Generalizations of Schöbi's Tetrahedral Dissection

N. J. A. Sloane
AT&T Shannon Labs
180 Park Ave., Florham Park, NJ 07932-0971

Vinay A. Vaishampayan
AT&T Shannon Labs
180 Park Ave., Florham Park, NJ 07932-0971

Email: njas@research.att.com, vinay@research.att.com

October 19, 2007; revised November 13, 2007 and May 13, 2008

Abstract

Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{R}^n$ such that $v_i \cdot v_j = -w$ for $i \neq j$ where $-1 < w < \frac{1}{n-1}$. The points $\sum_{i=1}^{n} \lambda_i v_i$ $(1 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ form a "Hill-simplex of the first type", denoted by $\mathcal{Q}_n(w)$. It was shown by Hadwiger in 1951 that $\mathcal{Q}_n(w)$ is equidissectable with a cube. In 1985, Schöbi gave a three-piece dissection of $\mathcal{Q}_3(w)$ into a triangular prism $c\mathcal{Q}_2(\frac{1}{2}) \times I$, where I denotes an interval and $c = \sqrt{2(w+1)/3}$. The present paper generalizes Schöbi's dissection to an n-piece dissection of $\mathcal{Q}_n(w)$ into a prism $c\mathcal{Q}_{n-1}(\frac{1}{n-1}) \times I$, where $c = \sqrt{(n-1)(w+1)/n}$. Iterating this process leads to a dissection of $\mathcal{Q}_n(w)$ into an n-dimensional rectangular parallelepiped (or "brick") using at most $n!$ pieces. The complexity of computing the map from $\mathcal{Q}_n(w)$ to the brick is $O(n^2)$. A second generalization of Schöbi's dissection is given which applies specifically in $\mathbb{R}^4$. The results have applications to source coding and to constant-weight binary codes.

Keywords: dissections, Hill tetrahedra, Schöbi, polytopes, Voronoi cell, source coding, constant-weight codes

2000 Mathematics Subject Classification: 52B45 (94A29, 94B60)

1. Introduction

We define $\mathcal{Q}_n(w)$ (where $n \geq 1$ and $-1 < w < \frac{1}{n-1}$) as in the Abstract, and let $\mathcal{O}_n := \mathcal{Q}_n(0)$, $\mathcal{P}_n := \mathcal{Q}_n(\frac{1}{n})$. Hadwiger [19] showed in 1951 (see also Hertel [20]) that $\mathcal{Q}_n(w)$ is equidissectable with a cube for all n. His proof is indirect and not constructive. The simplex $\mathcal{O}_n$ is especially interesting. The vectors v_i are now orthogonal, and the simplex has vertices

$$000\ldots00,\ 100\ldots00,\ 110\ldots00,\ 111\ldots00,\ \ldots,\ 111\ldots10,\ 111\ldots11\,. \qquad (1)$$

It is an *orthoscheme* in Coxeter's terminology [9]. Because of applications to encoding and decoding constant-weight codes [38], we are interested in algorithms that carry out the dissection of $\mathcal{O}_n$ in an efficient manner. In fact our question is slightly easier than the classical problem, because we only need to decompose $\mathcal{O}_n$ into pieces which can be reassembled to form a rectangular parallelepiped (or n-dimensional "brick"), not necessarily a cube[1].

For the case $n = 3$, Hill [21] had already shown in 1895 that the tetrahedra $\mathcal{Q}_3(w)$ are equidissectable with a cube. It appears that that the first explicit dissection of $\mathcal{O}_3$ into a cube was given by Sydler [37] in 1956. Sydler shows that $\mathcal{O}_3$ may be cut into four pieces which can be reassembled to form a prism with base an isosceles right triangle. One further cut then gives a brick. Sydler's dissection can be seen in a number of references (Boltianskii [4, p. 99], Cromwell [10, p. 47], Frederickson [15, Fig. 20.4], Sydler [37], Wells [39, p. 251]) and we will not reproduce it here. Some of these references incorrectly attribute Sydler's dissection to Hill.

In our earlier paper [38], we gave a dissection of $\mathcal{O}_n$ to a prism $\mathcal{O}_{n-1} \times I$ for all n that requires $(n^2 - n + 2)/2$ pieces. In three dimensions this uses four pieces, the same number as Sydler's, but is somewhat simpler than Sydler's in that all our cuts are made along planes perpendicular to coordinate axes. By iterating this construction we eventually obtain a dissection of $\mathcal{O}_n$ into an n-dimensional brick. The total number of pieces in the overall dissection is large (roughly $(n!)^2/2^n$), but the complexity of computing the coordinates of a point in the final brick, given a initial point in $\mathcal{O}_n$, is only $O(n^2)$.

In 1985, Schöbi [32][2] gave a dissection of $\mathcal{Q}_3(w)$ (where $-1 < w < \frac{1}{2}$; our w is Schöbi's $-\cos\omega$) into a prism with base an equilateral triangle that uses only three pieces (see Figs. 5, 6 below, also Frederickson [15, Fig. 20.5]). There is a way to cut $\mathcal{Q}_n(w)$ for any n into n pieces that is a natural generalization of Schöbi's dissection, but for a long time we were convinced that already for $n = 4$ these pieces could not be reassembled to form a prism $P \times I$ for any $(n-1)$-dimensional polytope P. In fact, we were wrong, and the main goal of this paper is to use the "Two Tile Theorem"

[1] For the problems of dissecting a rectangle into a square and a three-dimensional rectangular parallelepiped into a cube see Boltianskii [4, p. 52], Cohn [6], Frederickson [15, Page 236].

[2] According to Frederickson [15, Page 234], this construction was independently found by Anton Hanegraaf, unpublished.

(Theorem 1) to generalize Schöbi's dissection to all dimensions. We will show in Theorem 2 that $\mathcal{Q}_n(w)$ can be cut into n pieces that can be reassembled to form a prism $c\mathcal{P}_{n-1} \times I_\ell$, where $c = \sqrt{(n-1)(w+1)/n}$ and $\ell = \sqrt{(1-w(n-1))/n}$. The cross-section is always proportional to $\mathcal{P}_{n-1} = \mathcal{Q}_{n-1}(\frac{1}{n-1})$, independently of w.

By iterating this dissection we eventually decompose $\mathcal{Q}_n(w)$ (and in particular $\mathcal{O}_n$) into a brick. The total number of pieces is at most $n!$ and the complexity of computing the map from $\mathcal{Q}_n(w)$ to the brick is $O(n^2)$ (Theorem 3). Although this is the same order of complexity as the algorithm given in our earlier paper [38], the present algorithm is simpler and the number of pieces is much smaller.

The recreational literature on dissections consists mostly of *ad hoc* constructions, although there are a few general techniques, which can be found in the books of Lindgren [22] and Frederickson [15], [16]. The construction we have found the most useful is based on group theory. We call it the "Two Tile Theorem", and give our version of it in Section 2, together with several examples. In Section 3 we state and prove the main theorem, and then in Section 4 we study the overall algorithm for dissecting $\mathcal{O}_n$ into a brick.

Before finding the general dissection mentioned above, we found a different generalization of Schöbi's dissection which applies specifically to the 4-dimensional case. This is described in Section 5. It is of interest because it is partially (and in a loose sense) a "hinged" dissection (cf. Frederickson [16]). After two cuts have been made, the first two motions each leave a two-dimensional face fixed. We then make a third cut, giving a total of six pieces which can reassembled to give a prism $c\mathcal{P}_3 \times I$. This construction is also of interest because it is symmetrical, and it is the only *ad hoc* dissection we know of in four dimensions (the dissections found by Paterson [30] are all based on a version of the Two Tile Theorem).

