On Curling Numbers of Integer Sequences

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Based on joint work with Ben Chaffin, John Linderman and Allan Wilks (J. Integer Seqs., 2013)

Outline

- Curling numbers
- Curling number conjecture
- Gijswijt's sequence
- Sequences of 2's and 3's
- Enumeration of binary sequences by curling number
- Enum. of sequences of 2's and 3's by tail length
- Rotten sequences - do they exist?

The Curling Number Conjecture

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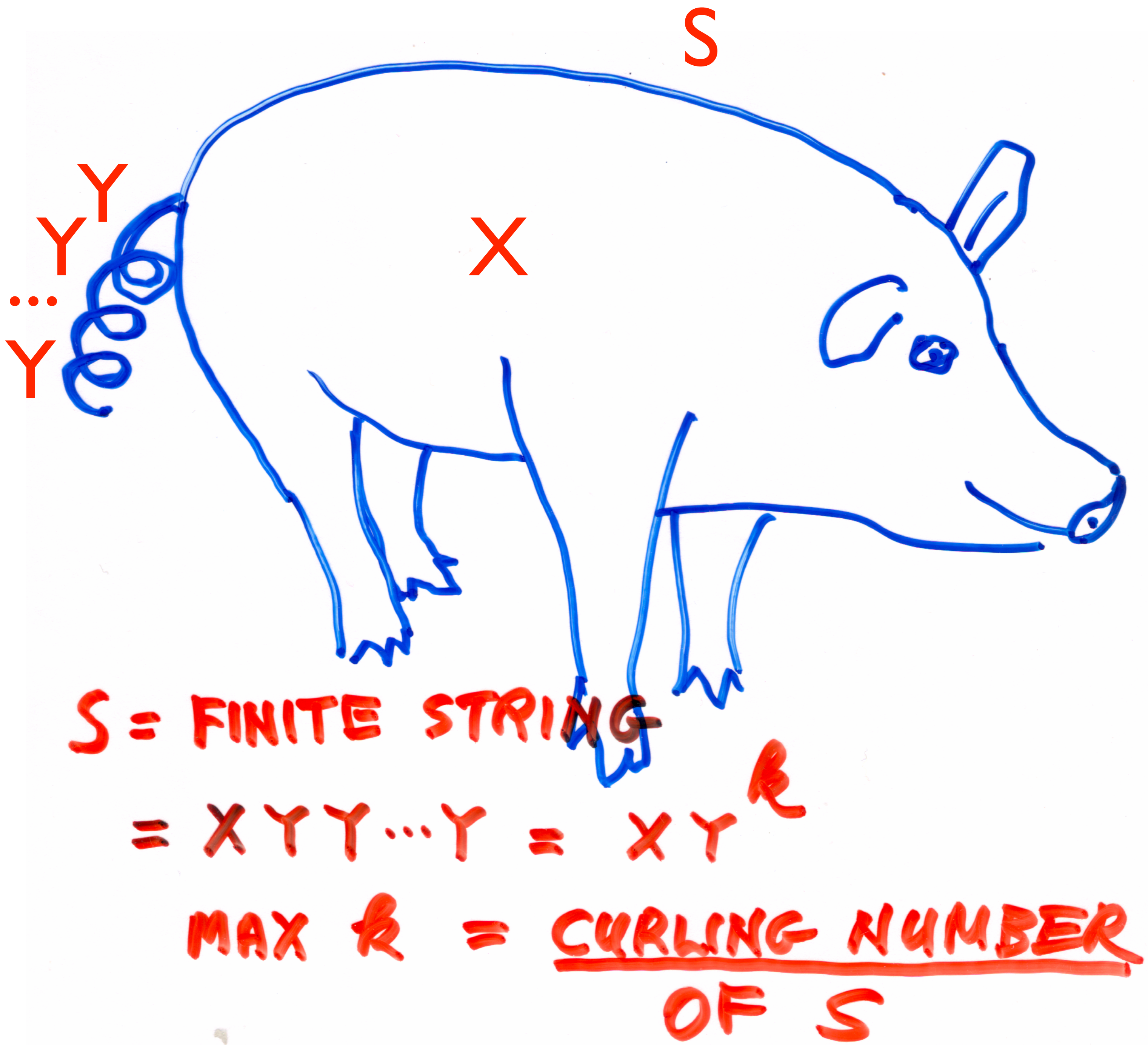
Definition of Curling Number



