Russell deserves credit for finding a paradox with the naïve theory of classes. Contrary to popular characterizations, there isn't just one paradox that he found. He found paradoxes for each of the three versions of the naïve theory of classes. The most well-known naïve theory, takes the relation sign "$\in$" of set membership is a primitive sign and adopts the following for the comprehension of classes:

(Naïve)

$(\exists w)(x \in w \equiv_x \mathrm{A}x)$,

where *w* is not free in the *wff* A. We have only to put $x \notin x$ for the *wff* A and we arrive at the Russell paradox for this naïve theory. We have a set *r* such that:

$x \in r \equiv_x x \notin x$ .

The following contradiction follows:

$r \in r \equiv r \notin r$ .

As long as "$\in$" is taken as a primitive elation sign, Russell paradox does not involve any impredicative notions and neither does it involve ontological self-reference unique to Russell's ontology of general propositions. The instance of (Naïve) that yields the Russell paradox is no more odd than the instance that yield the empty class— that is,

$x \in \Lambda \equiv_x x \neq x$ .

It is important to understand that showing that (Naïve) is inconsistent is not sufficient to have found a paradox in the naïve theory of classes. One must also show that there is no way to modify comprehension salvage the naïve notion.

In 1904 Russell set out to evaluate Frege's response to his reformulation of Russell paradox in the system of his *Grundgesetze der Arithmetik*. In a 1904 manuscript "Points about Meaning and Denotation," Russell reveals that he had discovered an important theorem. For all one-one functions *f* such that,

$fx = fy \supset_{x,y} x = y$

the following yields a contradiction:

$(\exists w)(x \in w \equiv_x (\exists z)( x = fz .\&. x \notin z))$

There are many instances of *f*. When $fz = z$, we have Russell's paradox. When $fz = \iota z$ we have a contradiction as well. Indeed, this general form for generating contradictions undermines all hope to save the naïve notion. It shows that the following fails (Geach 1956):

$(\text{Naïve})^{+1}$

$(\exists w)(x \in w .\equiv_x. x \neq w \bullet \mathrm{A}x))$,

where *w* is not free in A. It also shows that the following fails (Quine, 1955):

$(\text{Naïve})^{-1}$

$(\exists w)( x \neq w .\supset_x . x \in w \equiv \mathrm{A}x))$,

where *w* is not free in A. It is Russell's 1904 theorem that reveals that it is Russell alone that deserves the credit of having found the paradoxes that undermine the naïve notion of a class. (See Landini, 2013).

Now Russell knew that a simple type theory of sets blocks contradictions from arising from the general form

$(\exists w)(x \in w \equiv_x (\exists z)( x = fz .\&. x \notin z))$,

where *f* is a one-one function. Simple type regimented form yields the following:

$(\exists w^{(t)})(x^t \in w^{(t)} \equiv_{x^t} (\exists z^{(t)})( x^t = fz^{(t)} .\&. x^t \notin z^{(t)}))$.

No function *f* can be found in simple type-theory to produce a contradiction. The same point applies to a simple type theory of attributes in intension. No contradiction arises from

$(\exists \psi^{(t)})( \psi^{(t)}(x^t) \equiv_{x^t} (\exists \varphi^{(t)})( x^t = f(\varphi^{(t)}).\&. \sim \psi^{(t)}(x^t) )$.

But Russell's 1904 theorem was originally couched in his quantification theory of propositions, not in a modern quantification theory. Indeed, the theorem emerged from a paradox he had set out in Appendix B of *Principles.* In Appendix B of *Principles*, Russell observed that the following yields a contradiction:

$(\exists w)(x \in w \equiv_x \{(\exists m)( x = \{y \in m \supset_y y\} .\bullet. x \notin m)) \})$.

We arrive at the contradiction,

$\{p \in w \supset_p p\} \in w \equiv \{ p \in w \supset_p p\} \notin w$ .

Moreover, in 1904 Russell quickly found an analog for attributes, namely this:

$(\exists \varphi)(x \in w .\equiv_x . (\exists \psi)( x = \{ \psi p \supset_p p\}.\bullet. \sim\psi x))$.

This yields the contradiction

$\varphi\{ \varphi p \supset_p p\} \equiv \sim\varphi\{ \varphi p \supset_p p\}$.

As we see, a contradiction arises if we *pair* Russell's theory of propositions with a theory of sets (or attributes)—even a simple type-theory of sets (or attributes).

