The World of Blind Mathematicians

A visitor to the Paris apartment of the blind geometer Bernard Morin finds much to see. On the wall in the hallway is a poster showing a computer-generated picture, created by Morin's student François Apéry, of Boy's surface, an immersion of the projective plane in three dimensions. The surface plays a role in Morin's most famous work, his visualization of how to turn a sphere inside out. Although he cannot see the poster, Morin is happy to point out details in the picture that the visitor must not miss. Back in the living room, Morin grabs a chair, stands on it, and feels for a box on top of a set of shelves. He takes hold of the box and climbs off the chair safely—much to the relief of the visitor. Inside the box are clay models that Morin made in the 1960s and 1970s to depict shapes that occur in intermediate stages of his sphere eversion. The models were used to help a sighted colleague draw pictures on the blackboard. One, which fits in the palm of Morin's hand, is a model of Boy's surface. This model is not merely precise; its sturdy, elegant proportions make it a work of art. It is startling to consider that such a precise, symmetrical model was made by touch alone. The purpose is to communicate to the sighted what Bernard Morin sees so clearly in his mind's eye.

A sighted mathematician generally works by sitting around scribbling on paper: According to one legend, the maid of a famous mathematician, when asked what her employer did all day, reported that he wrote on pieces of paper, crumpled them up, and threw them into the wastebasket. So how do blind mathematicians work? They cannot rely on back-of-the-envelope calculations, half-baked thoughts scribbled on restaurant napkins, or hand-waving arguments in which "this" attaches "there" and "that" intersects "here". Still, in many ways, blind mathematicians work in much the same way as sighted mathematicians do. When asked how he juggles complicated formulas in his head without being able to resort to paper and pencil, Lawrence W. Baggett, a blind mathematician at the University of Colorado, remarked modestly, "Well, it's hard to do for anybody." On the other hand, there seem to be differences in how blind mathematicians perceive their subject. Morin recalled that, when a sighted colleague proofread Morin's thesis, the colleague had to do a long calculation involving determinants to check on a sign. The colleague asked Morin how he had computed the sign. Morin said he replied: "I don't know—by feeling the weight of the thing, by pondering it."

Blind Mathematicians in History

The history of mathematics includes a number of blind mathematicians. One of the greatest mathematicians ever, Leonhard Euler (1707–1783), was blind for the last seventeen years of his life. His eyesight problems began because of severe eyestrain that developed while he did cartographic work as director of the geography section of the St. Petersburg Academy of Science. He had trouble with his right eye starting when he was thirty-one years old, and he was almost entirely blind by age fifty-nine. Euler was one of the most prolific mathematicians of all time, having produced around 850 works. Amazingly, half of his output came after his blindness. He was aided by his prodigious memory and by the assistance he received from two of his sons and from other members of the St. Petersburg Academy.

The English mathematician Nicholas Saunderson (1682–1739) went blind in his first year, due to smallpox. He nevertheless was fluent in French, Greek, and Latin, and he studied mathematics. He was denied admission to Cambridge University and never earned an academic degree, but in 1728 King George II bestowed on Saunderson the Doctor of Laws degree. An adherent of Newtonian philosophy, Saunderson became the Lucasian Professor of Mathematics at Cambridge University, a position that Newton himself had held and that is now held by the physicist Stephen Hawking. Saunderson developed a method for performing arithmetic and algebraic calculations, which he called "palpable arithmetic". This method relied on a device that bears similarity to an abacus and also to a device called a "geoboard", which is in use nowadays in mathematics teaching. His method of palpable arithmetic is described in his textbook *Elements of Algebra* (1740). It is possible that Saunderson also worked in the area of probability theory: The historian of statistics Stephen Stigler has argued that the ideas of Bayesian statistics may actually have originated with Saunderson, rather than with Thomas Bayes [St].