A note about applications. If we have a dissection of a polytope P into a brick $I_{\ell_1} \times I_{\ell_2} \times \cdots \times I_{\ell_n}$, then we have a natural way to encode the points of P into n-tuples of real numbers. This bijection provides a useful parameterization of the points of P. It may be used for source coding, if we have a source that produces points uniformly distributed over P (for example, P might be the Voronoi cell of a lattice). Another application is in simulation: Stoyan et al. [36, pp. 29–30] mention three methods for generating points that are uniformly distributed over irregular regions. The dissection method provides a fourth solution for regions are equidissectable with a brick. For the application to constant-weight codes we refer the reader to [38]. (The dissection method thus provides a possible solution to Research Problem (17.3) in [25].)

Notation. A polytope in $\mathbb{R}^n$ is a union of a finite number of finite n-dimensional simplices. It need be neither convex nor connected. Let $P, P_1, \ldots, P_k$ be polytopes in $\mathbb{R}^n$. By $P = P_1 + \cdots + P_k$ we mean that the interiors of $P_1, \ldots, P_k$ are pairwise disjoint and $P = P_1 \cup \ldots \cup P_k$. Let Γ be a group of isometries of $\mathbb{R}^n$. Two polytopes P, Q in $\mathbb{R}^n$ are said to be Γ-*equidissectable* if there are polytopes $P_1, \ldots, P_k$, $Q_1, \ldots, Q_k$ for some integer $k \geq 1$ such that $P = P_1 + \ldots + P_k$, $Q = Q_1 + \ldots + Q_k$, and $P_1^{g_1} = Q_1, \ldots, P_k^{g_k} = Q_k$ for appropriate elements $g_1, \ldots, g_k \in \Gamma$. In case Γ is the full isometry group of $\mathbb{R}^n$ we write $P \sim Q$ and say that P and Q are *equidissectable*. Isometries may involve reflections: we do not insist that the dissections can be carried out using only transformations of determinant $+1$. I_ℓ denotes an interval of length ℓ, I is a finite interval of unspecified length, and $I_\infty = \mathbb{R}^1$.

For background information about dissections and Hilbert's third problem, and any undefined terms, we refer the reader to the excellent surveys by Boltianskii [4], Dupont [13], Frederickson [15], [16], Lindgren [22], McMullen [27], McMullen and Schneider [28], Sah [31] and Yandell [40]. It is worth remarking that the simplices $\mathcal{Q}_n(w)$ have another important property: they are 'space-fillers', prototiles for monohedral tilings of $\mathbb{R}^n$—see for example Debrunner [12].

2. The "Two Tile Theorem"

Let $A \subset \mathbb{R}^n$ be a polytope, Γ a group of isometries of $\mathbb{R}^n$ and Ω a subset of $\mathbb{R}^n$. If the images of A under the action of Γ have disjoint interiors, and $\Omega = \cup_{g \in \Gamma} A^g$, we say that A is a Γ-*tile* for Ω. This implies that Γ is discontinuous and fixed-point-free.

Versions of the following theorem—although not the exact version that we need—have been given by Aguiló, Fiol and Fiol [1, Lemma 2.2], Müller [29, Theorem 3] and Paterson [30]. It is a more precise version of the technique of "superposing tesselations" used by Macaulay [23], [24], Lindgren [22, Chap. 2] and Frederickson [15, p. 29], [16, Chap. 3].

Theorem 1. *If for some set $\Omega \subset \mathbb{R}^n$ and some group Γ of isometries of $\mathbb{R}^n$, two n-dimensional polytopes A and B are both Γ-tiles for Ω, then A and B are Γ-equidissectable.*

Proof. We have

$$A = A \cap \Omega = A \cap \bigcup_{g \in \Gamma} B^g = \bigcup_{g \in \Gamma} A \cap B^g ,$$

where only finitely many of the intersections $A \cap B^g$ are nonempty. The set of nonempty pieces $\{A \cap B^g \mid g \in \Gamma\}$ therefore gives a dissection of A, and by symmetry the set of nonempty pieces $\{A^g \cap B \mid g \in \Gamma\}$ gives a dissection of B. But $(A \cap B^g)^{g^{-1}} = A^{g^{-1}} \cap B$, so the two sets of pieces are the same modulo isometries in Γ. ▪

We give four examples; the main application will be given in the next section.

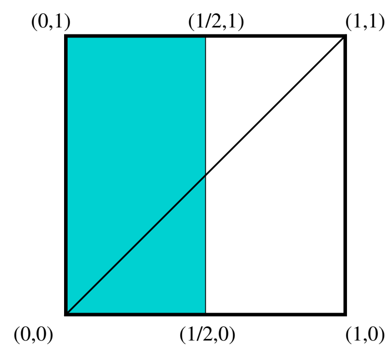

Figure 1: Illustrating the Two Tile Theorem: A is the triangle $(0,0),(1,0),(1,1)$, B (shaded) is the rectangle $(0,0),(\frac{1}{2},0),(\frac{1}{2},1),(0,1)$, Ω is the square $(0,0),(1,0),(1,1),(0,1)$ and Γ is generated by $\phi : (x,y) \mapsto (1-x,1-y)$.

Example 1. Let $A = \mathcal{O}_2$, the right triangle with vertices $(0,0),(1,0),(1,1)$, let B be the rectangle with vertices $(0,0),(\frac{1}{2},0),(\frac{1}{2},1),(0,1)$ and let ϕ be the map $(x,y) \mapsto (1-x,1-y)$. Let Γ be the group of order 2 generated by ϕ and let Ω be the square $(0,0),(1,0),(1,1),(0,1)$. Then A and B are both Γ-tiles for Ω. It follows from Theorem 1 that A and B are equidissectable (see Fig. 1). Alternatively, we could take the origin to be at the center of the square, and then the theorem applies with $\phi := (x,y) \mapsto (-x,-y)$.

Example 2. Again we take $A = \mathcal{O}_2$ to be the right triangle with vertices $(0,0),(1,0),(1,1)$, but now we take ϕ to be the map $(x,y) \mapsto (y+1,x)$. Note that ϕ involves a reflection. As mentioned in §1, this is permitted by our dissection rules. Let Γ be the infinite cyclic group generated by ϕ, and let Ω be the infinite strip defined by $x \geq y \geq x-1$. Then A is a Γ-tile for Ω (see Fig. 2). For B, the second tile, we take the square with vertices $(0,0),(\frac{1}{2},-\frac{1}{2}),(1,0),(\frac{1}{2},\frac{1}{2})$. This is also

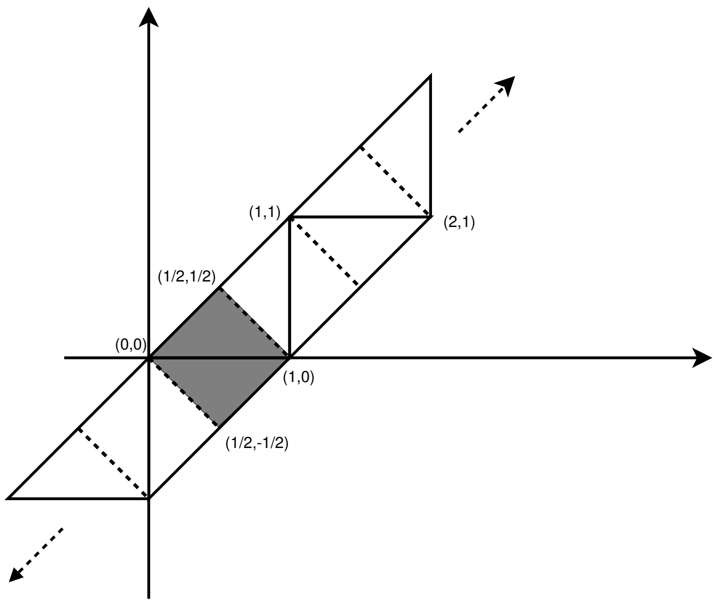

Figure 2: Another illustration of the Two Tile Theorem: A is the triangle $(0,0),(1,0),(1,1)$, B (shaded) is the square $(0,0),(\frac{1}{2},-\frac{1}{2}),(1,0),(\frac{1}{2},\frac{1}{2})$ and Ω is the strip $x \geq y \geq x-1$.

