# ON REGULAR POLYTOPE NUMBERS

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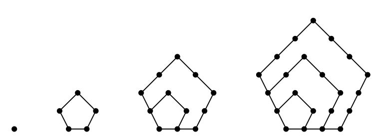
ABSTRACT. Lagrange proved a theorem which states that every nonnegative integer can be written as a sum of four squares. This result can be generalized in two directions. The one is a horizontal generalization which is known as polygonal number theorem, and the other is a higher dimensional generalization which is known as Hilber-Waring problem. In this paper, we shall generalize Lagrange's sum of four squares furthermore. To each regular polytope $V$ in an Euclidean space, we will associate a sequence of nonnegative integers which we shall call regular polytope numbers, and consider the problem of finding the order $g(V)$ of the set of regular polytope numbers associated to $V$. The construction of regular polytope numbers as well as numerical data for the order of the set of regular polytope numbers will be given.

In 1770, Lagrange proved a theorem which states that every nonnegative integer can be written as a sum of four squares. This theorem is known as the sum of four squares. There are two major generalizations of this beautiful result. The one is a horizontal generalization due to Cauchy which is known as polygonal number theorem, and the other is a higher dimensional generalization which is known as Hilber-Waring problem.

A nonempty subset $A$ of nonnegative integers is called a basis of order $g$ if $g$ is the minimum number with the property that every nonnegative integer can be written as a sum of $g$ elements in $A$. Lagrange's sum of four squares can be restated as the set $\{n^2 \mid n = 0, 1, 2 \ldots\}$ of nonnegative squares forms a basis of order 4. The polygonal numbers are sequences of nonnegative integers constructed geometrically from regular polygons. For example, pentagon numbers count the number of points in the following pentagonal array.



0 1 5 12 22

FIGURE 1. The pentagon numbers

The sequence of pentagon numbers is 0,1,5,12,22,35, $\cdots$. The sequence of $k$-gon numbers can be constructed in a similar manner. It is easy to check that formula for the $n$th $k$-gon number is $p_k^2(n) = n + (k-2)\frac{(n-1)n}{2}$. We now state Cauchy's polygonal number theorem (cf.[6]).

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**Theorem** (Cauchy) For every $k \geq 3$, the set $\{p_k^2(n) \mid n = 0, 1, 2, \cdots\}$ of $k$-gon numbers forms a basis of order $k$, i.e. every nonnegative integer can be written as a sum of $k$ $k$-gon numbers.

We note that polygonal numbers are two dimensional analogues of squares. Obviously, cubes, fourth powers, fifth powers, $\cdots$ are higher dimensional analogues of squares. In 1770, Waring stated without proof that every nonnegative integer can be written as a sum of 4 squares, 9 cubes, 19 fourth powers, and so on. In 1909, Hilbert proved that there is a finite number $g(d)$ such that every nonnegative integer is a sum of $g(d)$ $d$-th powers, i.e. the set $\{n^d | n = 0, 1, 2 \cdots\}$ of $d$th powers forms a basis of order $g(d)$. The Hilbert-Waring problem is concerned with the study of $g(d)$ for $d \geq 2$. This problem was one of the most important research topics in additive number theory in last 90 years, and it is still a very active area of research.

In this paper, we shall generalize Lagrange's sum of four squares furthermore. In fact, to each regular polytope $V$ in an Euclidean space, we will associate a sequence of nonnegative integers which we shall call regular polytope numbers, and consider the problem of finding the order $g(V)$ of the set of regular polytope numbers associated to $V$. The polygonal numbers can be considered as regular polytope numbers associated to regular polygons in $\mathbb{R}^2$ while the $d$th powers can be considered as regular polytope numbers associated to $d$-dimensional measure polytope. Therefore the theory of regular polytope numbers can be considered as a higher dimensional generalization of Cauchy's polygonal number theorem, or equivalently a horizontal generalization of Hilbert-Waring problem.

