sidestep relation (II) by establishing a one-to-one correspondence between the set of formal n-th powers of an element and the set of circular permutations of n objects of one sort with n-1 objects of another sort.

In Section 3 we prove that the integer $(2n-2)!/n!(n-1)!$ is odd if and only if n is a power of 2.

The author is indebted to his former students P. K. Davis and T. T. Cooper for their assistance in the work in Section 2, and also to J. R. Isbell for his suggestions leading to increased elegance in the present paper.

2. The powers problem. Lower-case Greek letters denote words in the alphabet $\{x, o\}$. The length of the word α is written $|\alpha|$, and the cardinal number of the set A is written $|A|$. For each positive integer $n \leqslant |\alpha|$ the symbol $\alpha(n)$ denotes the n-th letter in the word α. By $\alpha^{-1}(y)$ we mean $\{k: a(k) = y\}$ for $y \in \{x, o\}$. For each positive integer n the symbol S_n denotes the set $\{a: |a^{-1}(x)| = |a^{-1}(o)|+1 = n\}$.

By P_1 we mean $\{x\}$, and for each integer $n > 1$ we mean by P_n the set $\{\delta\beta o: \delta \in P_k$ and $\beta \in P_{n-k}$ for some positive integer $k < n\}$. Elements in P_n are called *explicit n-th powers of x*. It can be seen that $N(n) = |P_n|$ (1).

A word $a \in S_n$ is called *n-appropriate* if and only if $|\lambda^{-1}(x)| - |\lambda^{-1}(o)| \geqslant 1$ for every non-empty left segment λ of a. The set of all n-appropriate words is written A_n.

We omit the easy inductive proof that $P_n \subseteq S_n$ for every positive integer n.

THEOREM 2.1. *$P_n = A_n$ for each positive integer n.*

Proof. It is clear that $P_1 = \{x\} = A_1$. Let m be an arbitrary integer greater than 1. Suppose that the equality $P_k = A_k$ holds for every positive integer $k < m$. Let a be any element in P_m. It follows that $a = \mu\vartheta o$, where $\mu \in P_j = A_j$, and where $\vartheta \in P_{m-j} = A_{m-j}$, for some positive integer $j < m$. Let λ be a non-empty left segment of a. It is plain that $|\lambda^{-1}(x)| - |\lambda^{-1}(o)| \geqslant 1$ if $|\lambda| \leqslant |\mu|$, and also that $|\mu^{-1}(x)| - |\mu^{-1}(o)| = 1$. Therefore, we now suppose that $|\mu| < |\lambda| \leqslant |\mu\vartheta|$. It follows that $\lambda = \mu\tau$ for some non-empty left segment τ of ϑ, and that $|\lambda^{-1}(x)| - |\lambda^{-1}(o)| = |\mu^{-1}(x)| + |\tau^{-1}(x)| - |\mu^{-1}(o)| - |\tau^{-1}(o)| = 1 + |\tau^{-1}(x)| - |\tau^{-1}(o)| \geqslant 2$. Furthermore, $|(\mu\vartheta)^{-1}(x)| - |(\mu\vartheta)^{-1}(o)| = |\mu^{-1}(x)| - |\mu^{-1}(o)| + |\vartheta^{-1}(x)| - |\vartheta^{-1}(o)| = 2$. Thus we see that $|a^{-1}(x)| - |a^{-1}(o)| = 2-1$, and hence $a \in A_m$ and $P_m \subseteq A_m$.

Let β be an arbitrary element in A_m. The set $\{i: 0 < i < |\beta|$ and $|\lambda| = i$ for a left segment λ of β such that $|\lambda^{-1}(x)| - |\lambda^{-1}(o)| = 1\}$ is non-empty since it contains 1, and therefore this set contains a largest element p. There exist δ and γ such that $\beta = \delta\gamma$, and such that $|\delta| = p$. It is clear that $\delta \in A_{(p+1)/2} = P_{(p+1)/2}$.

We claim that $\gamma = \varphi o$, and that $\varphi \in A_{(2m-p-1)/2} = P_{(2m-p-1)/2}$. For, if σ is a left segment of γ, and if $1 \leqslant |\sigma| < |\gamma|$, then $|\sigma^{-1}(x)| - |\sigma^{-1}(o)| = |(\delta\sigma)^{-1}(x)| - |(\delta\sigma)^{-1}(o)| - 1 \geqslant 2-1$. And, furthermore, if φ is the left segment of γ such that $|\varphi| = |\gamma|-1$, then $|(\delta\varphi)^{-1}(x)| - |(\delta\varphi)^{-1}(o)| = |\delta^{-1}(x)| - |\delta^{-1}(o)| + |\varphi^{-1}(x)| - |\varphi^{-1}(o)| = 1 + |\varphi^{-1}(x)| - |\varphi^{-1}(o)| \geqslant 2$. But $|\beta^{-1}(x)| - |\beta^{-1}(o)| = 1$, and our claim is established. The theorem follows by mathematical induction.