$S = 7\ 5\ 2\ 2\ 5\ 2\ 2\ 5\ 2\ 2,\quad k = 3$

$cn(S) = k = 3$

The Curling Number Conjecture (continued)

Use cn to define a recurrence:

a(n) = cn(a(0), a(1), ..., a(n-1))

The Conjecture:

1. Given any k initial terms, a(n)=1 for some n >= k.
2. Every sequence eventually joins Gijswijt's sequence G (A90822)

Example: Start with 2 2 2 3 2 2

This continues 2 3 2 2 2 3 3 2 1 1 2 1 1 2 ...

Gijswijt's Sequence G

Fokko v. d. Bult, Dion Gijswijt, John Linderman,
N. J. A. Sloane, Allan Wilks (J. Integer Seqs., 2007)

Start with 1, always append curling number

1 1 <u>2</u>

1 1 2 <u>2</u> <u>2</u> <u>3</u>

1 1 2

1 1 2 2 2 3 <u>2</u>

1 1 2

1 1 2 2 2 3

1 1 2

1 1 2 2 2 3 2 <u>2</u> <u>2</u> <u>3</u> <u>2</u> <u>2</u> <u>2</u> <u>3</u> <u>3</u> <u>2</u>

1 1 2

. . . . . .

$a(220) = 4$

(A090822)

Gijswijt, continued

Is there a 5?

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300,000 terms: no 5

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300,000 terms: no 5

$2 \cdot 10^6$ terms: no 5

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10^{120} terms: no 5

Gijswijt, continued

Is there a 5?

300,000 terms: no 5

$2 \cdot 10^6$ terms: no 5

10^{120} terms: no 5

NJAS, FvdB: first 5 at about term $10^{10^{23}}$

Gijswijt, continued

First n appears at about term

$$2^{2^{3^{4^{5^{\cdot^{\cdot^{\cdot^{n-1}}}}}}}}$$

A tower of height n-1 (conjectured)