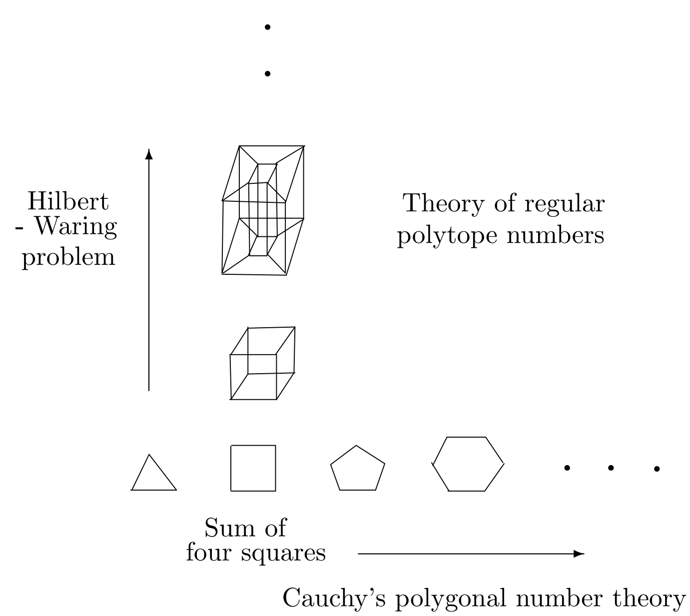



Figure 2. The address of regular polytope numbers

In section 1, we shall develop a method of constructing the sequence of regular polytope numbers associated to a regular polytope in an Euclidean space. We will obtain formulae for the $n$th regular polytope numbers. In section 2, we shall study relations between regular polytope numbers. As a result, we shall constitute an analogy between cross polytope numbers and measure polytope numbers. In section 3, we shall give numerical data for the order of the set of regular polytope numbers.

Throughout this paper, we shall use the following notations:

$\mathbb{Z}^{\geqslant}$ : the set of nonnegative integers
$V$ (resp. $V^d$) : a regular polytope (resp. of dimension $d$) in an Euclidean space
$V(n)$ : the $n$th polytope number associated to $V$
$\partial V(n)$ : the number of points in the $n$th array which lie on the boundary of $V$
$V(n)^\sharp$ : the number of points in the $n$th array which lie in the interior of $V$
$l^1$ : the 1-dimensional regular polytope, i.e. line segment
$p_k^2$ : the regular $k$-gon
$\alpha^d$ : the $d$-dimensional regular simplex, $d \geqslant 2$
$\beta^d$ : the $d$-dimensional cross polytope, $d \geqslant 2$
$\gamma^d$ : the $d$-dimensional measure polytope, $d \geqslant 2$

## 1. Construction of regular polytope numbers

Let $V^d$ be a $d$-dimensional regular polytope in an Euclidean space. In this section, we will develop the method of constructing the sequence $\{V(n)|n \in \mathbb{Z}^{\geqslant}\}$ of regular polytope numbers associated to $V$. This will be done by induction on the dimension $d$ of $V$. Therefore we start with the construction of 1-dimensional regular polytope numbers.

It is clear that a line segment $l$ is the only regular polytope in $\mathbb{R}^1$, and it is also clear that the $n$th regular polytope number associated to a line segment should be $n$, i.e. $l^1(n) = n$. Suppose now that the sequences of regular polytope numbers have been constructed for any regular polytopes of dimension less than $d$, and let $V$ be a regular polytope in $\mathbb{R}^d$. By convention we put $V(0) = 0$,$V(1) = 1$ and we define the sequence $\{V(n)|n \in \mathbb{Z}^{\geqslant}\}$ using induction once more on $n$. So we assume that $V(n-1)$ has been constructed, say on a regular polytope $X$ ( of the same shape as $V$). We take a vertex, say $x$, of $X$. We extend the edges of $X$ containing $x$ to form a lager regular polytope $\hat{X}$ containing $X$ which is similar to $X$ (You may refer to Figure 1). We next make the $n$th array of points associated to $V$ on $\hat{X}$ as follows. We first place the $(n-1)$th array of points on $X$ (Note that $X$ is contained in $\hat{X}$). Next, to each new $k$-dimensional face of $\hat{X}$, $0 \leqq k \leqq d-1$, we put $n$th array of points associated to the corresponding $k$-dimensional regular polytope. By convention, we put the $n$th 0-dimensional polytope number to be 1 if $n \geqslant 1$.We finally count all the points in $\hat{X}$ to define $V(n)$.

It follows easily from our construction that formula for the $n$th $k$-gon number is $n + (k-2)\frac{(n-1)n}{2}$. We can also easily check that the $n$th regular polytope number associated to the 3-dimensional cube is $n^3$ which coincides with our intuition.

We borrow a classical theorem from combinatorial geometry which classifies all the regular polytopes in Euclidean spaces(cf.[3]).

