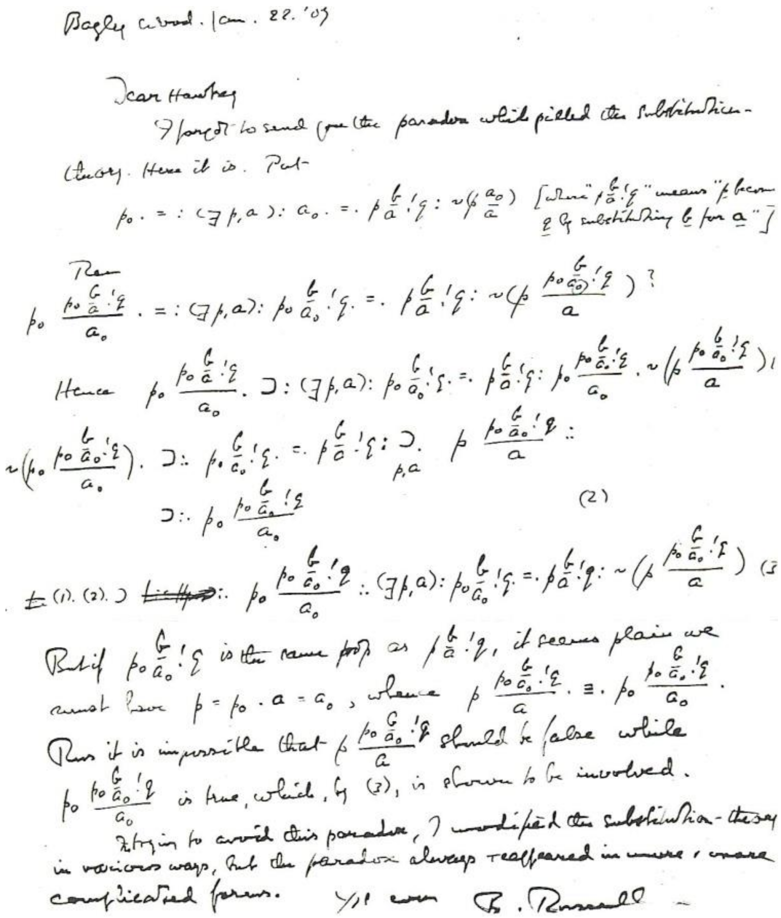
In April of 1906, Russell discovered a contradiction in his substitutional theory. Let us call this Russell's "$p_o/a_o$ paradox." It is not a paradox of the quantificational theory of propositions. We saw that in spite of its embracing ontological self-reference, Russell's quantification theory of propositions is consistent. The $p_o/a_o$ paradox arises only with the axioms of the substitutional theory—axioms added on to Russell's quantification theory of propositions. In 1907, Russell admitted to Ralph Hawtrey that this paradox "pilled" the substitutional theory.

In his letter (reproduced below), Russell explains that he tried several ways to solve the paradox but it kept on being resurrected in various new forms.


Bagley Wood. Jan. 22.'07

Dear Hawtrey

I forgot to send you the paradox which pilled the substitution-theory. Here it is. Put

$p_0 . =: (\exists p, a): a_0 . = . p\frac{b}{a}!q : \sim(p\frac{a_0}{a})$ [where "$p\frac{b}{a}!q$" means "$p$ becomes $q$ by substituting $b$ for $a$"]

Then

$p_0 \frac{p_0\frac{b}{a}!q}{a_0} . =: (\exists p,a): p_0\frac{b}{a_0}!q . = . p\frac{b}{a}!q : \sim(p\frac{p_0\frac{b}{a_0}!q}{a})$ ?

