The axioms of *Principles* do not separate the quantificational theory from a the quantification-free calculus. The primitive signs of Russell's formal language of propositions are the individual variables, $x_1, \ldots, x_n$ and the sign $\supset$, the braces (,) and brackets { , and }. The terms are defined thus: 1) individual variables are terms; (2) if *A* is a *wff*, then {*A*} is a term; (3) there are no other terms. The atomic *wffs* are of the form

$x \supset y$

where *x* and *y* are individual variables. The *wffs* are defined thus: (1) all atomic *wffs;* (2) if *x* is an individual variable occurring free in $\alpha \supset_{u_o, \ldots, u_n} \beta$ and α and β are terms, then the following are *wffs*:

$\alpha \supset_{x, u_o, \ldots, u_n} \beta$

$\alpha \supset \beta$;

(3) there are no other *wffs*. The axioms are as follows (Landini, 1996):

PoM(Ax 1) $x \supset y \,.\supset_{x,y}.\, x \supset y$

PoM(Ax 2) $x \supset y \,.\supset_{x,y}.\, x \supset x$

PoM(Ax 3) $x \supset y \,.\supset_{x,y}.\, y \supset y$

PoM(Ax 4) $\alpha \supset_{x, u_o, \ldots, u_n} \beta \,.\supset.\, \alpha[t/x] \supset_{u_o, \ldots, u_n} \beta[t/x]$,

where *t* is a term free for *x* in α and/or β.

Simplification *PoM*(Ax 5) $x \supset x :\supset_{x,y}: y \supset y \,.\supset.\, (x \bullet y \,.\supset.\, x)$

Syllogism *PoM*(Ax 6) $x \supset y \,.\bullet.\, y \supset z :\supset_{x,y}: x \supset z$

Composition *PoM*(Ax 7) $x \supset y \,.\bullet.\, x \supset z :\supset_{x,y,z}: (x \,.\supset.\, y \bullet z)$

Importation *PoM*(Ax 8) $y \supset y \,.\bullet.\, z \supset z :\supset_{x,y,z}: (x \,.\supset.\, y \supset z :\supset: x \bullet y \,.\supset.\, z)$

Exportation *PoM*(Ax 9) $x \supset x \,.\bullet.\, y \supset y :\supset_{x,y}: (x \bullet y \,.\supset.\, z :\supset: x \,.\supset.\, y \supset z)$

Reduction *PoM*(Ax 10) $x \supset x \,.\bullet.\, y \supset y :\supset_{x,y}: (x \supset y \,.\supset.\, x :\supset: x)$

Modus Ponens

From A and {A}) {B}, infer B

Conditional Proof

$\rightarrow \alpha$

β

$\alpha \supset \beta$

Universal Generalization

From $\alpha \supset_{u_o, \ldots, u_n} \beta$ infer $\alpha \supset_{x, u_o, \ldots, u_n} \beta$,

where *x* is does not occur free in the assumption line of any conditional proof in whose scope

$\alpha \supset_{x,u_o,\ldots,u_n} \beta$ lies. The following are definitions for the system:

$\sim\alpha =df\ x \supset x \,.\supset._{x}\ \alpha \supset x$

$\alpha \vee \beta =df\ \alpha \supset \beta \,.\supset.\ \beta$

$\alpha\beta =df\ x \supset x \,.\supset._{x}\ (\alpha\supset \beta \,.\supset.\ x :\supset: x)$

$\alpha \equiv \beta =df\ (\alpha \supset \beta) \bullet (\beta \supset \alpha)$

$(\exists x)\mathrm{A}x =df \sim(x) \sim\mathrm{A}x$

$(x)\mathrm{A}x =df \sim\mathrm{A}x \supset_{x} \mathrm{A}x$

$\alpha \bullet \beta =df\ \alpha\beta$

(This definition is added for convenience of exposition.)

This completes the system for the quantification theory.

Russell's system originates from his encounter with Peano at a congress in Paris in 1900. It was, he later recalled in *My Mental Development*, "the most important year of my intellectual life" (*MMD,* p. 12). Russell was struck by the techniques and logical apparatus demonstrated at the congress. A calculus for logic was developed by Schröder and it had been extended to form an *algebra of relatives* (independently developed by Peirce). Russell was not converted. He thought these methods "cumbrous and difficult" (*PoM,* p. 24). In contrast, Peano's work was a breakthrough though it has some correctable flaws. In *Principles.* Russell endeavored to correct the flaws. The presence of a rule of *conditional proof* in the system (as reconstructed above) may, at first, be something of a surprise. But there is ample evidence that such an inference rule was informally was adopted by Peano. In his 1889 paper *The Principles of Arithmetic*, Peano was struggling to formulate a viable inference rule that permits deduction under a hypothesis. He found the matter difficult because of the impact it has on the rule of universal generalization. Peano wrote that the laws of quantification theory remain "abstruse" and he wrote to Frege expressing his concerned about when it is legitimate to engage in a universal generalization within the scope of an assumption (Peano 1889, §18). In a letter to Peano, Frege explained that the rules are set out fully in his *Begriffsschrift* of 1879, and he went on to say that being that they are few in number he knew of no reason they should be said to be "abstruse" (Frege, 1969, p. 11). But in fairness to Peano, it must be understood that Frege's axiomatic methods entirely skirted the use of conditional proof and so Peano's question never arose for him.