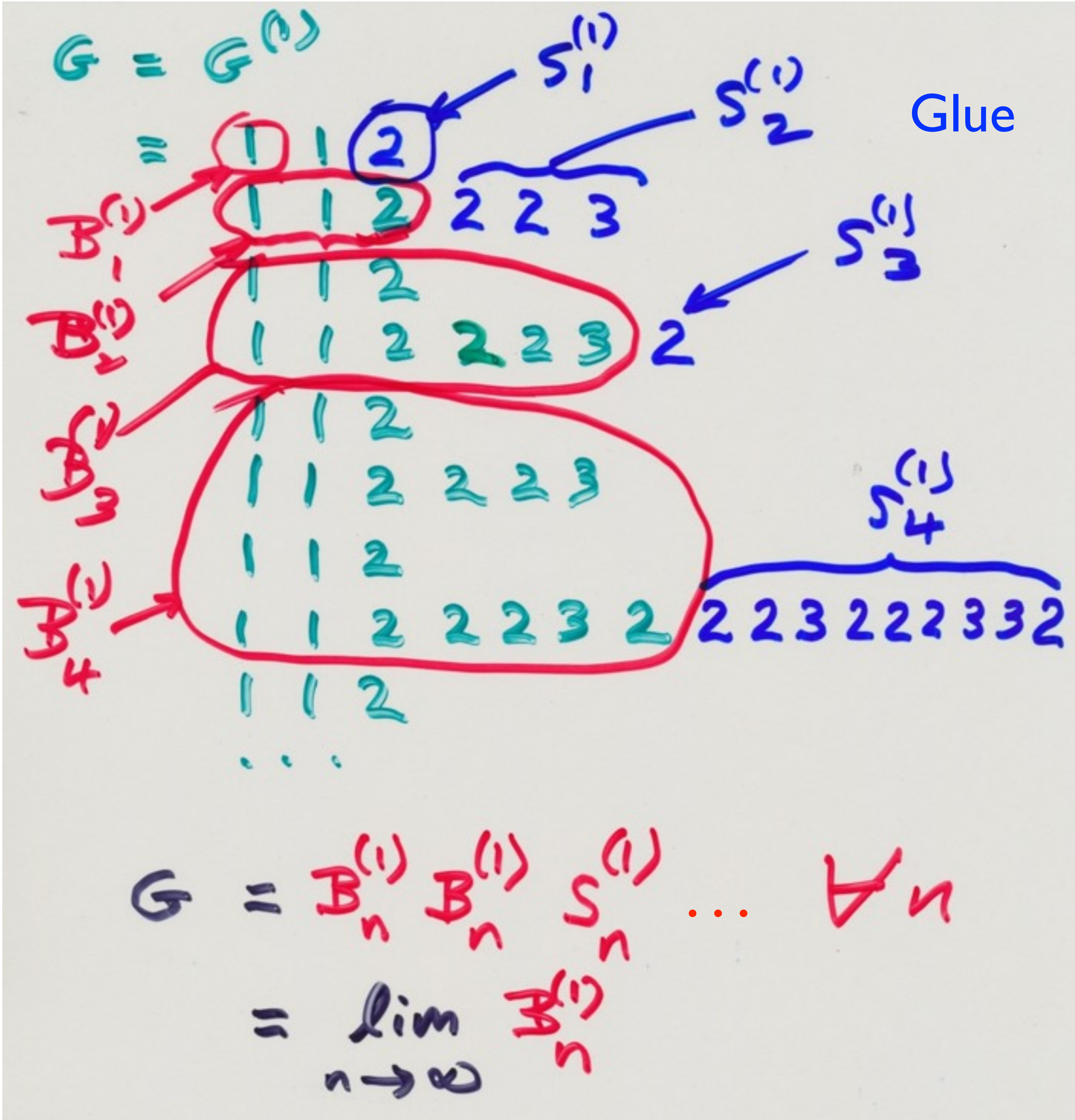
(F.v.d. Bult et al., J. Integer Sequences, 2007)

(A90822)

Gijswijt, continued

Blocks

Glue



Block : $B^{(1)}_{n+1} = B^{(1)}_n B^{(1)}_n S^{(1)}_n$

Glue : $S^{(1)}_n$ = longest run of 2's etc before the next 1.

Just need to understand $S^{(1)}_n$'s !

Define 2nd order Gijswijt

$$G^{(2)} := S^{(1)}_1 S^{(1)}_2 S^{(1)}_3 S^{(1)}_4 \dots$$

= 2 2 2 3 — $S^{(2)}_1$

B^2_1: 2 2 2 3

2 2 2 3 3 — S^2_2

B^2_2:

2 2 2 3

2 2 2 3

2 2 2 3 3

S^2_3

B^2_3:

2 2 2 3

2 2 2 3

2 2 2 3 3 3 4

...

$$G^{(2)} = B^{(2)}_n B^{(2)}_n B^{(2)}_n S^{(2)}_n \dots \ \forall n$$

However!

Define 2nd order curling number

$$cn^{(2)}(S) = \max\{2, cn(S)\}$$

Then $G^{(2)}$ = start with 2
append $cn^{(2)}$
repeat.

$\Rightarrow$ 2 2 2 (3)
2 2 2 3
2 2 2 3 (3) ...

$$\Rightarrow G^{(3)} = S_1^{(2)} S_2^{(2)} S_3^{(2)} S_4^{(2)} \ldots$$

$$= 3.3.334 \ldots$$

$$= B_n^{(3)} B_n^{(3)} B_n^{(3)} B_n^{(3)} S_n^{(3)} \ldots$$

AND SO ON!