**Theorem**(Schläfli) The only possible Schläfli symbols for a regular polytope in the Euclidean space in $\mathbb{R}^d$ are given by the following list:
$d = 2 :$ $\{n\}$, where $n \geqslant 3$ is an arbitrary integer;
$d = 3 :$ $\{3,3\}$,$\{3,4\}$,$\{4,3\}$,$\{3,5\}$,$\{5,3\}$;
$d = 4 :$ $\{3,3,3\}$,$\{3,3,4\}$,$\{4,3,3\}$,$\{3,4,3\}$,$\{3,3,5\}$,$\{5,3,3\}$;
$d \geqslant 5 :$ $\{3^{d-1}\}$,$\{3^{d-2},4\}$,$\{4,3^{d-2}\}$.
For each symbol in the list, there exists a regular polytope with that symbol, and two regular polytopes with the same symbols are similar.

Our aim is to construct a sequence of regular polytope numbers associated to each regular polytope listed in the above theorem. To accomplish this task we need information on the number $N_j$ of $j$-dimensional faces of a regular polytope. We tabulate this information as follows (cf.[3]).

$d = 3 :$

| name | Schläfli symbol | $N_0$ | $N_1$ | $N_2$ |
|---|---|---|---|---|
| tetrahedron | $\{3,3\}$ | 4 | 6 | 4 |
| octahedron | $\{3,4\}$ | 6 | 12 | 8 |
| cube | $\{4,3\}$ | 8 | 12 | 6 |
| icosahedron | $\{3,5\}$ | 12 | 30 | 20 |
| dodecahedron | $\{5,3\}$ | 20 | 30 | 12 |

$d = 4 :$

| name | Schläfli symbol | $N_0$ | $N_1$ | $N_2$ | $N_3$ |
|---|---|---|---|---|---|
| 5 cell | $\{3,3,3\}$ | 5 | 10 | 10 | 5 |
| 16 cell | $\{3,3,4\}$ | 8 | 24 | 32 | 16 |
| tessaract | $\{4,3,3\}$ | 16 | 32 | 24 | 8 |
| 24 cell | $\{3,4,3\}$ | 24 | 96 | 96 | 24 |
| 600 cell | $\{3,3,5\}$ | 120 | 720 | 1200 | 600 |
| 120 cell | $\{5,3,3\}$ | 600 | 1200 | 720 | 120 |

$d \geqslant 5 :$

| name | Schläfli symbol | $N_j$ $(0 \leqslant j \leqslant d-1)$ |
|---|---|---|
| regular simplex $\alpha^d$ | $\{3^{d-1}\}$ | $\binom{d+1}{j+1}$ |
| cross polytope $\beta^d$ | $\{3^{d-2},4\}$ | $2^{j+1}\binom{d}{j+1}$ |
| measure polytope $\gamma^d$ | $\{4,3^{d-2}\}$ | $2^{d-j}\binom{d}{j}$ |

**Theorem 1.1.** *The 3-dimensional regular polytope numbers are computed as follows:*

| *name of polytope* | *Schläfli symbol* | *nth polytope number* |
|---|---|---|
| *tetrahedron* | $\{3,3\}$ | $\frac{1}{6}n(n+1)(n+2)$ |
| *cube* | $\{4,3\}$ | $n^3$ |
| *octahedron* | $\{3,4\}$ | $\frac{1}{3}n(2n^2+1)$ |
| *dodecahedron* | $\{5,3\}$ | $\frac{1}{2}n(9n^2-9n+2)$ |
| *icosahedron* | $\{3,5\}$ | $\frac{1}{2}n(5n^2-5n+2)$ |

**Proof**. We only give a proof for dodecahedron numbers. The other cases can be treated similarly. Let $D$ denote a dodecahedron in $\mathbb{R}^3$. Then $N_0 = 20$, $N_1 = 30$, and $N_2 = 12$. By definition, $D(0) = 0$ and $D(1) = 1$. Let $x$ be a vertex of $D$. Then the number of edges (resp. pentagons) containing $x$ is 3 (resp. 3). It follows from our construction that

$$\begin{aligned} D(n) - D(n-1) &= (N_0 - 1) + (N_1 - 3)l^1(n)^\sharp + (N_2 - 3)p_5^2(n)^\sharp \\ &= 19 + 27(n-2) + 9\left(\tfrac{3n^2-n}{2} - 5(n-1)\right) \\ &= \tfrac{1}{2}(27n^2 - 45n + 20), n \geqslant 2. \end{aligned}$$

Therefore we have $D(n) = \frac{n}{2}(9n^2 - 9n + 2)$, $n \geqslant 0$.