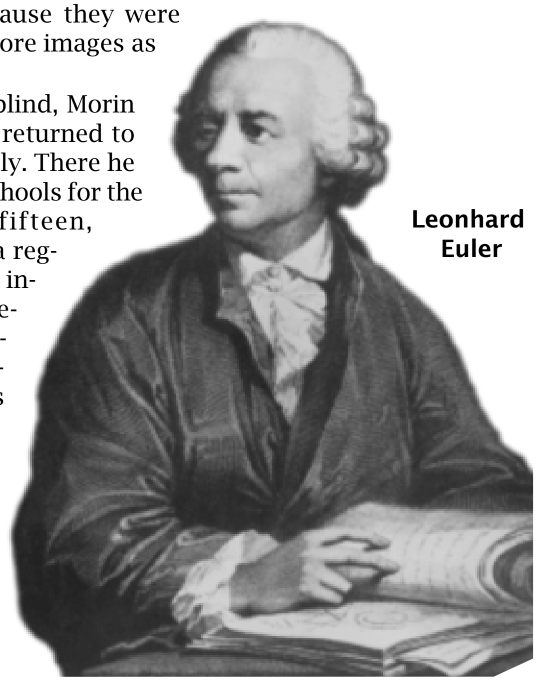
Several blind mathematicians have been Russian. The most famous of these is Lev Semenovich Pontryagin (1908–1988), who went blind at the age of fourteen as the result of an accident. His mother took responsibility for his education, and, despite her lack of mathematical training or knowledge, she could read scientific works aloud to her son. Together they fashioned ways of referring to the mathematical symbols she encountered. For example, the symbol for set intersection was "tails down", the symbol for subset was "tails right", and so forth. From the time he entered Moscow University in 1925 at age seventeen, Pontryagin's mathematical genius was apparent, and people were particularly struck by his ability to memorize complicated expressions without relying on notes. He became one of the outstanding members of the Moscow school of topology, which maintained ties to the West during the Soviet period. His most influential works are in topology and homotopy theory, but he also made important contributions to applied mathematics, including control theory. There is at least one blind Russian mathematician alive today, A. G. Vitushkin of the Steklov Institute in Moscow, who works in complex analysis.

France has produced outstanding blind mathematicians. One of the best known is Louis Antoine (1888–1971), who lost his sight at the age of twenty-nine in the first World War. According to [Ju], it was Lebesgue who suggested Antoine study two- and three-dimensional topology, partly because there were at that time not many papers in the area and partly because "dans une telle étude, les yeux de l'esprit et l'habitude de la concentration remplaceront la vision perdue" ("in such a study the eyes of the spirit and the habit of concentration will replace the lost vision"). Morin met Antoine in the mid-1960s, and Antoine explained to his younger fellow blind mathematician how he had come up with his best-known result. Antoine was trying to prove a three-dimensional analogue of the Jordan-Schönflies theorem, which says that, given a simple closed curve in the plane, there exists a homeomorphism of the plane that takes the curve into the standard circle. What Antoine tried to prove is that, given an embedding of the two-sphere into three-space, there is a homeomorphism of three-space that takes the embedded sphere into the standard sphere. Antoine eventually realized that this theorem is false. He came up with the first "wild embedding" of a set in three-space, now known as Antoine's necklace, which is a Cantor set whose complement is not simply connected. Using Antoine's ideas, J. W. Alexander came up with his famous horned sphere, which is a wild embedding of the two-sphere in three-space. The horned sphere provides a counterexample to the theorem Antoine was trying to prove. Antoine had proved that one could get the sphere embedding from the necklace, but when Morin asked him what the sphere embedding looked like, Antoine said he could not visualize it.

Everting the Sphere

Morin's own life story is quite fascinating. He was born in 1931 in Shanghai, where his father worked for a bank. Morin developed glaucoma at an early age and was taken to France for medical treatment. He returned to Shanghai, but then tore his retinas and was completely blind by the age of six. Still, he has a stock of images from his sighted years and recalls that as a child he had an intense interest in optical phenomena. He remembers being captivated by a kaleidescope. He had a book about colors that showed how, for example, red and yellow mix together to produce orange. Another memory is that of a landscape painting; he remembers looking at the painting and wondering why he saw three dimensions even though the painting was flat. His early visual memories are especially vivid because they were not replaced by more images as he grew up.