a Γ-tile for Ω, and so A and B are equidissectable. The two nonempty pieces are the triangles $A \cap B$ and $A \cap B^{\phi}$. The latter is mapped by ϕ^{-1} to the triangle with vertices $(0,0),(\frac{1}{2},-\frac{1}{2}),(1,0)$. This is a special case of the dissection given in Theorem 2. Of course in this case the dissection could also have been accomplished without using reflections.

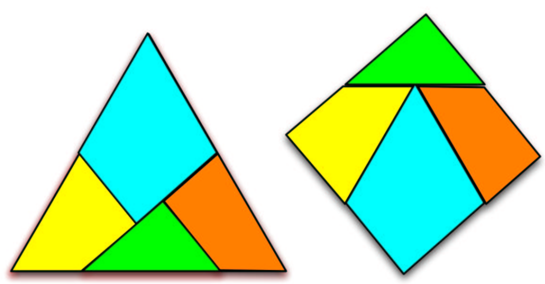

Figure 3: Four-piece dissection of an equilateral triangle to a square, usually attributed to Dudeney (1902)

Example 3. One of the most elegant of all dissections is the well-known four-piece dissection of an equilateral triangle to a square, shown in Fig. 3. This was published in 1902 by Dudeney, although Frederickson [15, Page 136], [16, pp. 8–10, 55–56, 90–92, 131–133, 175–176, 226–227] presents a convincing case that C. W. McElroy was the original discoverer. This dissection is described in many other references ([2], [5, Chap. 1], [11], [14, §5.5.1], [17], [18, Chap. 3], [34], [39, p. 61]), although not always correctly ([33], [35]: see the discussion in [11]). The usual construction of this dissection is by superimposing two strips, a technique that Lindgren [22, Fig. 5.2] calls a TT-dissection. The literature on dissections does not appear to contain a precise statement of conditions which guarantee that the TT construction produces a dissection. Such a theorem can be obtained as a corollary of the Two Tile Theorem, and will be published elsewhere, together with rigorous versions of other strip dissections. Both Gardner [18, Chap. 3] and Eves [14, §5.5.1] mention that L. V. Lyons extended Dudeney's dissection to cut the whole plane into a "mosaic of interlocking squares and equilateral triangles," and Eves shows this "mosaic" in his Fig. 5.5b (Fig. 4 below shows essentially the same figure, with the addition of labels for certain points). We will use this "mosaic," which is really a double tiling of the plane, to give an alternative proof that the dissection is correct from the Two Tile Theorem. Following Lyons, we first use the dissection to construct the double tiling. We then ignore how the double tiling was obtained, and apply the Two Tile Theorem to give an immediate certificate of proof for Dudeney's dissection. The double tiling also has some interesting properties that are not apparent from Eves's figure.

Let $\Omega = \mathbb{R}^2$, and take the first tile to be an equilateral triangle with edge length 1, area $c_1 := \frac{\sqrt{3}}{4}$ and vertices $A := (-1/4, -c_1)$, $B := -A$ and $C := (3/4, -c_1)$ (see Fig. 4), with the origin O at the midpoint of AB. The second tile is a square with edge length $c_2 := \sqrt{c_1}$. The existence of the dissection imposes many constraints, such as $|JB| = |JC| = |HI| = 1/2$, $|OD| = |OG| = c_2/2$, $2|LG| + |GK| = 2|JK| + |GK| = c_2$, etc., and after some calculation we find that the square should have vertices $D := (-c_1/2, c_3/2)$, $E := (c_3 - c_1/2, c_3/2 + c_1)$, $F := (c_3 + c_1/2, -c_3/2 + c_1)$ and $G := -D$, where $c_3 = c_2\sqrt{1 - c_1}$.

We now construct a strip of squares that replicates the square $DEFG$ in the southwest/northeast direction, and a strip of triangles replicating ABC (with alternate triangles inverted) in the horizontal direction. In order to fill the plane with copies of these strips, we must determine the offset of one strip of squares with respect to the next strip of squares, and of one strip of triangles with respect to the next strip of triangles. This implies the further constraints that $P - O = L - H = K - E$, etc., and in particular that P should be the point $(1 - 2c_3, -2c_1)$. Other significant points are $H := -L := (c_3 - 1/2, c_1)$, $I := (c_3, c_1)$, $J := (1/2, 0)$,

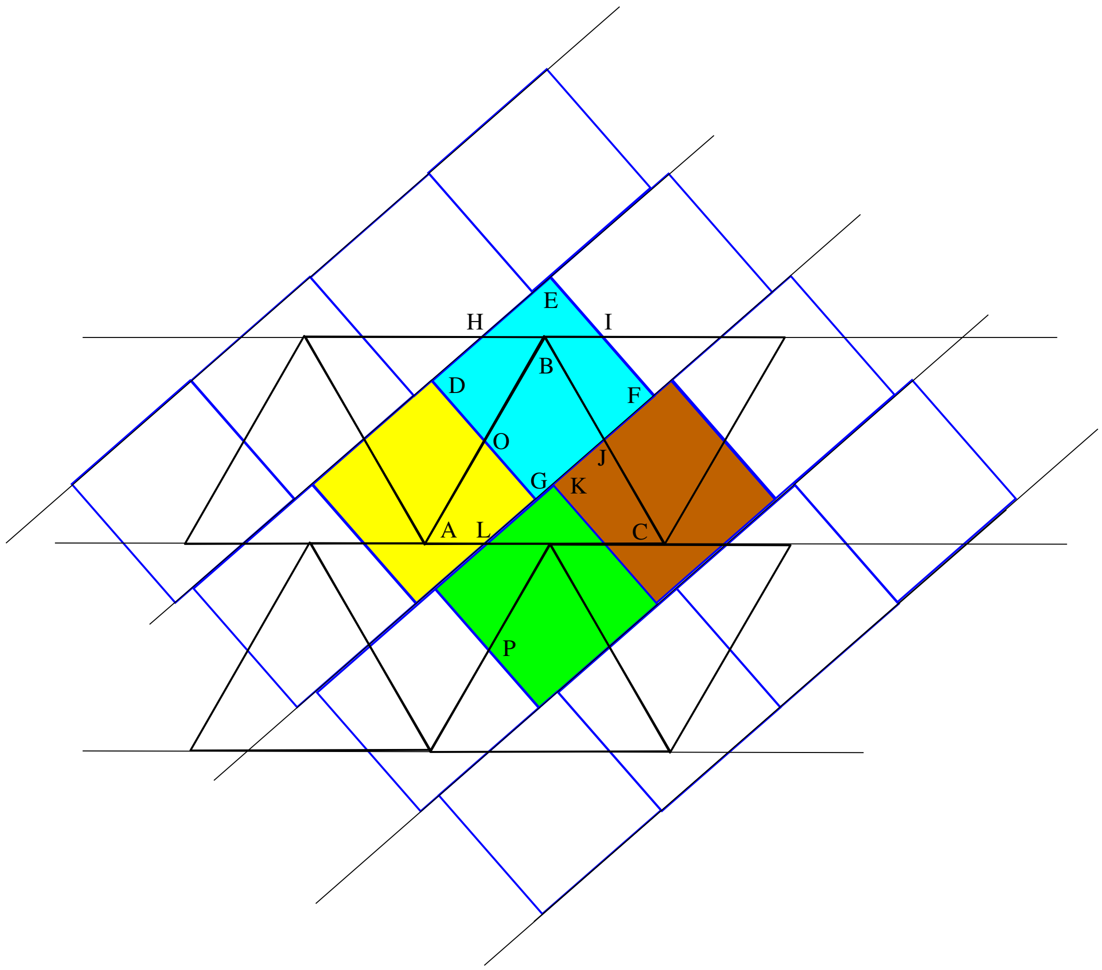

Figure 4: Lyons's "mosaic," a double tiling of the plane by triangles and squares.