Hence $p_0\frac{p_0\frac{b}{a}!q}{a_0} . \supset : (\exists p,a): p_0\frac{b}{a_0}!q . = . p\frac{b}{a}!q : p_0\frac{p_0\frac{b}{a_0}!q}{a_0} . \sim(p\frac{p_0\frac{b}{a_0}!q}{a})$ (1

$\sim(p_0\frac{p_0\frac{b}{a_0}!q}{a_0}) . \supset :: p_0\frac{b}{a_0}!q . = . p\frac{b}{a}!q : \supset_{p,a} . p\frac{p_0\frac{b}{a_0}!q}{a} ::$

$\supset :: p_0\frac{p_0\frac{b}{a_0}!q}{a_0}$ (2)

$\vdash (1).(2). \supset$ ~~[struck out]~~ $:: p_0\frac{p_0\frac{b}{a_0}!q}{a_0} :: (\exists p,a): p_0\frac{b}{a_0}!q . = . p\frac{b}{a}!q : \sim(p\frac{p_0\frac{b}{a_0}!q}{a})$ (3

But if $p_0\frac{b}{a_0}!q$ is the same prop as $p\frac{b}{a}!q$, it seems plain we must have $p = p_0 . a = a_0$, whence $p\frac{p_0\frac{b}{a_0}!q}{a} . \equiv . p_0\frac{p_0\frac{b}{a_0}!q}{a_0}$. Thus it is impossible that $p\frac{p_0\frac{b}{a_0}!q}{a}$ should be false while $p_0\frac{p_0\frac{b}{a_0}!q}{a_0}$ is true, which, by (3), is shown to be involved.

In trying to avoid this paradox, I modified the substitution-theory in various ways, but the paradox always reappeared in more & more complicated forms.

Yrs ever B. Russell

It is unfortunate that nothing was known of its existence for so many years. Historians were wholly unaware of the $p_o/a_o$ paradox and this misled them into thinking that Russell's quantification theory of propositions must engender some sort of *Liar* paradox—a paradox that eventually caused him to abandon the substitutional theory. Historians often speculated that it was efforts to solve some propositional Liar paradox that must have been what led Russell to his 1907 paper "Mathematical Logic as Based on the Theory of Types" (*ML*) and his dalliance with an ontology of *orders* of propositions—a ramified theory of propositions.

The $p_o/a_o$ is a diagonal paradox *unique* to the axioms of the substitutional theory. It does not arise in Russell's ontologically self-referential quantification theory of propositions. It arises because the substitutional axioms provide for the existence of functions that are in conflict with Cantor's power-theorem. Liar paradoxes are not diagonal paradoxes and they don't depend on the existence of functions violating Cantor's power-theorem. There is one reason and one reason alone that led Russell to abandon his substitutional theory: the diagonal $p_o/a_o$ paradox. The paradox posed a serious dilemma for Russell. He has to (1) preserve Cantor's work and this requires impredicative diagonal constructions, and yet (2) he has to prevent the method of Cantor's diagonal construction from generating the $p_o/a_o$ paradox and its variants. It is not easy to get between this Scylla and Charybdis.

Russell mentioned in his letter to Hawtrey that there are different "forms" (that is, versions) of the $p_o/a_o$. This happens because there are many different functions in substitutional theory that conflict with Cantor's power theorem. Cantor's power-theorem is proved by a diagonal method and it reveals that there can be no function from entities *onto* attributes of entities. In the substitutional theory, however, every pair *p*, *a* represents an attribute in intension. It is clear that there many different one-one functions expressible in the substitutional theory which assign a unique entity to every pair of entities *p* and *a*. One of the features of the substitutional theory is that we can prove theorems concerning the constituents of propositions. It is these theorems that give rise to one-one functions in violation of Cantor's diagonal results. Consider the following:

$$\{p\frac{z}{a} \supset_z z\} = \{r\frac{z}{c} \supset_z z\} :\supset: p = r .\bullet. a = c$$

$$\{p \supset q\} = \{r \supset s\} :\supset: p = r .\bullet. q = s$$

$$\{p\,\frac{b}{a}\,!q\} = \{r\,\frac{b}{c}\,!q\} :\supset: p = r .\bullet. a = c$$

Each of the above theorems yields (respectively) each of the following one-one functions:

$$f(p, a) = \{p\frac{z}{a} \supset_z z\}$$

$$f(p, a) = \{p \supset a\}$$

$$f(p, a) = \{p\,\frac{b}{a}\,!q\}.$$

The last one-one function is involved in the version of the $p_o/a_o$ discussed in the letter to Hawtrey. Consider the pair of entities $p_o$ and $a_o$ such that

$$(x)(\, p_o\,\frac{x}{a_o} \equiv \{(\exists\, p\,,a)(\, x = \{\, p\,\frac{b}{a}\,!q\} .\bullet. \sim(p\,\frac{x}{a}\,)\,)\}\,).$$