After he went blind, Morin left Shanghai and returned to France permanently. There he was educated in schools for the blind until age fifteen, when he entered a regular *lycée*. He was interested in mathematics and philosophy, and his father, thinking his son would not do well in mathematics, steered Morin toward philosophy. After studying at the École Normale Supérieure for a few years, Morin



Leonhard Euler

became disillusioned with philosophy and switched to mathematics. He studied under Henri Cartan and joined the Centre National de la Recherche Scientifique as a researcher in 1957. Morin was already well known for his sphere eversion and had spent two years at the Institute for Advanced Study by the time he finished his Ph.D. thesis in singularity theory in 1972, under the direction of René Thom. Morin spent most of his career teaching at the Université de Strasbourg and retired in 1999.

Photograph courtesy of John M. Sullivan, University of Illinois.

Bernard Morin with one of Stewart Dickson's models, at the International Colloquium on Art and Mathematics in Maubeuge, France, in September 2000.

It was in 1959 that Stephen Smale proved the surprising theorem that all immersions of the n-sphere into Euclidean space are regularly homotopic. His result implies that the standard embedding of the two-sphere into three-space is regularly homotopic to the antipodal embedding. This is equivalent to saying that the sphere can be everted, or turned inside out. However, constructing a sphere eversion following the arguments in Smale's paper seemed to be too complicated. In the early 1960s, Arnold Shapiro came up with a way to evert the sphere, but he never published it. He explained his method to Morin, who was already developing similar ideas of his own. Physicist Marcel Froissart was also interested in the problem and suggested a key simplification to Morin; it was for the collaboration with Froissart that Morin created his clay models. Morin first exhibited a homotopy that carries out an eversion of the sphere in 1967.

Charles Pugh of the University of California at Berkeley used photographs of Morin's clay models to construct chicken wire models of the different stages of the eversion. Measurements from Pugh's models were used to make the famous 1976 film *Turning a Sphere Inside Out*. Created by Nelson Max, now a mathematician at Lawrence Livermore National Laboratory, the film was a tour de force of computer graphics available at that time. Morin actually had two different renditions of his sphere eversion, and at first he was not sure which one appeared in the film. He asked some of his colleagues who had seen the film which rendition was depicted. "Nobody could answer," he recalled.

Since the making of Max's film, other sphere eversions have been developed, and new movies depicting them have been made. One eversion was created by William Thurston, who found a way to make Smale's original proof constructive. This eversion is depicted in the film *Outside In*, made at the Geometry Center [OI]. Another eversion originated with Rob Kusner of the University of Massachusetts at Amherst, who suggested that energy-minimization methods could be used to generate Morin's eversion. Kusner's idea is depicted in a movie called *The Optiverse*, created in 1998 by the University of Illinois mathematicians John M. Sullivan, George Francis, and Stuart Levy [O]. Sculptor and graphics animator Stewart Dickson used the *Optiverse* numerical data to make models of different stages of the optiverse eversion, for a project called "Tactile Mathematics" (one aim of the project is to create models of geometric objects for use by blind people). Some of the optiverse models were given to Morin during the International Colloquium on Art and Mathematics in Maubeuge, France, in September 2000. Morin keeps the models in his living room.

Far from detracting from his extraordinary visualization ability, Morin's blindness may have enhanced it. Disabilities like blindness, he noted, reinforce one's gifts and one's deficits, so "there are more dramatic contrasts in disabled people," he said. Morin believes there are two kinds of mathematical imagination. One kind, which he calls "time-like", deals with information by proceeding through a series of steps. This is the kind of imagination that allows one to carry out long computations. "I was never good at computing," Morin remarked, and his blindness deepened this deficit. What he excels at is the other kind of imagination, which he calls "space-like" and which allows one to comprehend information all at once.

One thing that is difficult about visualizing geometric objects is that one tends to see only the outside of the objects, not the inside, which might be very complicated. By thinking carefully about two things at once, Morin has developed the ability to pass from outside to inside, or from one "room" to another. This kind of spatial imagination seems to be less dependent on visual experiences than on tactile ones. "Our spatial imagination is framed by manipulating objects," Morin said. "You act on objects with your hands, not with your eyes. So being outside or inside is something that is really connected with your actions on objects." Because he is so accustomed to tactile information, Morin can, after manipulating a hand-held model for a couple of hours, retain the memory of its shape for years afterward.