$K := (1 - c_3 - c_1/2, c_3/2 - c_1)$. The angle CLG is $\arctan(c_1/c_3) = 41.15\ldots$ degrees.

We now have the desired double tiling of the plane. Both the triangle ABC and the square $DEFG$ are Γ-tiles for the whole plane, where Γ is the group (of type $p2$ in the classical notation, or type 2222 in the orbifold notation) generated by translation by $(1, 0)$, translation by $OP = (1 - 2c_3, -2c_1) = (0.009015\ldots, -0.8660\ldots)$, and multiplication by -1. We now ignore how this tiling was found, and deduce from Theorem 1 that Dudeney's dissection exists. The four pieces are $OBJG$, $ODHB$, HEI and $BIFJ$.

It is interesting that the horizontal strips of triangles do not line up exactly:[3] each strip is shifted to the left of the one below it by $1-2c_3 = 0.009015\ldots$. The group is correspondingly more

[3]This is responsible for the incorrect descriptions in the literature.

complicated than one might have expected from looking at Fig. 4, since the second generator for the group is not *quite* translation by $(0, -\sqrt{3}/2)$! There is an associated lattice, generated by the vectors OH and OJ, and containing the points I and L, which is *nearly* rectangular, the angle between the generators being $89.04\ldots$ degrees.

Incidentally, although Lindgren [22, p. 25] refers to this dissection as "minimal", we have never seen a proof that a three-piece dissection of an equilateral triangle to a square is impossible. This appears to be an open question.

Example 4. It is easy to show by induction that any lattice Λ in $\mathbb{R}^n$ has a brick-shaped fundamental region. The theorem then provides a dissection of the Voronoi cell of Λ into a brick. For example, the Voronoi cell of the root lattice D_n is described in [7, Chap. 21]. By applying the theorem, we obtain a dissection of the Voronoi cell into a brick that uses $2n$ pieces. For $n = 3$ this gives the well-known six-piece dissection of a rhombic dodecahedron into a $2 \times 1 \times 1$ brick (cf. [15, pp. 18, 242]).

3. The main theorem

We begin by choosing a particular realization of the simplex $\mathcal{Q}_n(w)$. Define the following vectors in $\mathbb{R}^n$:

$$v_1 := (a, b, b, \ldots, b),\ v_2 := (b, a, b, \ldots, b),\ v_3 := (b, b, a, \ldots, b),\ \ldots,\ v_n := (b, b, b, \ldots, a), \qquad (2)$$

where

$$\begin{aligned} b &:= (\sqrt{1 - w(n-1)} - \sqrt{1+w})/n\,, \\ a &:= b + \sqrt{1+w}\,. \end{aligned} \qquad (3)$$

Then $v_i \cdot v_i = 1$, $v_i \cdot v_j = -w$ for $i \neq j$, $i, j = 1, \ldots, n$. We take the convex hull of the vectors $0, v_1, v_1 + v_2, \ldots, v_1 + \cdots + v_n$, that is, the zero vector together with the rows of

$$\left[\begin{array}{ccccc} a & b & b & \ldots & b \\ a+b & a+b & 2b & \ldots & 2b \\ a+2b & a+2b & a+2b & \ldots & 3b \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a+(n-1)b & a+(n-1)b & a+(n-1)b & \ldots & a+(n-1)b \end{array} \right], \qquad (4)$$

to be our standard version of $\mathcal{Q}_n(w)$. This simplex has volume

$$(1+w)^{(n-1)/2}(1-w(n-1))^{1/2}/n!\,. \tag{5}$$

Setting $w=0, a=1, b=0$ gives our standard version of $\mathcal{O}_n$, as in (1), when the v_i are mutually perpendicular, and setting $w=1/n, a=n^{-3/2}(1+(n-1)\sqrt{n+1}), b=n^{-3/2}(1-\sqrt{n+1})$ gives our standard version of $\mathcal{P}_n$.

Two other versions of $\mathcal{P}_n$ will also appear. Let $p_i := 1/\sqrt{i(i+1)}$, and construct a $n \times n$ orthogonal matrix M_n as follows. For $i=1,\ldots,n-1$ the ith column of M_n has entries p_i (i times), $-ip_i$ (once) and 0 ($n-i-1$ times), and the entries in the last column are all $1/\sqrt{n}$. (The last column is in the $(1,1,\ldots,1)$ direction and the other columns are perpendicular to it.) For example,

$$M_3 := \left[\begin{array}{ccc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{array}\right].$$

The other two versions of $\mathcal{P}_n$ are: the convex hull of the zero vector in $\mathbb{R}^n$ together with the rows of

$$\sqrt{\frac{n+1}{n}}\left[\begin{array}{ccccc} p_1 & p_2 & p_3 & \ldots & p_n \\ 0 & 2p_2 & 2p_3 & \ldots & 2p_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & dp_n \end{array}\right], \tag{6}$$

and the convex hull of the zero vector in $\mathbb{R}^{n+1}$ together with the rows of

$$\sqrt{\frac{n+1}{n}}\left[\begin{array}{ccccc} \frac{n}{n+1} & -\frac{1}{n+1} & -\frac{1}{n+1} & \ldots & -\frac{1}{n+1} \\ \frac{n-1}{n+1} & \frac{n-1}{n+1} & -\frac{2}{n+1} & \ldots & -\frac{2}{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \ldots & -\frac{n}{n+1} \end{array}\right]. \tag{7}$$

To see that both of these simplices are congruent to the standard version of $\mathcal{P}_n$, note that multiplying (7) on the right by M_{n+1} produces (6) supplemented by a column of zeros, and then multiplying (6) on the right by $M_n^{\rm tr}$ (where tr denotes transpose) produces the standard version.

Remark. If we ignore for the moment the scale factor in front of (7), we see that its rows are the coset representatives for the root lattice A_n in its dual A_n^* [7, p. 109]. In other words, the rows of (7) contain one representative of each of the classes of vertices of the Voronoi cell for A_n. $\mathcal{P}_2$ is an equilateral triangle and $\mathcal{P}_3$ is a “Scottish tetrahedron” in the terminology of Conway and Torquato [8].

We can now state our main theorem.