**Theorem 1.2.** *The 4-dimensional regular polytope numbers are computed as follows:*

| *Schläfli symbol* | *nth polytope number* |
|---|---|
| $\{3,3,3\}$ | $\frac{1}{4!}n(n+1)(n+2)(n+3)$ |
| $\{3,3,4\}$ | $\frac{1}{3}n^2(n^2+2)$ |
| $\{4,3,3\}$ | $n^4$ |
| $\{3,4,3\}$ | $n^2(3n^2-4n+2)$ |
| $\{3,3,5\}$ | $\frac{n}{6}(145n^3-280n^2+179n-38)$ |
| $\{5,3,3\}$ | $\frac{n}{2}(261n^3-504n^2+283n-38)$ |

**Proof**. We only give the construction for four dimensional measure polytope $\{4,3,3\}$ since the other cases can be treated similarly. Our intuition says that the $n$th polytope number for this polytope should be $n^4$. Let $V$ be a measure polytope in $\mathbb{R}^4$ whose Schläfli symbol is $\{4,3,3\}$, and $x$ be a vertex of $V$. Then the number of edges (resp. squares, cubes) containing $x$ is 4 (resp. 6,4). It follows from our construction that

$$\begin{aligned} V(n) - V(n-1) &= (N_0 - 1) + (N_1 - 4)l^1(n)^\sharp + (N_2 - 6)p_4^2(n)^\sharp \\ &\quad + (N_3 - 4)\alpha^3(n)^\sharp \\ &= 4n^3 - 6n^2 + 4n - 1. \end{aligned}$$

Therefore we have $V(n) = n^4$ as we expected from our intuition.

**Theorem 1.3.** *The $d$-dimensional regular polytope numbers, $d \geqslant 5$, are computed as follows:*

| *name* | *Schläfli symbol* | *nth polytope number* |
|---|---|---|
| *regular simplex* $\alpha^d$ | $\{3^{d-1}\}$ | $\frac{1}{d!}n(n+1)\cdots(n+d-1)$ |
| *cross polytope* $\beta^d$ | $\{3^{d-2},4\}$ | $\sum_{r=0}^{d-1}(-1)^r\binom{d-1}{r}2^{d-1-r}\alpha^{d-r}(n)$ |
| *measure polytope* $\gamma^d$ | $\{4,3^{d-2}\}$ | $n^d$ |

*Here $\alpha^s(n)$ denotes the nth $s$-dimensional regular simplex number, i.e.* $\alpha^s(n) = \frac{1}{s!}n(n+1)\cdots(n+s-1) = \binom{n+s-1}{s}$.

**Proof**. Regular simplex numbers: We shall proceed by induction on $d$. So we assume that $\alpha^r(n) = \frac{n(n+1)\cdots(n+r-1)}{r!}$ for $r \leqslant d-1$. Let $\alpha^d$ denote a $d$-dimensional regular simplex in $\mathbb{R}^d$. Then $N_i = \binom{d+1}{i+1}$, $0 \leqslant i \leqslant d-1$. Let $x$ be a vertex of $\alpha^d$. It follows from a simple calculation that the number of $r$-dimensional faces, which are also $r$-dimensional regular simplexes, of $\alpha^d$ containing $x$ is $\binom{d}{r}$, $1 \leqslant r \leqslant d-1$. Hence it follows from our construction that

$$\begin{aligned}\alpha^d(n) - \alpha^d(n-1) &= (N_0 - 1) + \sum_{r=1}^{d-1}(N_r - \binom{d}{r})\alpha^r(n)^\sharp \\ &= \sum_{r=0}^{d-1}\binom{d}{r+1}\alpha^r(n)^\sharp,\end{aligned}$$

where $\alpha^0(n)^\sharp = 1$ by convention. Note that each face of a regular simplex is again a regular simplex of lower dimension. It follows easily from this fact and our construction that

$$\alpha^r(n) = \sum_{j=0}^{r}\binom{r+1}{j+1}\alpha^j(n)^\sharp, 1 \leqslant r \leqslant d-1.$$

Therefore we have

$$\begin{aligned}\alpha^d(n) - \alpha^d(n-1) &= \alpha^{d-1}(n) = \frac{n(n+1)\cdots(n+d-2)}{(d-1)!}. \\ &\quad \text{( by induction hypothesis )}\end{aligned}$$