LENGTHS OF $S_n^{(1)}, S_n^{(2)}, S_n^{(3)}, \dots$

n :	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	…
$\lvert S_n^{(1)} \rvert$:	1	3	1	9	4	24	1	3	1	9	4	67	1	3	1	9	…
$\lvert S_n^{(2)} \rvert$:	1	1	3	1	1	3	1	1	9	1	1	3	1	1	3	1	1 9 …

A 91579

…

Smooth first line: replace 4 by 3, 1; 9 by 8, 1; 25 by 24, 1; …

$\Rightarrow$ Ruler sequence

1, 3, 1, 8, 1, 3, 1, 24, 1, 3, 1, 8, 1, 3, 1, 67, …

$= \rho^{(1)}$ (2-adic valuation of n), where $\rho^{(1)} = 1, 3, 8, 24, 67, 195, \dots$

A 91588

……

5 in $G^{(2)}$ at $n = 77709404388415370160829246932345692180$.

5 in $G^{(1)}$ at $n = 10^{10^{23.09987\dots}}$

Sequences of 2’s and 3’s

Start: S = n 2’s and 3’s

Ω(n) = max extension (or tail) before 1 appears

2323.2223.1 Ω(4)=4

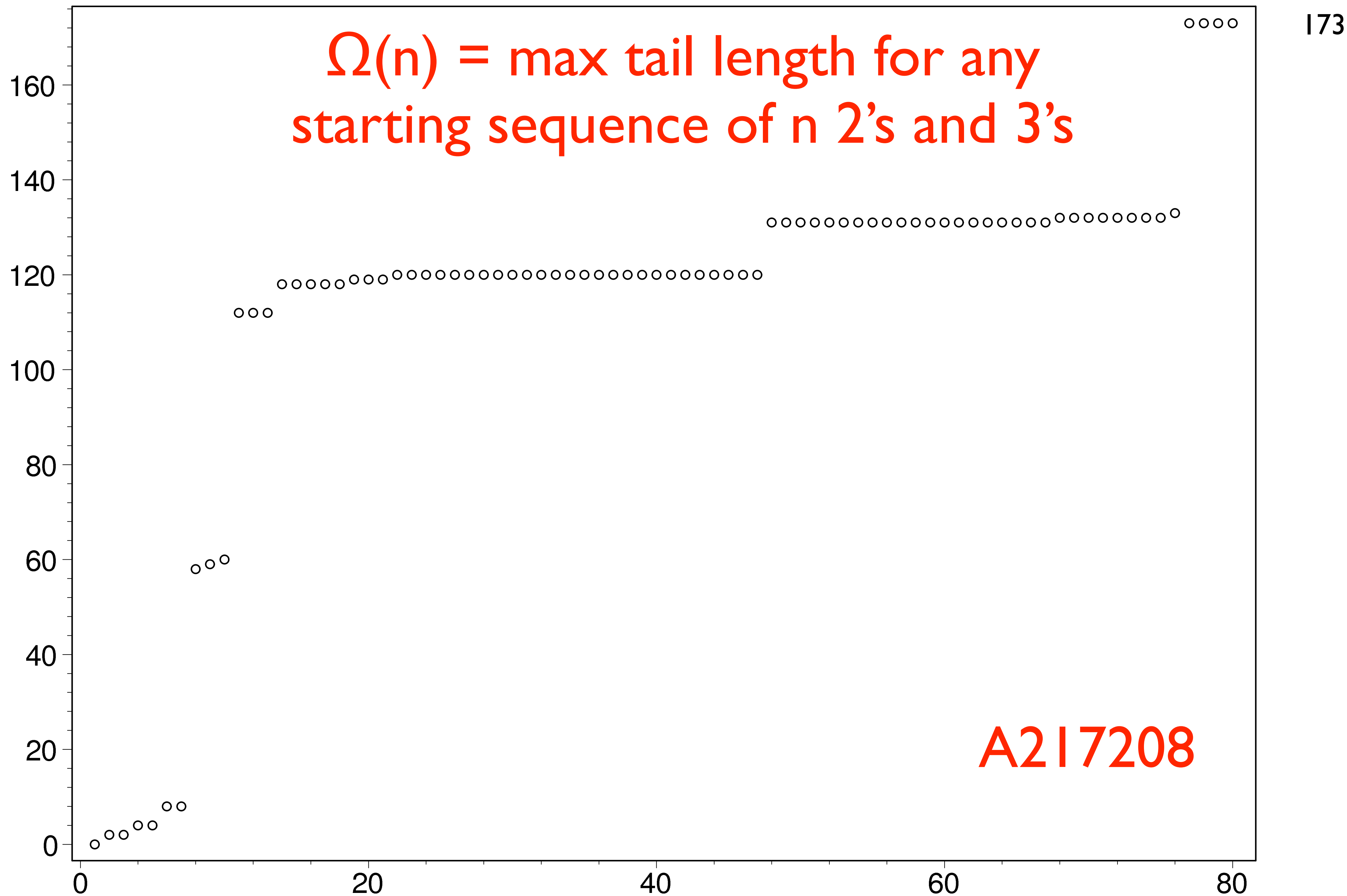
222322.23222332.1 Ω(6)=8

Know Ω(n) for n up to 48, conjectured for n up to 80

Lengths 22, 48 especially good! Ω(22)=120, Ω(48)=131.

Length 22: 23 223 223 23 2223 23 223 223

Explain! Generalize! More!



Ω(n) = max tail length for any starting sequence of n 2's and 3's
173
A217208
0
20
40
60
80
100
120
140
160

Properties of Good Starting Sequences

Sequences S which achieve $\Omega(n) > \Omega(n-1)$