We now have the contradiction:

$$p_{\text{o}} \frac{\{p_o \frac{b}{a_o}!q\}}{a_o} \equiv \sim ( p_{\text{o}} \frac{\{p_o \frac{b}{a_o}!q\}}{a_o} ).$$

But the others generate similar paradoxes. Consider the pair of entities *p* and *a* such that,

$$(x)( p \frac{x}{a} \equiv \{(\exists\, r, c)( x = \{r \supset c\} \,.\bullet.\, \sim (r \frac{x}{c} ) )\} ).$$

This readily yields the contradiction

$$p \frac{\{p \supset a\}}{a} \equiv \sim ( p \frac{\{p \supset a\}}{a} ).$$

Indeed, following the pattern of the paradox of Appendix B of *Principles*, one can arrive at this

$$(x)( p \frac{x}{a} \equiv \{(\exists\, r ,c)( x = \{r \frac{z}{c} \supset_z z\} \,.\bullet.\, \sim (r \frac{x}{c} ) )\} ).$$

This version occurs in Russell's manuscript "On Substitution" (dated April/May 1906) and is called "the pure form of the liar". Of course, it has little to do with the *liar* paradox. The $p_o/a_o$ paradox and its different versions, unlike any propositional Liar, stem from a fundamental tension between the substitutional theory and the diagonal method used by Cantor to generate his power- theorem. The propositional Liar does *not* require a diagonal construction and is not in conflict Cantor's power-theorem. The only relevant relationship between the two is that a hierarchy of orders of propositions (restricting variables to orders) blocks both. That is, Russell knew that restricting variables to orders of propositions affords a way (given he can adopt some mitigating axiom of Reducibility of propositions) of *preserving* Cantor's power-theorem while at the same time *blocking* its application to propositions.

It is important to realize that not all versions of the $p_o/a_o$ are alike. The one that follows the pattern of the Appendix B essentially involves identity with a general proposition. Others, as we shall see, do not. This is quite important. Indeed, Russell hoped in September of 1906 that the $p_o/a_o$ paradox can be solved by adopting a version of his substitutional theory that rejects the ontology of general propositions. In this version, which he set forth in his captivating paper "On 'Insolubilia' and Their Solution by Symbolic Logic" (I*nS*). This is his no-general propositions theory. In any case, the $p_o/a_o$ paradox came as a shock to Russell. Within the logic of substitution, it can be traced to theorem this

$$(\exists p, a)\ (x)( p \frac{x}{a}\ !\{\mathrm{A}x\}),$$

where *p* and *a* are not free in the *wff* A*x*. This seems to be logically true—given the ontology of propositions. And from this we readily arrive at

$$(\exists p, a)\ (x)( p \frac{x}{a} \equiv \{\mathrm{A}x\}),$$

where *p* and *a* are not free in the *wff* A*x*.

When Russell first formulated the appendix B version of the $p_o/a_o$ in April/May 1906, he quickly withdrew is paper "On the Substitutional Theory of Classes and Relations" from publication. But in a June of 1906 letter to Couturat, we find that Russell had by then approved the publication of *InS* because he thought he had solved the $p_o/a_o$ paradox. His plan, as we noted, was to abandon his ontology general propositions. This is not to say that there are no general *wffs*. It simply means that only those *wffs* A of the theory that are quantifier-free can be nominalized to form a term {A}. In order to implement this plan, Russell had to redo his base quantification theory for the logic of propositions. (This is the origins of the formal quantification theory of *9 of *Principia*.) Subordinate occurrences of quantifiers are to be defined in terms of *wffs* in which all the quantifiers are initially placed. Thus, for example,

$$q \supset (x)(x = x) =\text{df } (x)( q \supset \{x = x\}).$$

Russell's paper *InS* originally published only in French. Couturat helped in checking the translation of the English manuscript of the paper into French. The English manuscript awaited 1973 to reemerge and be studied by scholars of Russell's philosophy of logic. Unfortunately, no one then had much of any inkling of even the existence of Russell's substitutional theory to say nothing of the $p_o/a_o$ itself. The hint that there must be some hidden paradox that was driving Russell's work in the period only began to emerge with the work of Grattan-Guinness (1977), Cocchiarella (1980), Goldfarb (1989, written earlier) and Hilton (1980)—each offering importantly different accounts of what such a paradox might involve. But no one of them had then found the actual $p_o/a_o$ paradox. Landini found it in 1985 among Russell many manuscripts.