Theorem 2. *The simplex $\mathcal{Q}_n(w)$ is equidissectable with the prism $c\mathcal{P}_{n-1} \times I_\ell$, where $c := \sqrt{(n-1)(w+1)/n}$ and $\ell := \sqrt{(1-w(n-1))/n}$.*

Proof. Let Ω be the convex hull of the points $\{u_i \in \mathbb{R}^n \mid i \in \mathbb{Z}\}$, where $u_0 := (0,0,\ldots,0)$, $u_i := u_0^{\phi^i}$, ϕ is the map

$$\phi : (x_1, \ldots, x_n) \mapsto (x_n + a, x_1 + b, x_2 + b, \ldots, x_{n-1} + b)$$

and a, b are as in (3) (see Table 1).

Table 1: Points defining the infinite prism Ω. The convex hull of any $n+1$ successive rows is a copy of $\mathcal{Q}_n(w)$.

$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
u_{-1}	$=$	$-b$	$-b$	$-b$	$\ldots$	$-a$
u_0	$=$	0	0	0	$\ldots$	0
u_1	$=$	a	b	b	$\ldots$	b
u_2	$=$	$a+b$	$a+b$	$2b$	$\ldots$	$2b$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
u_n	$=$	$a+(n-1)b$	$a+(n-1)b$	$a+(n-1)b$	$\ldots$	$a+(n-1)b$
u_{n+1}	$=$	$2a+(n-1)b$	$a+nb$	$a+nb$	$\ldots$	$a+nb$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

We now argue in several easily verifiable steps.

(i) For any $i \in \mathbb{Z}$, the convex hull of $u_i, u_{i+1}, \ldots, u_{i+n}$ is a copy of $\mathcal{Q}_n(w)$, $\mathcal{Q}^{(i)}$ (say), with $\mathcal{Q}^{(i)} \subset \Omega$ and $(\mathcal{Q}^{(i)})^\phi = \mathcal{Q}^{(i+1)}$.

(ii) The simplices $\mathcal{Q}^{(i)}$ and $\mathcal{Q}^{(i+1)}$ share a common face, the convex hull of $u_{i+1}, \ldots, u_{i+n}$, but have disjoint interiors. More generally, for all $i \neq j$, $\mathcal{Q}^{(i)}$ and $\mathcal{Q}^{(j)}$ have disjoint interiors.

(iii) The points of Ω satisfy

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq x_1 - (a-b)\,. \tag{8}$$

(This is true for $\mathcal{Q}^{(0)}$ and the property is preserved by the action of ϕ.) The inequalities (8) define an infinite prism with axis in the $(1,1,\ldots,1)$ direction. We will show that every point in the prism belongs to Ω, so Ω is in fact equal to this prism.

(iv) The projection of $\mathcal{Q}^{(0)}$ onto the hyperplane perpendicular to the $(1, 1, \ldots, 1)$ direction is congruent to $c\mathcal{P}_{n-1}$, where $c := \sqrt{(n-1)(w+1)/n}$. (For multiplying (4) on the right by M_n gives a scaled copy of (6).) On the other hand, the intersection of the prism defined by (8) with the hyperplane $\sum_{i=1}^{n} x_i = 0$ consists of the points $(0, 0, \ldots, 0)$, $\sqrt{w+1}((n-1)/n, -1/n, \ldots, -1/n)$, $\sqrt{w+1}((n-2)/n, (n-2)/n, -2/n, \ldots, -2/n)$, $\ldots$, and—compare (7)—is also congruent to $c\mathcal{P}_{n-1}$. Since the projection and the intersection have the same volume, it follows that every point in the prism is also in Ω. (For consider a long but finite segment of the prism. The total volume of the copies of $\mathcal{Q}_n(w)$ is determined by the projection, and the volume of the prism is determined by the cross-section, and these coincide.) We have therefore established that Ω is the infinite prism

$$c\mathcal{P}_{n-1} \times I_\infty$$

with walls given by (8). Furthermore, $\mathcal{Q}_n(w)$ is a Γ-tile for Ω, where Γ is the infinite cyclic group generated by ϕ.

(v) For a second tile, we take the prism

$$B := c\mathcal{P}_{n-1} \times I_\ell \,,$$

where $\ell := \sqrt{(1 - w(n-1))/n}$. The length ℓ is chosen so that B has the same volume as $\mathcal{Q}_n(w)$ (see (5)). We take the base of B to be the particular copy of $c\mathcal{P}_{n-1}$ given by the intersection of Ω with the hyperplane $\sum_{i=1}^{n} x_i = 0$, as in (iv). The top of B is found by adding $\ell/\sqrt{n}$ to every component of the base vectors. To show that B is also a Γ-tile for Ω, we check that the image of the base of B under ϕ coincides with the top of B. This is an easy verification.

Since $\mathcal{Q}_n(w)$ and B are both Γ-tiles for Ω, the desired result follows from Theorem 1. ■

Remarks. (i) The prism B consists of the portion of the infinite prism Ω bounded by the hyperplanes $\sum x_i = 0$ and $\sum x_i = \sqrt{1 - w(n-1)}$. The "apex" of $\mathcal{Q}_w(n)$ is the point $(a + (n-1)b, a + (n-1)b, \ldots, a + (n-1)b)$, which—since $a + (n-1)b = \sqrt{1 - w(n-1)}$—lies on the hyperplane $\sum x_i = n\sqrt{1 - w(n-1)}$. There are therefore n pieces $\mathcal{Q}_n(w) \cap B^{\phi^k}$ $(k = 0, 1, \ldots, n-1)$ in the dissection, obtained by cutting $\mathcal{Q}_w(n)$ along the hyperplanes $\sum x_i = k\sqrt{1 - w(n-1)}$ for $k = 1, \ldots, n-1$. To reassemble them to form B, we apply ϕ^{-k} to the kth piece.

(ii) The case $n = 2$, $w = 0$ of the theorem was illustrated in Fig. 2. In the case $n = 3$, $-1 < w < \frac{1}{2}$, the three pieces are exactly the same as those in Schöbi's dissection [32]. However, it is interesting that we reassemble them in a different way to form the same prism $c\mathcal{P}_2 \times I_\ell$, with

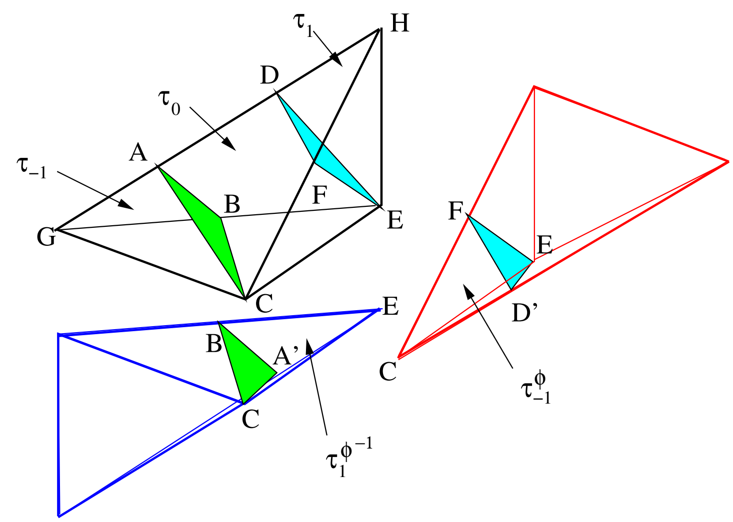

Figure 5: Exploded view showing three adjacent copies of $\mathcal{Q}_3(w)$ and their intersections with the two cutting planes.