- S is unique
- S begins with 2
- S does not contain 33
- S does not contain TTTT (including 2222)

True for n up to 48. Assumed true for n = 49 ... 80

Unavoidable Regularities ?

The problem: Start with S = n 2's and 3's.
Repeatedly extend using curling number.

Eventually must reach state where have:
- either no final repeat: not equal to XYY
- or equal to XYYYY

Shirshov's Theorem, Lyndon's Theorem ???

CURLING NUMBERS OF BINARY SEQUENCES

S	cn(S)
0000	4
0001	1
0010	1
0011	2
0100	2
0101	2
0110	1
0111	3

1	2	3	4
6	6	2	2

n\k	1	2	3	4	5	...
1	2					
2	2	2				
3	4	2	2			
4	6	6	2	2		
5	12	12	4	2	2	
6	20	26	10	...		
7	40	52	20	...		
8	74	110	38	...		
9	148	214	...			

↓

A122536

c(n, 1)

c(n, k)

LENGTH n

CURLING NUMBER k

A216955

(triangle for n < 105)

Notation

IF $cn(S) = k$,

$$S = X Y^k \quad \text{(in many ways)}$$

SHORTEST Y IS PRIMITIVE

$(\neq T^i,\ i > 1)$

AND UNIQUE

$|Y| =: \pi$ (PERIOD OF S)

$p(n, k)$ = # primitive seqs with $cn = k$

$$q(n, k) = \sum_{i \leq k} p(n, i)$$

Notation, continued

S with cn k is **ROBUST** if no proper suffix of S^{k+1} has cn $\geq k+1$.

Example $S = 32232$ (cn 1) NOT ROBUST:

$S^2 = \not{3}2232.32232$ (cn $= 2$)