Russell never set out the system of *InS* formally, but it can be readily recovered from the pieces of it that remain in his writings. It is presented here in contrast to the original system of 1905. The language of the theory adopts as new primitives the sign ∃ as well as the identity sign = and the sign *f*. The language allows that only quantifier-free *wffs* A can be nominalized to form terms {A}. However, as we shall see, *wffs* involving quantifiers may flank the sign ) in virtue of definitions which initially place the quantifiers. Where $\alpha$ , $\beta$ , $\delta$ are quantifier free terms the axioms are as follows:

${}^{1906}S_1$ $\alpha .\supset. \beta \supset \alpha$

${}^{1906}S_2$ $\alpha .\supset. \beta \supset \delta :\supset: \beta .\supset. \alpha \supset \delta$

${}^{1906}S_3$ $\alpha \supset \beta :\supset: \beta \supset \delta .\supset. \alpha \supset \delta$

${}^{1906}S_4$ $\alpha \supset \beta .\supset. \alpha :\supset: \alpha$

${}^{1906}S_{5a}$ $A[\alpha | x] .\supset. (\exists x)Ax$,

where α is free for free occurrences of *x* in A.

${}^{1906}S_{5b}$ $A[\alpha | x] \vee A[\beta | x] .\supset. (\exists x)Ax$,

where *x* is not free in $\alpha$.

${}^{1906}S_{6a}$ $\alpha = \alpha$

${}^{1906}S_{6b}$ $\alpha = \beta .\supset. A\alpha \supset A[\beta/\alpha]$,

where free $\beta$ replaces one or more free occurrence of $\alpha$ in A.

$^{1906}S_7$ $\alpha$ in $\{A\alpha\}$

$^{1906}S_8$ $\alpha$ in $\{A\beta_1,\dots,\beta_n\}$ .). $\alpha = \{A\beta_1,\dots,\beta_n\}$ .∨. $\alpha$ in $\beta_1$ . ∨.,…,.∨. $\alpha$ in $\beta_n$,

where A is any *wff* all of whose distinct free terms are $\beta_1,\dots,\beta_n$.

$^{1906}S_9$ $(x, y)(x$ in $y$ .&. $y$ in $x$ :Ↄ: $x = y)$

$^{11906}S_{10}$ $(x, y, z)(x$ in $y$ .•. $y$ in $z$ :Ↄ: $x$ in $z)$

$^{1906}S_{11}$ $(p, a)(q)(x, y)( p \frac{x}{a} !q$ .•. $p \frac{y}{a} !q$ .•. $a$ in $p$ :Ↄ: $x = y)$

$^{1906}S_{12}$ $(p, a)(z)(q)( p \frac{z}{a} !q$ .•. $a$ in $p$ .•. $a \neq p$ :Ↄ: $z$ in $q$ .•. $z \neq q)$

$^{1906}S_{13}$ $(x, y)( x \frac{y}{x} !y )$

$^{1906}S_{14}$ $(x, y)( x \frac{y}{y} !x )$

$^{1906}S_{15}$ $(p, a)(x)(\exists q)( p \frac{z}{a} !q$ .•. $(r)( p \frac{z}{a} !r$ :Ↄ: $q = r))$

$^{1906}S_{16}$ $(p)(\exists q)(q$ ex $p)$

$^{1906}S_{17a}$ $(\exists u)( u$ out $\{Au|v\})$ Ↄ $(u)( \{A\alpha|v\} \frac{u}{a} !\{Au|v\})$,

where *a* and *u* are free for *v* in A.

Mitigating Axiom Schema

$^{1906}S_{17b}$ $(\exists\, p, a_1, \dots, a_n)(\ p \frac{x_1,\dots,x_n}{a_1,\dots,a_n} !\ r \equiv_r r = q$ .•. $q \equiv A(x_1, \dots, x_n))$,

where $p, a_1, \dots, a_n$ are not free in A.