$c := \sqrt{2(w+1)/3}$, $\ell := \sqrt{(1-2w)/3}$. First we describe our dissection, which is illustrated in Fig. 5. The figure shows an exploded view of three adjacent copies of $\mathcal{Q}_3(w)$, namely $\mathcal{Q}_3(w)^{\phi^{-1}}$ (the lower left tetrahedron), $\mathcal{Q}_3(w)$ (the upper left tetrahedron) and $\mathcal{Q}_3(w)^{\phi}$ (the tetrahedron on the right), and their intersections with the two cutting planes. The three pieces in the dissection can be seen in the upper left tetrahedron $\mathcal{Q}_3(w)$. They are $\tau_{-1} = \mathcal{Q}_3(w) \cap B^{\phi^{-1}}$ (the piece $ABCG$ on the left of this tetrahedron), $\tau_0 = \mathcal{Q}_3(w) \cap B$ (the central piece $ABCDEF$) and $\tau_1 = \mathcal{Q}_3(w) \cap B^{\phi}$ (the piece $DEFH$ on the right). In Fig. 5 we can also see an exploded view of these three pieces reassembled to form the triangular prism: $\tau_1^{\phi^{-1}}$ is the right-hand piece $A'BCE$ of the lower figure and τ_{-1}^{ϕ} is the left-hand piece $CD'EF$ of the figure on the right. The fully assembled prism is shown in Fig. 6: the tetrahedron $A'BCE$ is $\tau_1^{\phi^{-1}}$ and the tetrahedron $CD'EF$ is τ_{-1}^{ϕ}.

On the other hand, Schöbi reassembles the same pieces by rotating τ_1 about the edge EF (which acts as a hinge), sending D to D' and giving the tetrahedron $CD'EF$, and rotating τ_{-1} about the hinge BC, sending A to A' and giving the tetrahedron $A'BCE$. This is strictly different from our construction, since ϕ has no fixed points. The pieces are the same and the end result is the same, but the two outer pieces τ_1 and τ_{-1} have been interchanged!

(iii) By repeated application of Theorem 2 we can dissect $\mathcal{Q}_n(w)$ into an n-dimensional brick.

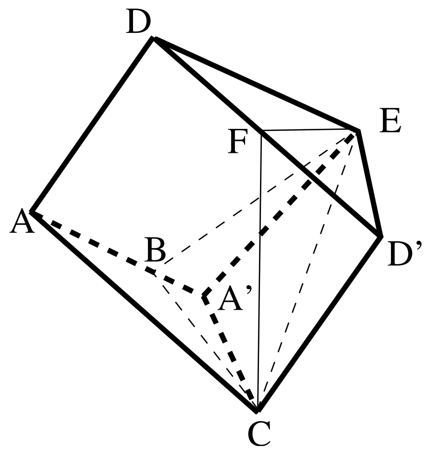

Figure 6: Three-piece dissection of the triangular prism. Our construction and Schöbi's use the same three pieces but assemble them in a different way. In this view the points A' and B are at the back of the figure.

Each of the n pieces from the first stage is cut into at most $n-1$ pieces at the second stage, and so on, so the total number of pieces in the final dissection is at most $n!$. (It could be less, if a piece from one stage is not intersected by all of the cutting planes at the next stage. It seems difficult to determine the exact number of pieces.)

4. Dissecting $\mathcal{O}_n$ into a brick

In this section we discuss in more detail the recursive dissection of $\mathcal{Q}_n(w)$ into a rectangular parallelepiped or "brick" in the case of greatest interest to us, when we start with perpendicular vectors $v_1, \ldots, v_n$ and the simplex is the orthoplex $\mathcal{O}_n = \mathcal{Q}_n(0)$.

From Theorem 2 we have

$$\begin{aligned} \mathcal{O}_n &\sim \sqrt{\frac{n-1}{n}}\mathcal{P}_{n-1} \times I_{\frac{1}{\sqrt{n}}}\,, \\ \mathcal{P}_n &\sim \frac{\sqrt{n^2-1}}{n}\mathcal{P}_{n-1} \times I_{\frac{1}{n}}\,, \end{aligned} \tag{9}$$

and so (since $\mathcal{P}_1 = I_1$)

$$\mathcal{O}_n \sim \frac{1}{\sqrt{2}} I_1 \times I_{p_2} \times I_{p_3} \times \cdots \times I_{p_{n-1}} \times I_{\frac{1}{\sqrt{n}}},\ \text{for } n > 1\,. \tag{10}$$

The right-hand side of (10) is our final brick; we will denote it by Π (for $n = 1$ we set $\Pi = I_1$). Note that $\mathrm{vol}(\mathcal{O}_n) = \mathrm{vol}(\Pi) = 1/n!$.

Let Θ denote the map from $\mathcal{O}_n$ to Π associated with the dissection (10). We will show that given $x := (x_1, \ldots, x_n) \in \mathcal{O}_n$, $(y_1, \ldots, y_n) := \Theta(x) \in \Pi$ can be computed in $O(n^2)$ steps.

The algorithm for computing Θ breaks up naturally into two parts. The first step involves dissecting $\mathcal{O}_n$ into n pieces and reassembling them to form the prism

$$B := \sqrt{\frac{n-1}{n}}\, \mathcal{P}_{n-1} \times I_{\frac{1}{\sqrt{n}}} .$$

All later steps start with a point in $\lambda_k \mathcal{P}_k$ for $k = n-1, n-2, \ldots, 2$ and certain constants λ_k, and produce a point in $\lambda_{k-1}\mathcal{P}_{k-1} \times I$.

For the first step we must determine which of the pieces $\mathcal{O}_n \cap B^{\phi_1^r}$ $(r = 0, 1, \ldots, n-1)$ x belongs to, where ϕ_1 is the map $(x_1, \ldots, x_n) \mapsto (x_n + 1, x_1, x_2, \ldots, x_{n-1})$. This is given by $r := \lfloor \sum_{i=1}^n x_i \rfloor$, and then mapping x to $x' := x^{\phi_1^{-r}}$ corresponds to reassembling the pieces to form B. However, x' is expressed in terms of the original coordinates for $\mathcal{O}_n$ and we must multiply it by M_n to get coordinates perpendicular to the $(1, 1, \ldots, 1)$ direction, getting $x'' := (x''_1, \ldots, x''_{n-1}, y_n) = x' M_n$. The final component of x'' is the projection of x' in the $(1, 1, \ldots, 1)$ direction. The other components of x'', $(x''_1, \ldots, x''_{n-1})$ define a point in $\sqrt{\frac{n-1}{n}}\, \mathcal{P}_{n-1}$, but expressed in coordinates of the form shown in (6), and before we proceed to the next stage, we must reexpress this in the standard coordinates for $\sqrt{\frac{n-1}{n}}\, \mathcal{P}_{n-1}$, which we do by multiplying it by M_{n-1}^{tr} (see the beginning of §3), getting x'''.

The following pair of observations shorten these calculations. First, y_n can be computed directly once we know r, since each application of ϕ_1^{-1} subtracts 1 from the sum of the coordinates. If $s := \sum_{i=1}^n x_i$, then $r := \lfloor s \rfloor$ and $y_n = (s - r)/\sqrt{n}$. Second, the product of M_n-with-its-last-column-deleted and M_{n-1}^{tr} is the $n \times (n-1)$ matrix

$$N_n := \begin{bmatrix} 1 - p_n & -p_n & \ldots & -p_n \\ -p_n & 1 - p_n & \ldots & -p_n \\ \ldots & \ldots & \ldots & \ldots \\ -p_n & -p_n & \ldots & 1 - p_n \\ -\frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n}} & \ldots & -\frac{1}{\sqrt{n}} \end{bmatrix} .$$

Multiplication by N_n requires only $O(n)$ steps.