Gummer [1] showed that $|A_n| = (2n-2)!/n!(n-1)!$. We present a simple and intuitive alternate proof of this fact. To this end we remark that there are exactly

$$\frac{1}{2n-1} \cdot \frac{(2n-1)!}{n!(n-1)!}$$

(1) Treat o as a binary operation symbol and P_n as a set of terms generated by o and one variable x in a bracket free notation.

distinguishable circular permutations of n occurrences of the letter x with $n-1$ occurrences of the letter o. It therefore suffices to demonstrate a one-to-one correspondence between A_n and the collection of all such circular permutations. This follows from our next theorem.

Let Γ be any circular arrangement of occurrences of x and o. Let y be such an occurrence in Γ. We call y *worthy in* Γ if and only if every non-empty word β, spelled counterclockwise around Γ by consecutive letters the first of which is y, satisfies the inequality $|\beta^{-1}(x)| > |\beta^{-1}(o)|$.

THEOREM 2.2. *Let Γ be any circular arrangement of n occurrences of o with $n+k$ occurrences of x, where n and k are any non-negative integers. Then there occur in Γ exactly k worthy letters.*

Proof. Fixing an arbitrary non-negative integer k, we employ induction on n. The basis $(n = 0)$ is trivial. Let m be any non-negative integer, and suppose that the theorem holds when $n = m$.

Let Σ be any circular arrangement of $m+1+k$ occurrences of x with $m+1$ occurrences of o. Reading counterclockwise around Σ we encounter at least once the two-letter word xo. Isolating this word xo from Σ we obtain a circular arrangement Δ of $m+k$ occurrences of x with m occurrences of o. By the inductive hypothesis there are in Δ exactly k worthy occurrences $y_1, y_2, \ldots, y_k$ of the letters x or o. It is now evident that $y_1, y_2, \ldots, y_k$ are worthy also in Σ, and that these are the only occurrences of letters worthy in Σ. The theorem follows.

It is an obvious corollary of 2.2 that each circular arrangement of n occurrences of x with $n-1$ occurrences of o unwraps, from exactly one letter in the arrangement, into an n-appropriate word. The desired equality (I) now follows from 2.1.

It is another obvious corollary of 2.2 that the number of distinct paths in two-space, from the lattice point $\langle 0, 0\rangle$ to the lattice point $\langle n-1, n\rangle$, is $(2n-2)!/n!(n-1)!$, when each of these paths is subject to the following three conditions:

If the lattice point $\langle k, j\rangle$ lies in the path, and if $0 \leqslant k < n-1$, then exactly one of the lattice points $\langle k+1, j\rangle$ or $\langle k, j+1\rangle$ lies in the path.

The path is $2n-1$ units long.

No point $\langle a, b\rangle$ lies in the path if $b < a$.

Let G be any set. Let n and k be any positive integers. Let $g_1, g_2, \ldots, g_n$ be any n not necessarily distinct elements in G. And let $*_1, *_2, \ldots, *_k$ be k distinct and independent, non-commutative and non-associative binary operations on G (2). Then a routine argument from 2.1 and 2.2 shows that there are exactly

$$k^{n-1}N(n)$$

(2) I.e. $\langle G; *_1, \ldots, *_k\rangle$ is a completely free algebra.

distinct elements, in the algebra $\langle G; *_1, *_2, \dots, *_k\rangle$, associated with the ordered n-tuple $\langle g_1, g_2, \dots, g_n\rangle$, where each such element is obtained by a judicious insertion, among the terms of the n-tuple, of $n-1$ operation symbols.

3. The parity of $N(n)$. For each real number y the symbol $\operatorname{int} y$ denotes the greatest integer not exceeding y.

We acknowledge the assistance of H. S. Zuckerman in the proof of the following lemma:

LEMMA 3.1. *Let n be any positive integer. Let f be the non-negative integer such that $n! = 2^f p$ for some odd integer p. Then:*

(3.1.1) *If n is a power of 2, then $f = n-1$.*

(3.1.2) *If n is not a power of 2, then $f < n-1$.*

Proof. It is folklore (e.g., p. 29 of [4]) that

$$f = \sum_{i=1}^{\infty} \operatorname{int}(n/2^i).$$

First, we suppose that $n = 2^m$ for some non-negative integer m. Then it is obvious that $\operatorname{int}(n/2^i) = n/2^i$ for every positive integer $i \leqslant m$, and also that $\operatorname{int}(n/2^j) = 0$ for every integer $j > m$. Therefore we have

$$f = \sum_{i=1}^{m} n/2^i = n-1.$$

The assertion (3.1.1) follows.