Unfortunately, there is a serious flaw in the system of Principles of Mathematics. The flaw might be aptly called "the conjunction problem." Nothing in the system permits the formulation of conjunctions. Accordingly, it is often impossible to use the axioms when one wants them. For example, one cannot arrive at the theorem:

$\vdash x \supset x :\supset_{x,y}: y \supset y .\supset. (x .\supset. y \supset x)$

One way to rectify this problem is to reformulate Russell's axioms 5-10 as follows:

Simplification *PoM*(Ax 5) $x \supset x :\supset_{x,y}: y \supset y .\supset. (x \bullet y . \supset. x)$

Syllogism *PoM*(Ax 6) $x \supset y :\supset_{x,y}: y \supset z . \supset. x \supset z$

Composition *PoM*(Ax 7) $x \supset y :\supset_{x,y}: x \supset z . \supset. (x .\supset. y \bullet z)$

Importation *PoM*(Ax 8) $y \supset y :\supset_{x,y}: z \supset z . \supset. (x . \supset. y \supset z : \supset: x \bullet y . \supset. z)$

Exportation *PoM*(Ax 9) $x \supset x :\supset_{x,y}: y \supset y .\supset. (x \bullet y .\supset. z :\supset: x . \supset. y \supset z)$

Reduction *PoM*(Ax 10) $x \supset x :\supset_{x,y}: y \supset y .\supset. (x \supset y . \supset. x : \supset: x)$

This seems more than is really needed. It is viable to alter only Russell's axiom 7. But a still better solution is wanted.

The most sympathetic way to salvage the quantification theory of propositions of Russell's *Principles* is to leave the system's axiom intact while simply amalgamating his Axiom 2 and Axiom 3. This yields the following:

PoM(Ax 2/3) $x \supset y .\supset_{x,y}. (x \supset x) \bullet (y \supset y)$.

This solves the conjunction problem of the logic of *Principles*. But one may naturally wonder how it is that Russell might have failed to notice the conjunction problem. The natural place to look to build an error theory is definitions of conjunction and tilde. Indeed, it is interesting to note the the conjunction problem would not have arisen if Russell had offered the following:

$\sim\alpha =df\ \alpha \supset f$

$f =df\ (x)(x \supset x . \supset. x)$.

This is because his adopting of conditional proof would enable him to arrive at a derived rule of Conjunction. Where *A* and *B* stand in for *wffs* for implications, quantified or otherwise, consider the following derivation:

DR(conj): From *A*, *B*, infer *A* • *B.*

1. A

2. B

3. $A .\supset. B \supset f$

4. $B \supset f$ *1, 3, mp*

5. $A .\supset. B \supset f ;\supset; f$ *3-5, conditional proof*

6. AB 5, df

This avoids the conjunction problem entirely. It seems clear, therefore, that Russell thought he could get such a derived rule of conjunction in a similar way from his definition of tilde. He probably thought he could get it from the quantification derive rules. That is:

DR(conj): From A, B, infer $A \bullet B$.

1. A

2. B

3. $A \mathbin{.\supset.} (r)(B \supset r)$

3a. $(r)(B \supset r)$ *1, 3, mp*

4. $B \supset (r)r$ *quant derived rule 1*

5. $A \mathbin{.\supset.} (r)(B \supset r) \supset (r)r$ *3-5, conditional proof*

5a. $(r)(A \mathbin{.\supset.} (r)(B \supset r) \supset r)$ *quant derived rule* 2

AB 5a, df

Unfortunately, this cannot work. The problem is not in getting the needed derived quantification rules. The problem is that the expression $(r)r$ is illicit unless it were to abbreviate

$(r)(r \supset r \mathbin{.\supset.} r)$.

But if it is an abbreviation, then the above derivation cannot go through. One can arrive at:

$A \mathbin{.\supset.} (r)(r \supset r \mathbin{.\supset.} B \supset r) \supset (r)(r \supset r \mathbin{.\supset.} r)$

But even with our quantification rules 1 and 2, we need to use importation and exportation to get from here to arrive at the conjunction $A \bullet B$, that is, AB, defined as:

$(r)(r \supset r \mathbin{:\supset:} A \supset (r)(r \supset r \mathbin{.\supset.} B \supset r)$.

Russell's infelicitous expression "$(r)r$" may perhaps look innocuous, but it undermine the proof.

Once the conjunction problem of *Principles* is rectified, all is well. In order to develop the system, however, it is central to first prove as theorems analogs of the more usual axioms of propositional calculus Thus where P and Q and R stand in for *wffs* for implications, quantified (formal) or otherwise, one can readily prove the following:

*simp $P \bullet Q \mathbin{.\supset.} P$

*syll $P \supset Q \mathbin{:\supset:} Q \supset R \mathbin{.\supset.} P \supset R$

*comp $P \supset Q \mathbin{:\supset:} P \supset R \mathbin{.\supset.} (P \mathbin{.\supset.} Q \bullet R)$

*import $P \mathbin{.\supset.} Q \supset R \mathbin{:\supset:} P \bullet Q \mathbin{.\supset.} R$

*export $P \bullet Q \mathbin{.\supset.} R \mathbin{:\supset:} P \mathbin{.\supset.} Q \supset R$

*reduct $P \supset Q \mathbin{.\supset.} P \mathbin{:\supset:} P$

This makes the propositional system look more normal and it helps in seeing how it is that this system reaches semantic completeness with respect to analogs, in the language of the theory, of logical truths of the modern quantification theory of the first-order predicate calculus.