Geometry: Pure Thinking

At a meeting at the Mathematisches Forschungsinstitut Oberwolfach in July 2001, Emmanuel Giroux presented a lecture on his latest work entitled

"Contact structures and open book decompositions". Despite Giroux's blindness—or maybe because of it—he gave what was probably the clearest and best organized lecture of the week-long meeting. He sat next to an overhead projector, and as he put up one transparency after another, it was apparent that he knew exactly what was on every transparency. He used his hands to schematically illustrate his precise description of how to attach one geometric object to the boundary of another. Afterwards some in the audience recalled other lectures by Giroux, in which he described, with great clarity, certain mathematical phenomena as evolving like the frames in a film. "In part it's my way of doing things, my style" to try to be as clear as possible, Giroux said. "But also I'm often extremely frustrated because other mathematicians don't explain what they are doing at the board and what they write." Thus the clarity of his lectures is in part a reaction against hard-to-understand lectures by sighted colleagues, who can get away with being less organized.

Giroux has been blind since the age of eleven. He notes that most blind mathematicians are or were working in geometry. But why geometry, the most visual of all areas of mathematics? "It's pure thinking," Giroux replied. He explained that, for example, in analysis, one has to do calculations in which one keeps track line-by-line of what one is doing. This is difficult in Braille: To write, one must punch holes in the paper, and to read one must turn the paper over and touch the holes. Thus long strings of calculations are hard to keep track of (this burden may ease in the future, with the development of "paperless writing" tools such as refreshable Braille displays). By contrast, "in geometry, the information is very concentrated, it's something you can keep in mind," Giroux said. What he keeps in mind is rather mysterious; it is not necessarily pictures, which he said provide a way of representing mathematical objects but not a way of thinking about them.

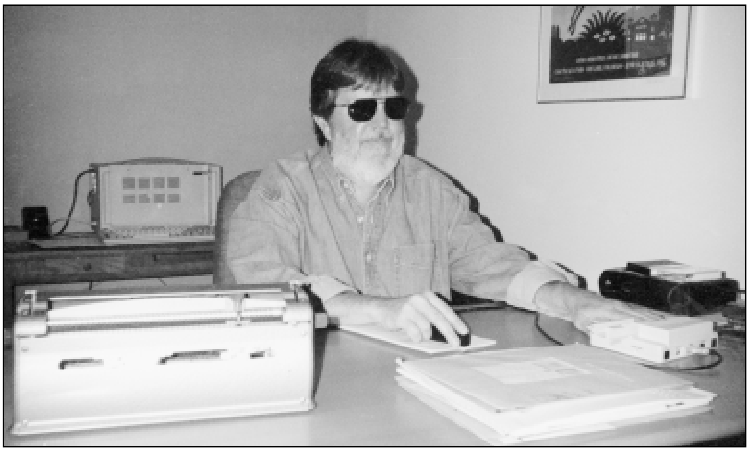
In [So], Alexei Sossinski points out that it is not so suprising that many blind mathematicians work in geometry. The spatial ability of a sighted person is based on the brain analyzing a two-dimensional image, projected onto the retina, of the three-dimensional world, while the spatial ability of a blind person is based on the brain analyzing information obtained through the senses of touch and hearing. In both cases, the brain creates flexible methods of spatial representation based on information from the senses. Sossinski points out that studies of blind people who have regained their sight show that the ability to perceive certain fundamental topological structures, like how many holes something has, are probably inborn. "So a blind person who has regained his eyesight can at first not distinguish between a square and a circle," Sossinski writes. "He just sees their topological equivalence. On the other hand he sees immediately that a torus is not a ball." In a private communication, Sossinski also noted that sighted people sometimes have misconceptions about three-dimensional space because of the inadequate and misleading two-dimensional projection of space onto the retina. "The blind person (via his other senses) has an undeformed, directly 3-dimensional intuition of space," he said.