From this it follows that

$$\alpha^d(n) = \frac{n(n+1)\cdots(n+d-1)}{d!}, n = 0, 1, \cdots .$$

Cross polytope numbers: Let $\beta^d$ denote a $d$-dimensional cross polytope in $\mathbb{R}^d$. Then $N_i = 2^{i+1}\binom{d}{i+1}, 0 \leqslant i \leqslant d-1$. Let $x$ be a vertex of $\beta^d$. It follows from a simple calculation that the number of $r$-dimensional faces, which are $r$-dimensional regular simplexes, of $\beta^d$ containing $x$ is $2^r\binom{d-1}{r}, 1 \leqslant r \leqslant d-1$. Hence it follows from our construction that

$$\begin{aligned}\beta^d(n) - \beta^d(n-1) &= \sum_{r=0}^{d-1}\left(N_r - 2^r\binom{d-1}{r}\right)\alpha^r(n)^\sharp \\ &= \sum_{r=0}^{d-1}\left(2^{r+1}\binom{d}{r+1} - 2^r\binom{d-1}{r}\right)\alpha^r(n)^\sharp \\ &\quad \text{(replacing r by d-1-r)} \\ &= \sum_{r=0}^{d-1}\left(2^{d-r}\binom{d}{r} - 2^{d-1-r}\binom{d-1}{r}\right)\alpha^{d-1-r}(n)^\sharp.\end{aligned}$$

By a simple manipulation on binomial coefficients, we can verify that

$$2^{d-r}\binom{d}{r} - 2^{d-1-r}\binom{d-1}{r} = \sum_{u=0}^{r}(-1)^u\binom{d-1}{u}2^{d-1-u}\binom{d-u}{r-u}.$$

Therefore we have

$$\begin{aligned}
\beta^d(n)-\beta^d(n-1) &= \sum_{r=0}^{d-1}\left(\sum_{u=0}^{r}(-1)^u\binom{d-1}{u}2^{d-1-u}\binom{d-u}{r-u}\right)\alpha^{d-1-r}(n)^\sharp \\
&= \sum_{u=0}^{d-1}(-1)^u\binom{d-1}{u}2^{d-1-u}\left(\sum_{r=u}^{d-1}\binom{d-u}{r-u}\alpha^{d-1-r}(n)^\sharp\right) \\
&= \sum_{u=0}^{d-1}(-1)^u\binom{d-1}{u}2^{d-1-u}\sum_{s=0}^{d-u-1}\binom{d-u}{s+1}\alpha^{s}(n)^\sharp \\
&= \sum_{u=0}^{d-1}(-1)^u\binom{d-1}{u}2^{d-u-1}\alpha^{d-u-1}(n) \\
&= \sum_{r=0}^{d-1}(-1)^r\binom{d-1}{r}2^{d-r-1}\left(\alpha^{d-r}(n)-\alpha^{d-r}(n-1)\right).
\end{aligned}$$

This proves that

$$\beta^d(n)=\sum_{r=0}^{d-1}(-1)^r\binom{d-1}{r}2^{d-r-1}\alpha^{d-r}(n).$$

Measure polytope numbers: Let $\gamma^d$ be a $d$-dimensional measure polytope in $\mathbb{R}^d$. Then $N_i=2^{d-i}\binom{d}{i}, 0\leqslant i\leqslant d-1$. Let $x$ be a vertex of $\gamma^d$. It follows from a simple calculation that the number of $r$-dimensional faces, which are also $r$-dimensional measure polytopes, of $\gamma^d$ containing $x$ is $\binom{d}{r}, 1\leqslant r\leqslant d-1$. We now use induction on $d$. So we assume that $\gamma^r(n)=n^r$ for $0\leqslant r\leqslant d-1$. Note that each face of a measure polytope is again a measure polytope of lower dimension. From this fact, it is easy to observe that

$$\gamma^r(n)=\sum_{s=0}^{r}2^{r-s}\binom{r}{s}\gamma^s(n)^\sharp, 1\leqslant r\leqslant d-1.$$