Next, we suppose that the integer n is not a power of 2. It follows that $n = 2^k q$ for some non-negative integer k, and for some odd integer $q > 2$. Let s be the largest integer i such that $\operatorname{int}(n/2^i) > 0$. It follows that $\operatorname{int}(n/2^{s+1}) = 0$, and hence $1 > n/2^{s+1} = 2^k q/2^{s+1} = 2^{k-s-1} q > 2^{k-s}$. Therefore, since the integer q is odd, we now see that $2^{k-s} q$ is not an integer, and hence $n/2^s - \operatorname{int}(n/2^s) = 2^{k-s} q - \operatorname{int}(2^{k-s} q) > 0$.

Therefore, from the choice of the integer s it is now clear that $1 \leqslant \operatorname{int}(n/2^s) < n/2^s$, and hence

$$f = \sum_{i=1}^{s} \operatorname{int}(n/2^i) < \sum_{i=1}^{s} n/2^i = n - (n/2^s) < n-1.$$

The assertion (3.1.2) follows, and the lemma is proved.

COROLLARY 3.2. *Let n be any positive integer. Let f be the non-negative integer such that $n! = 2^f d$ for some odd integer d. Let g be the non-negative integer such that $(2n)! = 2^g e$ for some odd integer e. Then:*

(3.2.1) *If n is a power of 2, then $g - 2f - 1 = 0$.*

(3.2.2) *If n is not a power of 2, then $g - 2f - 1 > 0$.*

Proof. The lemma is trivial in the case that $n = 1 = 2^0$. Suppose that $n > 1$. Then the set $\{i\colon i \text{ is a positive integer such that } \operatorname{int}(n/2^i) > 0\}$ is both finite and non-empty, and contains a largest element k. The set $\{i\colon i \text{ is a positive integer such that } \operatorname{int}(2n/2^i) > 0\}$ also is both finite and non-empty, and the integer $k+1$ is its largest element. We now have

$$\begin{aligned} g-2f-1 &= \sum_{i=1}^{\infty}(2n/2^i) - 2\sum_{i=1}^{\infty}\operatorname{int}(n/2^i)-1 \\ &= \sum_{i=1}^{k+1}\operatorname{int}(n/2^{i-1}) - 2\sum_{i=1}^{k}\operatorname{int}(n/2^i)-1 \\ &= \sum_{v=0}^{k}\operatorname{int}(n/2^v) - 2\sum_{i=1}^{k}\operatorname{int}(n/2^i)-1 \\ &= \operatorname{int}(n/2^0) - \sum_{i=1}^{k}\operatorname{int}(n/2^i)-1 \\ &= n - \sum_{i=1}^{\infty}\operatorname{int}(n/2^i)-1 = n-f-1. \end{aligned}$$

Therefore, when n is a power of 2, then by (3.1.1) we have $f = n-1$, and hence $g-2f-1 = n-(n-1)-1 = 0$. Likewise, on the other hand, when n is not a power of 2, then by (3.1.2) we have $f < n-1$, and hence $g-2f-1 > n-(n-1)-1 = 0$.

THEOREM 3.3. *The integer $N(n)$ is odd if and only if n is a power of 2.*

Proof. It is easy to see that

$$N(n) = \frac{(2n)!}{(n!)^2}\cdot\frac{1}{2p_1},$$

where p_1 is the odd integer $2n-1$. Let f be the non-negative integer such that $n! = 2^f p_2$ for some odd integer p_2. Let g be the non-negative integer such that $(2n)! = 2^g p_3$ for some odd integer p_3. It now follows that

$$N(n) = 2^{g-2f-1}p_3/p_2^2p_1.$$

From this equation together with (3.2.1) we see that if n is a power of 2, then $N(n) = p_3/p_2^2p_1$. Therefore, since in this case the integer $N(n)$ is a quotient of products of odd integers, $N(n)$ itself is odd.

Likewise, we see by (3.2.2) that if n is not a power of 2, then $N(n) = 2^t p_3/p_2^2p_1$, where t is the positive integer $g-2f-1$. Therefore, since $p_2^2p_1$ is an odd integer, it follows in this case that the integer $N(n)$ is even.

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