$^{1906}S_{18}$ $(\exists u_1,\dots,u_n)( \alpha$ out $\{Au_1|v_1,\dots, u_n|v_n\})$ :•: $\alpha \neq \{Au_1|v_1,\dots, u_n|v_n\}$

.•. $\alpha \neq \beta_1$ .•. ,…, .•. $\alpha \neq \beta_n$ .:):.

$(x)(\exists u_1,\dots, u_n)( \{A\sigma_1|v_1,\dots,\sigma_n|v_n\} \frac{x}{a} !\{Au|v\}$/∀;x!$\{Au_1|v_1,\dots, u_n|v_n\}$ :•:

$\sigma_1 \frac{x}{a} u_1$ .•. ,…, .•. $\sigma_n \frac{x}{a} u_n )$

where each $u_i$ and $\sigma_i$, $1 \leq i \leq N$, are free for their respective $v_i$ in A, and $\beta_1,\dots,\beta_n$ are all the terms occurring free in A.

*Modus Ponens* (1)

From A and {A} Ↄ {B}, infer B

*Modus Ponens* (2)

From A and A Ↄ B, infer B.

*Universal Generalization*

From A*x* infer (∀*x*)A*x*.

*Switch*

From (∀*x*)(∃*y*)A(*x* ,*y*) infer (∃*y*) (∀*x*)A(*x*, *y*) where there is a logical particle in A on one side of which all free occurrences of *x* occur and on the other side of which all free occurrences of *y* occur.

The logical particles of the quantification theory of propositions are defined in terms of the relation of implication. As expected, Russell has:

$\sim\alpha =df\ \alpha \supset f$

$\alpha \vee \beta =df\ \ \alpha \supset \beta .\supset. \beta$

$\alpha \bullet \beta =df\ \sim (\alpha \supset \sim\beta)$

$\alpha \equiv \beta =df\ (\alpha \supset \beta) \bullet (\beta \supset \alpha)$

$\mathrm{A} \vee \mathrm{B} =df\ \mathrm{A} .\supset. \mathrm{B} \supset \mathrm{B}$

$\mathrm{A} \bullet \mathrm{B} =df \sim(\mathrm{A} \supset \sim\mathrm{B})$

$\mathrm{A} \equiv \mathrm{B} =df\ (\mathrm{A} \supset \mathrm{B}) \bullet (\mathrm{B} \supset \mathrm{A})$

$\alpha\ \mathrm{out}\beta =df\ (x)(\ \beta\frac{x}{\alpha}\ !\beta)$

$\alpha\ \mathrm{in}\ \beta =df \sim (\alpha\ \mathrm{out}\ \beta)$

$\alpha\ \mathrm{ind}\ \beta =df\ \alpha\ \mathrm{out}\ \beta .\bullet. \beta\ \mathrm{out}\ \alpha$

$\alpha\ \mathrm{ex}\ \beta =df \sim (\exists x)(\ x\ \mathrm{in}\ \alpha .\bullet. x\ \mathrm{in}\ \beta)$

Where α is quantifier-free and *x* does not occur free in the formula A and *y* does not occur free in the formula B, definitions include the following (in analogy with Principia's section *9):

$\sim (x)\mathrm{A}x =df\ (\exists x) \sim\mathrm{A}x$

$\sim (\exists x)\mathrm{A}x =df\ (x) \sim\mathrm{A}x$

$(x)\ \mathrm{A}x \supset p =df\ (\exists x)(\ \mathrm{A}x \supset p)$

$p \supset (x)\ \mathrm{A}x\ \ =df\ (x)(p \supset \mathrm{A}x)$

$(\exists x)\ \mathrm{A}x \supset p =df\ (x)(\ \mathrm{A}x \supset p)$

$p \supset (\exists x)\ \mathrm{A}x =df\ (\exists x)(p \supset \mathrm{A}x)$

$(x)\mathrm{A}x \supset (\exists y)\mathrm{B}y =df\ (\exists x)(\exists y)(\mathrm{A}x \supset \mathrm{B}y\ )$

$(\exists x)\mathrm{A}x \supset (y)\mathrm{B}\ y =df\ (y)(\exists x)\ (\ \mathrm{A}x \supset \mathrm{B}y)$

$(x)\mathrm{A}x \supset (y)\mathrm{B}y\ =df\ (\exists x)(y)\ (\ \mathrm{A}x \supset \mathrm{B}y)$

$(\exists x)\mathrm{A}x \supset (\exists y)\mathrm{B}y\ =df\ (x)(\exists y)\ (\ \mathrm{A}x \supset \mathrm{B}y)$

This completes the system.