$p'(n,k)$: # robust, primitive, cn k

Thm 1

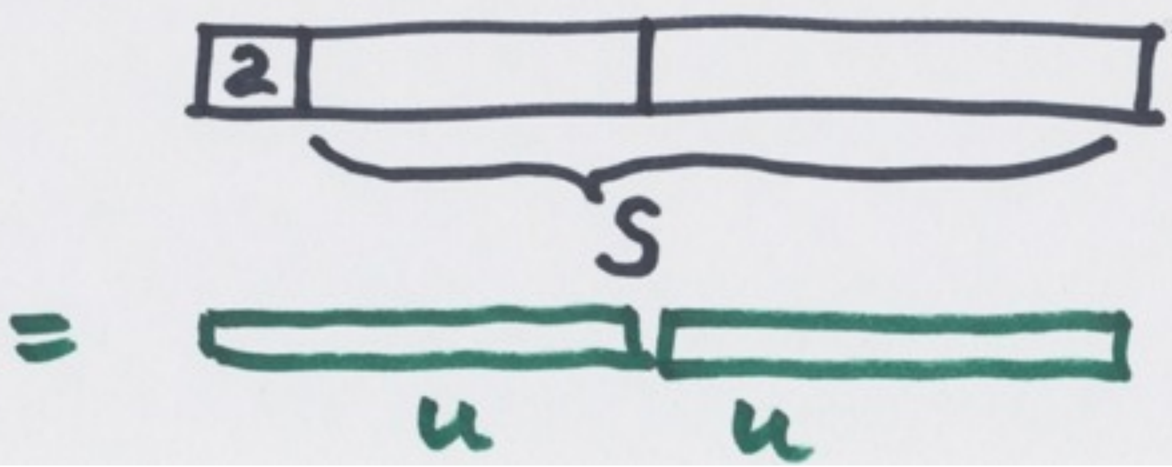
$$c(n,k) = 2\,c(n-1,k) + [k|n]\left\{p'\left(\tfrac{n}{k},k-1\right) + q\left(\tfrac{n}{k},k-2\right)\right\} - [k+1|n]\left\{p'\left(\tfrac{n}{k+1},k\right) + q\left(\tfrac{n}{k+1},k-1\right)\right\}.$$

In particular:

$$c(n,1) = 2\,c(n-1,1) - [2|n]\,p'\left(\tfrac{n}{2},1\right).$$

Proof: S even length, cn 1 ⇒ 2S cn 1

S odd length, cn 1:



⇒ u primitive & robust!

Allan Wilks : Structure of non-robust sequences of cn 1

$\Rightarrow c(n,1)$ for $n \leq 200$.

But no explicit formula.

Conjecture

$$\lim_{n \to \infty} \frac{c(n,1)}{2^n} = 0.27004433\ldots$$

TAIL LENGTHS OF 2,3 - SEQUENCES

		LENGTH
222	3 1	1
223	1	0
232	1	0
233	2 1	1
322	2 3 1	2
323	1	0
332	1	0
333	3 4 1	2

0	1	2
4	2	2

n\i	0	1	2	3	4	5	6	7	8	9–55	56	57	58	59
1	2													
2	2	1	1											
3	4	2	2											
4	6	5	3	1	1									
5	12	9	6	2	3									
6	20	18	12	6	7	0	0	0	1					
7	40	34	25	11	14	1	0	1	2					
8	74	71	47	24	28	1	3	2	3	00…0	0	2	1	