It now follows from our construction that

$$\begin{aligned}
\gamma^d(n)-\gamma^d(n-1) &= \sum_{r=0}^{d-1}(N_r-\binom{d}{r})\gamma^r(n)^\sharp \\
&= \sum_{r=0}^{d-1}(2^{d-r}-1)\binom{d}{r}\gamma^r(n)^\sharp \\
&= \sum_{r=0}^{d-2}(2^{d-r}-1)\binom{d}{r}\gamma^r(n)^\sharp+\binom{d}{d-1}\gamma^{d-1}(n)^\sharp.
\end{aligned}$$

It follows from our observation that

$$\gamma^{d-1}(n)^\sharp=n^{d-1}-\sum_{r=0}^{d-2}2^{d-1-r}\binom{d-1}{r}\gamma^r(n)^\sharp.$$

Thus we have

$$
\begin{aligned}
&\gamma^d(n) - \gamma^d(n-1) \\
= \quad & \binom{d}{d-1} n^{d-1} + \sum_{r=0}^{d-2} \left[ (2^{d-r}-1)\binom{d}{r} - 2^{d-1-r}\binom{d-1}{r}\binom{d}{d-1} \right] \gamma^r(n)^\sharp \\
= \quad & \binom{d}{d-1} n^{d-1} - \binom{d}{d-2}\gamma^{d-2}(n)^\sharp + \\
& \sum_{r=0}^{d-3} \left[ (2^{d-r} - \binom{d-r}{1} 2^{d-1-r} - 1)\binom{d}{r} \right] \gamma^r(n)^\sharp \\
= \quad & \binom{d}{d-1} n^{d-1} - \binom{d}{d-2} n^{d-2} + \\
& \sum_{r=0}^{d-3} \left[ (2^{d-r} - \binom{d-r}{1} 2^{d-1-r} + \binom{d-r}{2} 2^{d-r-2} - 1)\binom{d}{r} \right] \gamma^r(n)^\sharp.
\end{aligned}
$$

Continuing this process, we finally have

$$\gamma^d(n) - \gamma^d(n-1) = \sum_{i=1}^{d} (-1)^{i-1} \binom{d}{i} n^{d-i}.$$

Thus we have $\gamma^d(n) = n^d, n = 0, 1, \cdots$.

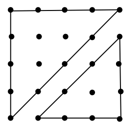
**Remark**: We compute the $n$th cross polytope number of dimension $d$, $2 \leqslant d \leqslant 10$, as follows:

| $d$ | $\beta^d(n)$ |
|---|---|
| 2 | $n^2$ |
| 3 | $\frac{1}{3}n(2n^2+1)$ |
| 4 | $\frac{1}{3}n^2(n^2+2)$ |
| 5 | $\frac{n}{15}(2n^4+10n^2+3)$ |
| 6 | $\frac{1}{45}n^2(2n^4+20n^2+23)$ |
| 7 | $\frac{n}{315}(4n^6+70n^4+196n^2+45)$ |
| 8 | $\frac{n^2}{315}(n^6+28n^4+154n^2+132)$ |
| 9 | $\frac{n}{2835}(2n^8+84n^6+798n^4+1636n^2+315)$ |
| 10 | $\frac{n^2}{14175}(2n^8+120n^6+1806n^4+7180n^2+5067)$ |

## 2. Relation between regular polytope numbers

In the first section we have developed the concept of regular polytope numbers and computed them. The formulae for regular simplex numbers and measure polytope numbers are simple and they coincide with our intuition. However formulae for cross polytope numbers are complicated and look unnatural. The purpose of this section is to give an analogy between cross polytope numbers and measure polytope numbers. By doing this, we give naturality on cross polytope numbers.

We start with the following simple geometric figure.



$$n^2 = \frac{n(n+1)}{2} + \frac{n(n-1)}{2}$$

$n$th square number = $n$th triangle number + $(n-1)$th triangle number

FIGURE 3. Decomposition of a square

Figure 3 can be generalized in two directions, i.e. horizontally and vertically. As a horizontal generalization we can easily verify that $p_k(n) = p_3(n)+(k-3)p_3(n-1)$, i.e. $n$th $k$-gon number equals $n$th triangle number plus $(k-3)$times $(n-1)$th triangle number. As a vertical generalization, we can show that $n^3 = \alpha^3(n) + 4\alpha^3(n-1) + \alpha^3(n-2)$, i.e. $n$th cube number equals $n$th pyramid number plus 4 times $(n-1)$th pyramid number plus $(n-2)$th pyramid number. More generally, we have the following theorem.