In 1906 Poincaré published a paper entitled "La Paradoxes et de Logistic". It contained a diatribe against Cantor whose diagonal arguments, in Poincaré's opinion, contain viciously circular (self-referential) definitions of sets or attributes. Poincaré did not make distinctions between semantic paradoxes such as Berry's "least integer not nameable in less that nineteen syllables," Cantor's paradox of the greatest Cardinal, Burali-Forti's paradox of the greatest ordinal, and Russell's paradoxes of classes and attributes. He maintained that all the paradoxes, and Cantor's work in general, derive from viciously circular self-reference. There is a long tradition of scholars interpreting Russell as agreeing with Poincaré. But the historical facts, revealed only by a careful study of Russell's substitutional theory, are quite to the contrary.

Russell defended Cantor against Poincaré. Russell endeavors to preserve Cantor's diagonal power-class argument and he maintains that they do not derive from viciously circular definitions. Impredicative comprehension is perfectly legitimate. Russell was eager to herald his substitutional theory as a solution designed so that it "... avoids all known contradictions, while at the same time preserving nearly the whole of Cantor's work on the transfinite" (*InS*, p. 213).

Indeed, Russell lampooned Poincaré's vicious circle principle (VCP) admonition to avoid viciously circular definitions. He observes that in stating his adherence to the VCP, Poincaré is more viciously circular than ever (*InS*, p. 197). The lampoon of Poincaré's view occurs again in Russell's paper "Mathematical Logic as Based on the Theory of Types" and this is the occasion for the humorous remarks in Philip Jourdain's amusing book *The Philosophy of B*rtr*nd R*ss*ll*.
The way to avoid awkward subjects is not to mention that they are not to be mentioned. As Russell put it: "One might as well, in talking to a man with a long nose, say: "When I speak of noses, I except such as are inordinately long," which would not be a very successful effort to avoid a painful topic" (Jourdain (1919), p. 77; Russell, *ML*, p. 63). Russell maintained that the genuine (non-semantically equivocal) paradoxes of classes and attributes require a radical reformulation of the first principles of the calculus for logic as the theory of propositional structure. That reformulation, Russell thinks, is done by his substitutional theory of *InS*. In stark contrast, Berry's paradox and other paradoxes involving "nameability" and "definability" are to be dismissed as confusions due to equivocation. They do not require a reformulation of logical first principles.

In *InS,* and more saliently in "On the Substitutional Theory of Classes and Relatons" (*STCR*), which was written before Russell had discovered his the $p_o/a_o$ paradox, Russell dismisses the semantic paradoxes of "nameability" and "definability." The Paradoxes of "nameability" and "definability" are *not*, in Russell's view, genuine paradoxes calling for a reconstruction of logical first principles. They are easily dismissed by the observation that notions of "nameability" and "definability" are intelligible only if a fixed list of symbols I is first set forth and we have "nameable-in-I" and "definable-in-I" (*STCR*, p. 185; *InS*, p. 209). There can be no debate over this matter since Russell is perfectly explicit. The explanation

Russell gives is perfectly sound. They are not, it should be noted, applied to the notion of *truth* (and *falsehood*) which, because of Russell's commitment to non-general propositions. have to be primitive indefinable properties of propositions. The long-standing tradition of interpreting Russell as advocating that there must be a common solution for both logical and semantic paradoxes does not fit Russell's writings. It is little more than a myth (Landini 2004*b*). We see clearly that, at least during the era of substitution, Russell separated them. Russell never accepted Poincaré's VCP as the solution of anything. It lampooned it and dismissed it (*InS*, p. 205). In Russell's view, the structure of a theory with special variables restricted to simple (impredicative) types of attributes must be emulated in a "no-comprehension-of-attributes/classes," and "no-general-propositions" theory whose only genuine variables are the individual variables of pure logic.