The first stage in the computation of Θ can therefore be summarized as follows:

Step A. Given $x := (x_1, \ldots, x_n) \in \mathcal{O}_n$. Let $s := \sum_{i=1}^{n} x_i$, $r := \lfloor s \rfloor$.

Compute $x' := x^{\phi_1^{-r}}$.

Pass $x''' := x' N_n$ to the next stage, and output $y_n :=$ (fractional part of $s)/\sqrt{n}$.

In all the remaining steps we start with a point x in $\lambda_k \mathcal{P}_k$ for some constant λ_k, where $k = n-1, n-2, \ldots, 2$. Instead of ϕ_1 we use the map $\phi_2 : (x_1, \ldots, x_k) \mapsto (x_k + a, x_1 + b, x_2 + b, \ldots, x_{k-1} + b)$, where $a = k^{-3/2}(1 + (k-1)\sqrt{k+1}), b = k^{-3/2}(1 - \sqrt{k+1})$. Each application of ϕ_2^{-1} subtracts $1/\sqrt{k}$ from the sum of the coordinates. We omit the remaining details and just give the summary of this step (for simplicity we ignore the constant λ_k):

Step B_k. Given $x := (x_1, \ldots, x_k) \in \mathcal{P}_k$. Let $s := \sum_{i=1}^{k} x_i$, $r := \lfloor \sqrt{k} s \rfloor$.

Compute $x' := x^{\phi_2^{-r}}$.

Pass $x''' := x' N_k$ to the next stage, and output $y_k :=$ (fractional part of $\sqrt{k} s)/k$.

Since the number of computations needed at each step is linear, we conclude that:

Theorem 3. *Given $x \in \mathcal{O}_n$, $\Theta(x) \in \Pi$ can be computed in $O(n^2)$ steps.*

Remarks. The inverse map Θ^{-1} is just as easy to compute, since each of the individual steps is easily reversed. Two details are worth mentioning. When inverting step B_k, given x''' and y_k, we obtain x' by multiplying x''' by N_k^{tr} and adding $y_k/\sqrt{k}$ to each component. For the computation of r, it can be shown (we omit the proof) that for inverting step A, to go from x' to x, r should be taken to be the number of strictly negative components in x'. For step B_k, r is the number of indices i such that

$$x'_i \; < \; b\sqrt{k} \sum_{j=1}^{k} x'_j \, .$$

5. An alternative dissection of $\mathcal{O}_4$

In this section we give a six-piece dissection of $\tau := \mathcal{O}_4$ into a prism $\sqrt{\frac{3}{4}} \mathcal{P}_3 \times I_{\frac{1}{2}}$. Although it requires two more pieces than the dissection of Theorem 2, it still only uses three cuts. It also has an appealing symmetry.

We start by subtracting $\frac{1}{2}$ from the coordinates in (1), in order to move the origin to the center of τ. That is, we take τ to be the convex hull of the points $A := (-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$,

$B := (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $C := (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$, $D := (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, $E := (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (see Fig. 7). We use (w, x, y, z) for coordinates in $\mathbb{R}^4$. Note that τ is fixed by the symmetry $(w, x, y, z) \mapsto (-z, -y, -x, -w)$.

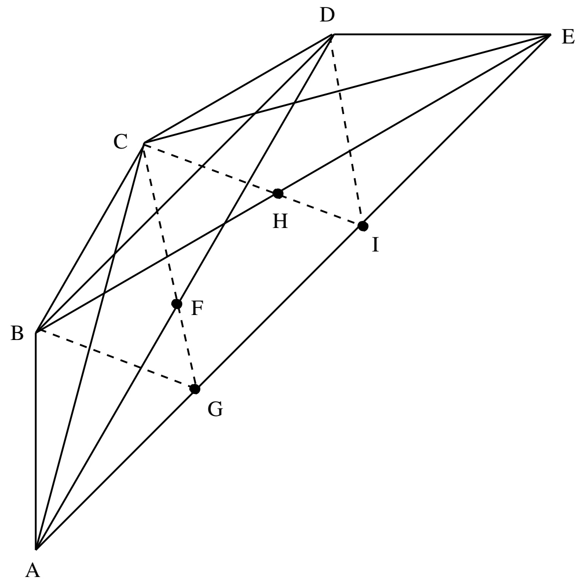

Figure 7: $\tau := \mathcal{O}_4$ is the convex hull of A, B, C, D, E; the first two cuts are made along the hyperplanes containing B, C, F, G and C, D, H, I, respectively.

We make two initial cuts, along the hyperplanes $w + y + z = -\frac{1}{2}$ and $w + x + z = \frac{1}{2}$. The first cut intersects the edges of τ at the points B, C, $F := (0, 0, 0, -\frac{1}{2})$ and $G := (-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$; the second at the points C, D, $H := (\frac{1}{2}, 0, 0, 0)$ and $I := (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. The three pieces resulting from these cuts will be denoted by τ_1 (containing A), τ_2 (the central piece), and τ_3 (containing E).

We apply the transformation $\alpha := (w, x, y, z) \mapsto (-y, -x, -w, -1 - z)$ to τ_1 and $\beta := (w, x, y, z) \mapsto (1 - w, -z, -y, -x)$ to τ_3. α fixes the triangle BCF, although not pointwise, and similarly β fixes the triangle CDH, and so these transformations may be regarded as hinged, in a loose sense of that word[4]. This is what led us to this dissection—we were attempting to generalize Schöbi's hinged three-dimensional dissection.

After applying α and β, the resulting polytope $\tau_4 := \tau_1^\alpha + \tau_2 + \tau_3^\beta$ is a convex body with seven

[4]Since a hinged rod in the plane has a fixed point, and a hinged door in three dimensions has a fixed one-dimensional subspace, a hinged transformation in four dimensions should, strictly speaking, have a two-dimensional region that is *pointwise* fixed.

vertices and six faces. (This and other assertions in this section were verified with the help of the programs Qhull [3] and MATLAB [26].) The seven vertices are $B, C, D, G, I,\ J := (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{5}{6})$ and $K := (\frac{5}{6}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$, which are shown schematically in Fig. 8. This figure is realistic in so far as it suggests that the edges BK, GI and JD are equal and parallel (in fact, $K-B = I-G = D-J = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$).

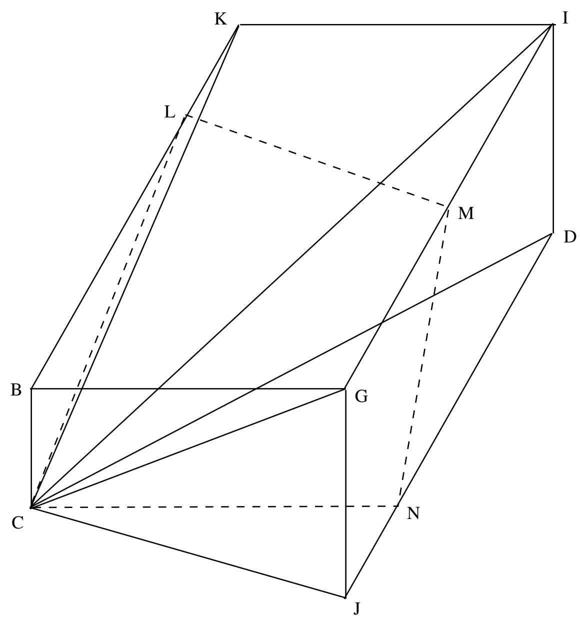

Figure 8: After the first two motions, we have a polytope τ_4 with seven vertices B, G, J, C, K, I, D. The third cut is along the hyperplane containing C, L, M, N.