As noted in [Ja], attempts to understand spatial ability have a long history going back at least to the time of Plato, who believed that all people, blind or sighted, have the same ability to understand spatial relations. Based on the ability of the visually impaired to learn shapes through touch, Descartes, in *Discours de la méthode* (1637), argued that the ability to create mental representational frameworks is innate. In the late eighteenth century, Diderot, who involved blind people in his research, concluded that people can gain a good sense of three-dimensional objects through touch alone. He also found that changes in scale presented few problems for the blind, who "can enlarge or shrink shapes mentally. This spatial imagination often consisted of recalling and recombining tactile sensations [Ja]." In recent decades, much research has been devoted to investigating the spatial abilities of blind people. The prevailing view was that the blind have weaker or less efficient spatial abilities than the sighted. However, research such as that presented in [Ja] challenges this view and appears to indicate that, for many ordinary tasks such as remembering a walking route, the spatial abilities of blind and sighted people are the same.

Challenges of Analysis

Not all blind mathematicians are geometers. Despite the formidable challenges analysis presents to the blind, there are a number of blind analysts, such as Lawrence Baggett, who has been on the faculty of the University of Colorado at Boulder for thirty-five years. Blind since the age of five, Baggett liked mathematics as a youngster and found he could do a lot in his head. He never learned the standard algorithm for long division because it was too clumsy to carry out in Braille. Instead, he figured out his own ways of doing division. There were not many textbooks in Braille, so he depended on his mother and his classmates reading to him. Initially he planned to become a lawyer "because that's what blind people did in those days." But once he was in college, he decided to study mathematics.

Baggett says he has never been very good in geometry and cannot easily visualize complicated topological objects. But this is not because he is blind; in visualizing, say, a four-dimensional sphere, he said, "I don't know why being able to see makes it any easier." When he does mathematics, he



Lawrence Baggett.

sometimes visualizes formulas and schematic, suggestive pictures. When he is tossing around ideas in his head, he sometimes makes Braille notes, but not very often. "I try to say it aloud," he explained. "I pace and talk to myself a lot." Working with a sighted colleague helps because the colleague can more easily look up references or figure out what a bit of notation means; otherwise, Baggett said, collaboration is the same as between two sighted mathematicians. But what about, say, going to the blackboard to draw a picture or to do a little calculation? "They do that to me too!" Baggett said with a laugh. The collaborators simply describe in words what is on the board.

Baggett does not find his ability to calculate in his head to be extraordinary. "My feeling is that sighted mathematicians could do a lot in their heads too," he remarked, "but it's handy to write on a piece of paper." A story illustrated his point. At a meeting Baggett attended in Poland in the dead of winter, the lights in the lecture hall suddenly died. It was completely dark. Nevertheless, the lecturer said he would continue. "And he did integrals and Fourier transforms, and people were following it," Baggett recalled. "It proved a point: You don't need the blackboard, but it's just a handy device."

Blind mathematics professors have to come up with innovative methods for teaching. Some write on the blackboard by writing the first line at eye level, the next at mouth level, the next at neck level, and so on. Baggett uses the blackboard, but more for pacing the lecture than for systematically communicating information the students are expected to write down. In fact, he tells them not to copy what he writes but rather to write down what he says. "My boardwork is just an attempt to make the class as much like a normal lecture as possible," he remarked. "Many of [the students] decide they have to learn a different way in my class, and they do." He makes up exams in TeX and has a Web page for homework problems and other information. For grading, he can use graders "but I lose personal feedback," so he uses a variety of schemes, such as having students present oral reports on their work. It is clear that Baggett's devotion to teaching and concern for students overcome any limitations imposed by his disability.

Means of Communicating

When he was growing up in Argentina in the 1950s, Norberto Salinas, who has been blind since age ten, found, just as Baggett did, that the standard profession for blind people was assumed to be law. As a result, there was no Braille material in mathematics and physics. But his parents would read aloud and record material for him. His father, a civil engineer, asked friends in mathematics and physics at the University of Buenos Aires whether his son could take the examination to enter the university. After Salinas got the maximum grade, the university agreed to accept him. In a contribution to a Historia-Mathematica online discussion group about blind mathematicians, Eduardo Ortiz of Imperial College, London, recalled examining Salinas in an analysis course at University of Buenos Aires. Salinas communicated graphical information by drawing pictures on the palm of Ortiz's hand, a technique that Ortiz himself later used when teaching blind students at Imperial. Salinas taught mathematics in Peru for a while and then went to the United States to get his Ph.D. at the University of Michigan. Today he is on the faculty of the University of Kansas.