Unfortunately, Russell came to abandon *InS*. Speculations about the reason *why* should now be at an end. He discovered that the system of *InS* didn't work. It allows one to revive a new and more complicated version of the $p_o/a_o$ paradox. It is not presently known when he discovered this. In a letter dated 1 June 1907, he explained that to Jourdain that he no longer finds valid his proof (which was set out in *InS*) of the infinity of non-general propositions. That seems to be the upper limit. In January of 1907 we saw that he wrote the letter to Hawtrey noting that he had modified the substitutional theory in various ways but it was "pilled" by "more and more complicated forms" of the $p_o/a_o$ paradox. That may not be the lower limit. *InS* appeared in September of 1906, but was completed by June. We know this paper was completed by June because it was translated into French by Louis Couturat whose letters of correspondence establish the date conclusively (Galaugher (2013). In any case, it is rather straightforward to trace what specifically enabled the resurrection of the $p_o/a_o$ paradox in *InS*. Observe that in *InS t*he following is illicit:

$$(x)( p\,\frac{x}{a} \equiv \{(\exists\, r ,c)( x = \{r\,\frac{b}{c}\,!q\} \,.\!\bullet\!.\, \sim (r\,\frac{x}{c}\,) )\} \,).$$

One cannot nominalize a general *wff A* to make a term $\{A\}$ for a proposition. But the following is admissible as an instance of Russell's mitigating axiom $^{1906}S_{17b}$

$$(x)( p\,\frac{x}{a} \equiv (\exists\, r ,c)( x = \{r\,\frac{b}{c}\,!q\} \,.\!\bullet\!.\, \sim (r\,\frac{x}{c}\,) ) \,).$$

Of course, this requires the application of the definitions set out in the analog of system *9 for *InS*. That is, we have

$$(x)\,(r',c')(\exists\, r ,c)(\,(p\,\frac{x}{a} :\supset:\; x = \{r\,\frac{b}{c}\,!q\} \,.\!\bullet\!.\, \sim (r\,\frac{x}{c}\,)\,) \bullet$$

$$(x = \{r'\,\frac{b}{c'}\,!q\} \,.\!\bullet\!.\, \sim (r\,\frac{x}{c}\,) :\supset: p\,\frac{x}{a}\,)\,).$$

Further definitions are needed as well to treat the definite descriptions in secondary scope. But the quantification theory of *9 will assure that a contradiction can be derived in *InS*. Unfortunately, there is no known manuscript in which Russell carries out the derivation. Perhaps a manuscript had been sent to Whitehead and was accidently destroyed with his papers shortly after his death in 1947.

Russell must have been chagrined by the failure of *InS*. After a short dalliance in his paper "Mathematical Logic as Based on the Theory of Types" (*ML*) which was completed in July 1907, and which entertains and admittedly *ad hoc* substitutional theory of *orders* of general propositions, he abandoned his substitutional theory sometime in 1908. But there is a straightforward way fix the mitigating axioms for *InS* to avoid the $p_o/a_o$ paradox. The plan is quite simple. Every *wff* in primitive notation of the language of *Principia's* simple impredicative type theory can, as we saw, be translated into the substitutional language of *InS*. Hence, any such *wff* in translation, and no others, are quite legitimately allowed as instances of the mitigating axiom schemata of *InS*. This makes the mitigating axiom schemas of *InS* perfectly safe from any resurrection of the $p_o/a_o$ paradox, for any such resurrection must involve expressions in the language of substitution that have no analog in the primitive language of *Principia's* simple impredicative type theory. Moreover, every result of the mathematical logic of Principia will, quite obviously, be expressible in the system of *InS.*

Russell gave up on *InS* too soon! The solution of the $p_o/a_o$ paradox was at hand. He was very close to seeing it for himself. In *ML,* he had explicitly imagined the language of *Principia's* simple impredicative type scaffolded bindable predicate variables would be accepted for convenience. A recipe for translation into the propositional language of substitution would then be given, perhaps in the appendix of the work. Unfortunately, he clearly imagined that the parameters governing the convenience of the use of *Principia's* bindable simple impredicative type regimented predicate variables would be constrained by first setting out the propositional language of his substitutional theory together with its axioms. This prevented him from discovering the complete solution of the $p_o/a_o$ paradox for *InS*. The convenient language of *Principia's* simple impredicative type regimented bindable predicate variables must be given *first* and independently of the language of substitution, and then it used to find the parameters of safety for the mitigating axiom schemas of *InS,*