$T(n, i)$

LENGTH n

TAIL LENGTH i

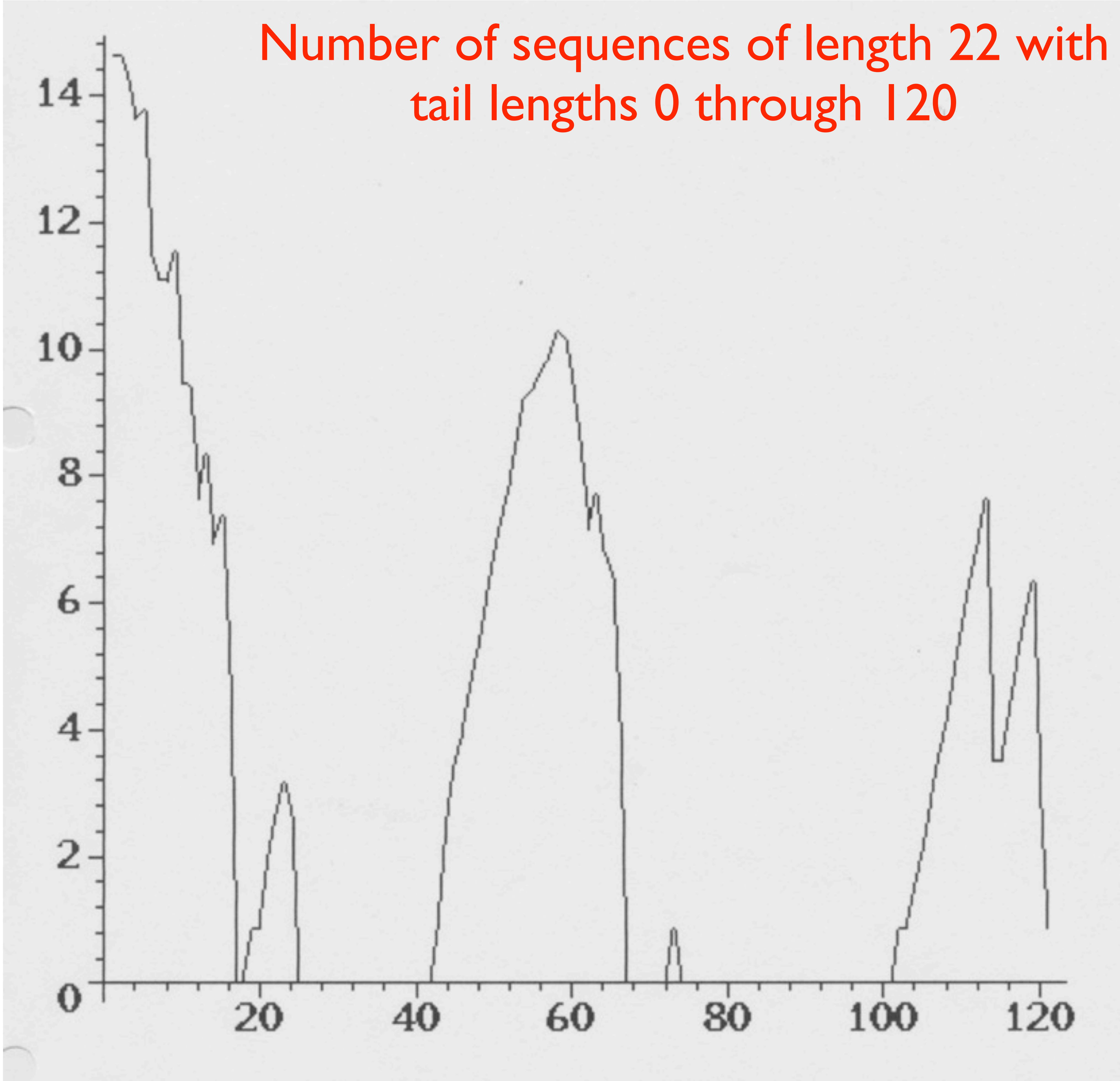
↓

A122536 again

A217209 (48 ROWS)