We now make one further cut, along the hyperplane $w + x + y + z = 0$, which separates τ_4 into two pieces τ_5 (containing B) and τ_6 (containing K). This hyperplane meets the edge BK at the point $L := (\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$, GI at the point $M := (0, 0, 0, 0)$, and JD at the point $N := (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4})$. The point L is three-quarters of the way along BK, M bisects GI, and N is one-quarter of the way along JD,

The final motion is to apply $\gamma := (w, x, y, z) \mapsto (x, y, z, w-1)$ to τ_6, and to form $\tau_7 := \tau_5 + \tau_6^\gamma$. The convex hull of τ_7 involves three new points, the images of L, M and N under γ, namely $P := (-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$, $Q := (0, 0, 0, -1)$ and $R := (\frac{1}{4}, \frac{1}{4}, -\frac{3}{4}, -\frac{3}{4})$, respectively. Then τ_7 is the convex hull of the eight points C, L, M, N and R, B, P, Q, and it may be verified that the first four and the last four of these points define copies of $\sqrt{\frac{3}{4}}\mathcal{P}_3$, and that τ_7 is indeed congruent to

$\sqrt{\frac{3}{4}}\mathcal{P}_3 \times I_{\frac{1}{2}}$, as claimed.

We end with a question: can this construction be generalized to higher dimensions?

Acknowledgments. We thank G. N. Frederickson for helpful comments and discussions, and especially for drawing our attention to reference [14]. We also thank the referees for some helpful suggestions.

References

[1] F. Aguiló, M. A. Fiol and M. L. Fiol, Periodic tilings as a dissection method, *Amer. Math. Monthly*, **107** (2000), 341–352.

[2] J. Akiyama and G. Nakamura, Dudeney dissection of polygons, in J. Akiyama, M. Kano and M. Urabe, eds., *Discrete and Computational Geometry (Tokyo, 1998)*, *Lecture Notes in Comput. Sci.*, **1763** (2000), 14–29.

[3] C. B. Barber, D. P. Dobkin and H. T. Huhdanpaa, The Quickhull algorithm for convex hulls, *ACM Trans. Mathematical Software*, **22** (1996), 469–483. Available from http://www.qhull.org.

[4] V. G. Boltianskii, *Hilbert's Third Problem*, Translated from the Russian by R. A. Silverman, Wiley, NY, 1978.

[5] S. T. Coffin, *The Puzzling World of Polyhedral Dissections*, Oxford Univ. Press, 1961.

[6] M. J. Cohn, Economical triangle-square dissection, *Geometriae Dedicata*, **3** (1974–1975), 447–467.

[7] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, Springer, NY, 1988; 3rd. ed., 1998.

[8] J. H. Conway and S. Torquato, Packing, tiling, and covering with tetrahedra, *Proc. Natl. Acad. Sci. USA*, **103** (2006), 10612–10617.

[9] H. S. M. Coxeter, *Regular Polytopes*, 3rd. ed., Dover, NY, 1973.

[10] P. R. Cromwell, *Polyhedra*, Cambridge Univ. Press, 1997.

[11] D. W. Crowe and I. J. Schoenberg, On the equidecomposability of a regular triangle and a square of equal areas, *Mitt. Math. Sem. Giessen*, No. 164 (1984), 59–64.

[12] H. E. Debrunner, Tiling Euclidean d-space with congruent simplexes, in J. E. Goodman et al., eds., *Discrete Geometry and Convexity (New York, 1982)*, *Ann. New York Acad. Sci.*, **440** (1985), 230–261.

[13] J. L. Dupont, *Scissors Congruences, Group Homology and Characteristic Classes* , World Scientific, Singapore, 2001.

[14] H. Eves, *A Survey of Geometry*, Allyn and Bacon, Boston, MA, Vol. 1, 1966.

[15] G. N. Frederickson, *Dissections: Plane and Fancy*, Cambridge Univ. Press, 1997.

[16] G. N. Frederickson, *Hinged Dissections: Swinging & Twisting*, Cambridge Univ. Press, 2002.

[17] J.-P. Friedelmeyer, Dissections et puzzles, *Pour la science*, Dossier 59, April/June, 2008.

[18] M. Gardner, *The Second* Scientific American *Book of Mathematical Puzzles and Diversions*, Simon and Schuster, NY, 1961.

[19] H. Hadwiger, Hillsche Hypertetraeder, *Gazeta Matemática (Lisboa)*, **12** (No. 50, 1951), 47–48.

[20] E. Hertel, *Verallgemeinerte Hadwiger-Hill Simplexe*, Reports on Algebra and Geometry, Mathematisches Institut, Friedrich-Schiller-Universität Jena, Jena, Germany, 2003.

[21] M. J. M. Hill, Determination of the volumes of certain species of tetrahedra without employment of the method of limits, *Proc. London Math. Soc.*, **27** (1895–1896), 39–53.

[22] H. Lindgren, *Geometric Dissections*, Van Nostrand, Princeton, NJ, 1964; revised edition with an appendix by G. N. Frederickson, Dover, NY, 1972.

[23] W. H. Macaulay, The dissection of rectilinear figures, *Math. Gazette*, **7** (1914), 381–388; **8** (1915), 72–76 and 109–115.

[24] W. H. Macaulay, The dissection of rectilinear figures, *Messenger Math.*, **48** (1919), 159–165; **49** (1919), 111–121; **52** (1922), 53–56.

[25] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam, 1977.

[26] *MATLAB User's Guide*, MathWorks, Inc., Natick, MA, 2007.

[27] P. McMullen, Valuations and dissections, in P. M. Gruber and J. M. Wills, eds., *Handbook of Convex Geometry*, North-Holland, Amsterdam, Vol. B, 1993, pp. 933–988.

[28] P. McMullen and R. Schneider, Valuations on convex bodies, in P. M. Gruber and J. M. Wills, eds., *Convexity and its Applications*, Birkhäuser, Basel, 1983, pp. 170–247.

[29] C. Müller, Equidecomposability of polyhedra with reference to crystallographic groups, *Discrete Comput. Geom.*, **3** (1988), 383–389.

[30] D. A. Paterson, Geometric dissections in 4-D, *J. Recreational Math.*, **28** (1996–1997), 22–37.

[31] C.-H. Sah, *Hilbert's Third Problem: Scissors Congruence*, Pitman, London, 1979.

[32] P. Schöbi, Ein elementarer und konstruktiver Beweis für die Zerlegungsgleichheit der Hill'schen Tetraeder mit einer Quader, *Elem. Math.*, **40** (1985), 85–97.

[33] I. J. Schoenberg, *Mathematical Time Exposures*, Math. Assoc. America, 1982.

[34] H. Steinhaus, *Mathematical Snapshots*, Stechert, NY, 1938.

[35] H. Steinhaus, *Mathematical Snapshots*, 3rd American Edition, Oxford, 1983.

[36] D. Stoyan, W. S. Kendall and J. Mecke, *Stochastic Geometry and Its Applications*, John Wiley & Sons, Chichester, 1987.

[37] J.-P. Sydler, Sur les tétraèdres équivalents à un cube, *Elem. Math.*, **11** (1956), 78–81.

[38] C. Tian, V. A. Vaishampayan and N. J. A. Sloane, Constant weight codes: a geometric approach based on dissections, Preprint, 2007.

[39] D. Wells, *The Penguin Dictionary of Curious and Interesting Geometry*, Penguin Books, London, 1991.

[40] B. H. Yandell, *The Honors Class: Hilbert's Problems and Their Solvers*, A. K. Peters, Natick, MA, 2002.