**Theorem 2.1.** *Every $d$-dimensional regular polytope number can be written as a linear combination of $d$-dimensional regular simplex numbers with nonnegative integer coefficients.*

**Proof**. For the cases of dimension 2,3, and 4, the theorem is an immediate consequence of direct computations. We only give the results for these cases.

$d = 2$;

$$p_k(n) = p_3(n) + (k-3)p_3(n-1), k \geqslant 3.$$

$d = 3$;

$$\begin{aligned}
n\text{th cube number} &= \alpha^3(n) + 4\alpha^3(n-1) + \alpha^3(n-2), \\
n\text{th octahedron number} &= \alpha^3(n) + 2\alpha^3(n-1) + \alpha^3(n-2), \\
n\text{th dodecahedron number} &= \alpha^3(n) + 16\alpha^3(n-1) + 10\alpha^3(n-2), \\
n\text{th icosahedron number} &= \alpha^3(n) + 8\alpha^3(n-1) + 6\alpha^3(n-2).
\end{aligned}$$

$d = 4$;

$$\begin{aligned}
n\text{th } \{3,3,4\} \text{ number} &= \alpha^4(n) + 3\alpha^4(n-1) + 3\alpha^4(n-2) + \alpha^4(n-3), \\
n\text{th } \{4,3,3\} \text{ number} &= \alpha^4(n) + 11\alpha^4(n-1) + 11\alpha^4(n-2) + \alpha^4(n-3), \\
n\text{th } \{3,4,3\} \text{ number} &= \alpha^4(n) + 19\alpha^4(n-1) + 43\alpha^4(n-2) + 9\alpha^4(n-3), \\
n\text{th } \{3,3,5\} \text{ number} &= \alpha^4(n) + 115\alpha^4(n-1) + 357\alpha^4(n-2) + 107\alpha^4(n-3), \\
n\text{th } \{5,3,3\} \text{ number} &= \alpha^4(n) + 45\alpha^4(n-1) + 1993\alpha^4(n-2) + 543\alpha^4(n-3).
\end{aligned}$$

$d \geqslant 5$;

Cross polytope numbers: It follows from Theorem 1.3. that

$$\beta^d(n) = \sum_{r=0}^{d-1} (-1)^r \binom{d-1}{r} 2^{d-1-r} \alpha^{d-r}(n).$$

Note that $\alpha^{d-1}(n)=\alpha^d(n)-\alpha^d(n-1)$. By a successive application of this relation, we have

$$\alpha^{d-r}(n)=\sum_{i=0}^{r}(-1)^i\binom{r}{i}\alpha^d(n-i).$$

Therefore we have

$$\begin{aligned}
\beta^d(n) &= \sum_{r=0}^{d-1}(-1)^r\binom{d-1}{r}2^{d-1-r}\sum_{i=0}^{r}(-1)^i\binom{r}{i}\alpha^d(n-i)\\
&= \sum_{i=0}^{d-1}\sum_{r=i}^{d-1}(-1)^r\binom{d-1}{r}2^{d-1-r}(-1)^i\binom{r}{i}\alpha^d(n-i)\\
&= \sum_{i=0}^{d-1}a_i\alpha^d(n-i),
\end{aligned}$$

where $a_i=(-1)^i\sum_{r=i}^{d-1}(-1)^r\binom{d-1}{r}\binom{r}{i}2^{d-1-r}$.

We now have

$$\begin{aligned}
a_i &= (-1)^i\sum_{r=i}^{d-1}(-1)^r\binom{d-1}{r}\binom{r}{i}2^{d-1-r}\\
&= (-1)^i\sum_{r=i}^{d-1}(-1)^r\binom{d-1}{i}\binom{d-1-i}{d-1-r}2^{d-1-r}\\
&\quad \text{(replacing d-1-r by s)}\\
&= \binom{d-1}{i}\sum_{s=0}^{d-1-i}(-1)^{d-1-i-s}\binom{d-1-i}{s}2^s\\
&= \binom{d-1}{i}.
\end{aligned}$$

This proves that

$$\beta^d(n)=\sum_{i=0}^{d-1}\binom{d-1}{i}\alpha^d(n-i).$$

Measure polytope numbers: It is well-known, from Worpitzky's identity(cf.[9]), that

$$x^n=\sum_{k=0}^{n-1}\left\langle \begin{matrix} n \\ k \end{matrix}\right\rangle\binom{x+k}{n},$$

where $\left\langle \begin{matrix} n \\ k \end{matrix}\right\rangle$ denotes the Eulerian number which counts the number of permutations in $S_n$ which has exactly $k$ ascents. It is easy to verify that this identity can be reformulated as

$$\gamma^d(n)=\sum_{i=0}^{d-1}\left\langle \begin{matrix} d \\ i \end{matrix}\right\rangle\alpha^d(n-i).$$