Salinas said that he would often translate taped material into Braille, a step that helped him to absorb the material. He developed his own version of a Braille code for mathematical symbols and in the 1960s helped to design the standard code for representing such symbols in Spanish Braille. In the United States, the standard code for mathematical symbols in Braille is the Nemeth code, developed in the 1940s by Abraham Nemeth, a blind mathematics and computer science professor now retired from the University of Detroit. The Nemeth code employs the ordinary six-dot Braille codes to express numbers and mathematical symbols, using special indicators to set mathematical material off from literary material. Standard Braille was clearly not intended for technical material, for it does not provide representations for even the most common technical symbols; even integers must be represented by the codes for letters ($a = 1, b = 2, c = 3$, etc.). The Nemeth code can be difficult to learn because the same characters that mean one thing in literary Braille have different meanings in Nemeth. Nevertheless it has been extremely important in helping blind people, especially students, gain access to scientific and technical materials. Salinas and John Gardner, a blind physicist at Oregon State University, have developed a new code called GS8, which uses eight dots instead of the usual six. The two additional dots, which are reserved for mathematical notation, provide the possibility of representing 255 characters rather than the sixty-three that are possible in standard Braille. In addition, the syntax of GS8 is based on LaTeX, making it feasible to convert GS8 documents into LaTeX, and vice versa.

Computers have opened up a whole world of communication possibilities for blind people. Screen reader programs, such as Jaws or SpeakUp, translate text on-screen into spoken words using speech synthesizers. Unfortunately, these programs generally do not work well with text containing mathematical symbols, and some blind mathematicians tend to use the programs only for reading email or surfing the Web (which is becoming more complicated for the blind due to the heavy use of graphics). A blind computer scientist at Cornell University, T. V. Raman, has developed a program called AsTeR, which accepts a TeX file as input and as output produces an audio file that contains a synthesized vocalization of the document, mathematics and all. Gardner has developed a program called TRIANGLE that has a speech synthesizer that is more basic than AsTeR and also includes a program for converting between LaTeX and the GS8 code.

Some blind mathematicians actually read TeX source files directly; Giroux does so using a refreshable Braille touch-screen. He said it is more comfortable to have an audio recording of a paper, but before having a recording made, he wants to know whether the paper is really interesting to him. Reading TeX files provides quick and direct access to the documents. Of course, TeX files are meant to be read by computers rather than humans and are therefore cumbersome and verbose. Nevertheless, Giroux said that their easy availability through electronic preprint servers and journals represents "huge progress" in his ability to stay in touch with current research. Books are a bigger problem than papers; although TeX is the standard way of publishing mathematics books, obtaining the TeX files from publishers is not a straightforward process.

Mathematics Accessible to the Blind

It is easy to understand how well-meaning people who know little about mathematics might assume that the subject's technical notation would create an insurmountable barrier for blind people. But in fact, mathematics is in some ways more accessible for the blind than other professions. One reason is that mathematics requires less reading because mathematical writing is compact compared to other kinds of writing. "In mathematics," Salinas noted, "you read a couple of pages and get a lot of food for thought." In addition, blind people often have an affinity for the imaginative, Platonic realm of mathematics. For example, Morin remarked that sighted students are usually taught in such a way that, when they think about two intersecting planes, they see the planes as two-dimensional pictures drawn on a sheet of paper. "For them, the geometry is these pictures," he said. "They have no idea of the planes existing in their natural space." Because blind students do not use drawings, it is natural for them to think about the planes in an abstract way.

The most famous blind American mathematician right now may be Zachary J. Battles, whose extraordinary story was even covered in *People* magazine. Blind almost from birth and adopted from a South Korean orphanage when he was three years old, Battles went on to earn a bachelor's degree in mathematics and a bachelor's and master's in computer science from Pennsylvania State University. He also traveled to Ukraine twice to teach English as a second language and worked as a mentor for other disabled students. He is now studying mathematics at the University of Oxford on a Rhodes Scholarship. Like so many other blind mathematicians, Battles is an inspiration to the sighted and the blind alike.

—*Allyn Jackson*

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