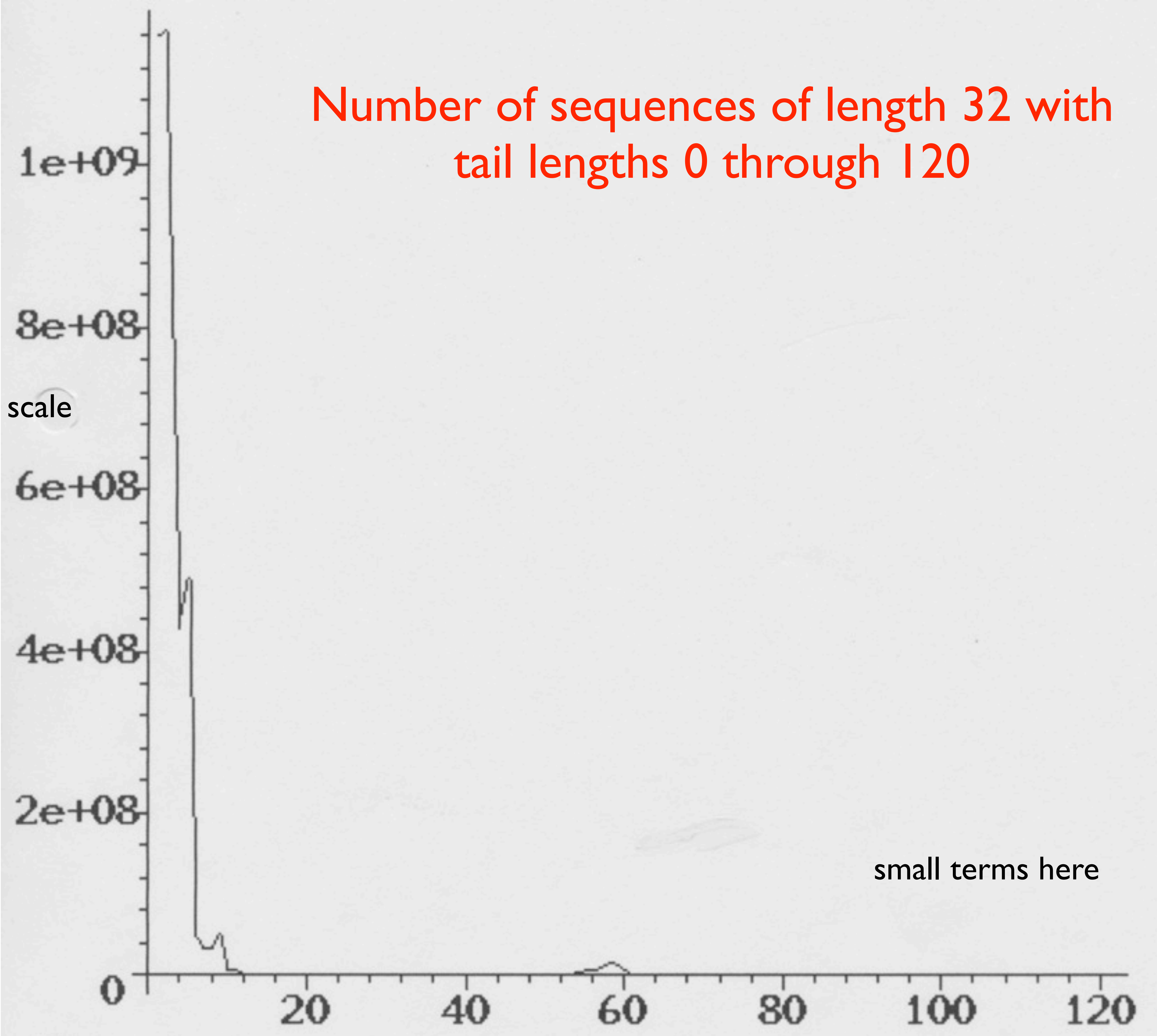
log scale!

(base e)



Number of sequences of length 32 with tail lengths 0 through 120

linear scale



DO NOT HAVE A GOOD MODEL FOR $T(n,i)$

A RANDOM n-TUPLE HAS Ω n | with probab.y

1	.270
2	.434
3	.162
≥ 4	.134

"2 or 3" : .596
HEADS

"1 or ≥ 4" : .404
TAILS

2^n experiments, max time to first tail

$$\approx \frac{n \log 2}{-\log .596} = 1.34\,n$$

MIGHT SUGGEST $\Omega(n) \approx 1.34\,n$

(Not valid of course)

DO DOUBLY ROTTEN SEQUENCES EXIST?

32323 IS ROTTEN

32323.23321 |TAIL| = 4

BUT 232323.321 |TAIL| = 2

S IS DOUBLY ROTTEN IF

|TAIL 2S| AND |TAIL 3S| BOTH < |TAIL S|

Conjecture: DO NOT EXIST

WOULD IMPLY $\Omega(n+1) \geq \Omega(n)$

(OPEN)

For more information, see

On Curling Numbers of Integer Sequence,

B. Chaffin, J. P. Linderman, N. J. A. Sloane, A. R. Wilks,

J. Integer Sequences, Vol. 16 (2013), #13.4.3.

Many related sequences are in the OEIS:

http://oeis.org

The OEIS needs more editors!

- contact me (njasloane@gmail.com)