**Remark**: The Eulerian number $\left\langle \begin{matrix} d \\ i \end{matrix} \right\rangle$ shares similar properties with the binomial coefficient $\binom{d-1}{i}$. For example, we have identities

$$\begin{aligned} \left\langle \begin{matrix} d \\ i \end{matrix} \right\rangle &= \left\langle \begin{matrix} d \\ d-1-i \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} d \\ i \end{matrix} \right\rangle &= (i+1)\left\langle \begin{matrix} d-1 \\ i \end{matrix} \right\rangle + (d-i)\left\langle \begin{matrix} d-1 \\ i-1 \end{matrix} \right\rangle, d > 0. \end{aligned}$$

This constitutes an analogy between cross polytope numbers and measure polytope numbers.

## 3. Numerical results

In this section, we give numerical data for the order $g$ of the set of regular polytope numbers which are obtained by computer search. Since Euler's polygonal number theorem gives the exact order for the set of regular polytope numbers of dimension 2, we only consider the cases of dimension $\geqslant 3$.

$d = 3$

| Schläfli symbol | $n$th polytope number | $g$ |
|---|---|---|
| $\{3,3\}$ | $\frac{1}{6}n(n+1)(n+2)$ | 5 |
| $\{4,3\}$ | $n^3$ | 9* |
| $\{3,4\}$ | $\frac{1}{3}n(2n^2+1)$ | 7 |
| $\{5,3\}$ | $\frac{1}{2}n(9n^2-9n+2)$ | 22 |
| $\{3,5\}$ | $\frac{1}{2}n(5n^2-5n+2)$ | 15 |

$d = 4$

| Schläfli symbol | $n$th polytope number | $g$ |
|---|---|---|
| $\{3,3,3\}$ | $\frac{1}{4!}n(n+1)(n+2)(n+3)$ | 8 |
| $\{3,3,4\}$ | $\frac{1}{3}n^2(n^2+2)$ | 11 |
| $\{4,3,3\}$ | $n^4$ | 19* |
| $\{3,4,3\}$ | $n^2(3n^2-4n+2)$ | 28 |
| $\{3,3,5\}$ | $\frac{n}{6}(145n^3-280n^2+179n-38)$ | 125 |
| $\{5,3,3\}$ | $\frac{n}{2}(261n^3-504n^2+283n-38)$ | 606 |

$d = 5$

| Schläfli symbol | $n$th polytope number | $g$ |
|---|---|---|
| $\{3,3,3,3\}$ | $\frac{1}{5!}n(n+1)(n+2)(n+3)(n+4)$ | 10 |
| $\{3,3,3,4\}$ | $\frac{1}{15}n(2n^4+10n^2+3)$ | 14 |
| $\{4,3,3,3\}$ | $n^5$ | 37* |

$d = 6$

| Schläfli symbol | $n$th polytope number | $g$ |
|---|---|---|
| $\{3^5\}$ | $\frac{1}{6!}n(n+1)(n+2)(n+3)(n+4)(n+5)$ | 13 |
| $\{3^4, 4\}$ | $\frac{1}{45}n^2(2n^4+20n^2+23)$ | 19 |
| $\{4, 3^4\}$ | $n^6$ | 73* |

$d = 7$

| Schläfli symbol | $n$th polytope number | $g$ |
|---|---|---|
| $\{3^6\}$ | $\frac{1}{7!}n(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)$ | 15 |
| $\{3^5, 4\}$ | $\frac{1}{315}n(4n^6+70n^4+196n^2+45)$ | 21 |
| $\{4, 3^5\}$ | $n^7$ | 143 |

**Remark**: In the table, * denotes that the given value is exact. That $g(\{4,3\}) = 9$ was proved jointly by Wieferich[8] and Kempner[5], $g(\{4,3,3\}) = 19$ was proved in joint work of Balasubramanian[1] and Deshouillers and Dress[4], $g(\{4,3,3,3\}) = 37$ was proved by Chen[2], and $g(\{4,3,3,3,3\}) = 73$ was proved by Pillai[7].

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