Now it is easy to generalize and arrive at the result that, when couched in a theory of Russellian propositions, and when *f* is a one-one function, both of the following generate contradictions:

$(\exists w)(x \in w .\equiv_x . (\exists z)( x = fz .\bullet. x \notin z))$

$(\exists \varphi)(\varphi x .\equiv_x . (\exists \psi)( x = f\psi .\bullet. \sim\psi x))$.

This holds even in simple type theory. Russell drew precisely this conclusion. The contradictions that now arise from pairing his theory of propositions with a simple type theory of sets or with a simple type theory of attributes are not blocked. The reason they arise, however, is that the one-one function *f* involved here is a function whose values are Russellian propositions. In the Appendix B paradox, *f* is a one-one function such that

$fm = \{y \in m \supset_y y\}$.

The proof of the Appendix B rests on the fact that identical Russellian propositions have identical constituents. In particular, Russell's theory of propositional identity assures,

$$\{y \in w \supset_y y\}=\{y \in m \supset_y y\} .\supset. w = m.$$

This assures that the function *f* in question is one-one. The same point applies where the one-one function *f* involved is this

$$f\psi = \{\psi p \supset_p p\}.$$

Russell's ontology of propositions assures that

$$\{\varphi p \supset_p p\} = \{\psi p \supset_p p\}.\supset. \varphi x \equiv_x \psi x .$$

This makes the function *f* one-one. The source of these paradoxes, unlike Russell's paradox (when couched in a theory which takes membership "∈" as a primitive), is a violation of Cantor's theorem that there can be no function from any subset of entities *onto* sets/attributes of those entities. Russell's ontology of propositions, when paired with (a simple type) ontology of sets or of attributes, assures the existence of functions in violation of Cantor's theorem.

Russell rightly concluded that his theory of propositions is incompatible with a type regimented ontology of attributes (or classes/sets). But, at the time, this seemed to him to pose no telling difficulty. Russell advanced a substitutional theory of propositional structure that showed the way to formulate a theory of $^{cp}$Logic that avoids the adoption of any comprehension principles of attributes (or classes/sets) The theory accepts the existence of some universals, and indeed it takes some attributes (universals) to exist of logical necessity. Indeed, the sign "⊃" stands for a relation, and it is wholly type free. But the substitutional theory offers an approach to $^{cp}$Logic that is a "no-comprehension principles for attributes (classes/sets). The substitutional theory endeavors to *emulate* a simple impredicative type theory of attributes in intension. Using scope distinctions, it constructs extensional contexts from the intensional contexts of propositions and thereby emulates a simple-type theory of classes and relations-in-extension.

The language of the substitutional theory adheres to Russell's fundamental principle that the calculus for logic should adopt only individual variables. All lower-case letters of the English alphabet are employed as individual variables. All variables are of the substitutional theory are genuine individual variables, and quantification is intended as objectual. There are no special propositional variables in the language. Thus, *p, q, r*, and so on, are individual variables no less so than *x, y, z* etc. The theory is built upon the language of the quantificational calculus of propositions by adding a new atomic *wff*

$$p \, \frac{x}{a} \, ! \, q$$

which says that *q* is exactly like *p* but containing entity *x* wherever *p* contains entity *a*. More conveniently, Russell reads it as saying that *q* results by substituting *x* for *a* in *p*. But it is important to keep in mind that substitution is not something we do. It is about the entities involved and it makes no distinction between entities that are propositions and those that are not propositions. Russell was well aware that the replacement of the letters "*x*" and "*a*" in the *wff* "*p* " may well *not* track the ontological matter of substitution. For example, if the variables "*a*" and "*v*" are assigned to the same entity, then

$$\{a = v\} \frac{x}{a} !\{x = v\}$$

is wrong. The proper outcome is $\{x = x\}$.

The substitutional theory has nothing to do with a substitutional semantics for quantification theory. It assumes that we are to accept a semantics for quantification theory that fully Realist (objectual). Moreover, the substitutional theory of propositional structure is not a theory of types of entities. It is an agnosticism about types of entities. It is an agnosticism about classes and relations-in-extension. The substitutional theory endeavors to show how to capture mathematics within the ontology of logic without adopting comprehension axioms schemas for universals and without comprehension axiom schemas for classes (and relations-in-extension). It *emulates* a theory of types of attributes (and thus it emulates a theory of classes as their extensions) in a wholly type-free theory. In this way, Russell intended it to offer a genuine *solution* of just those paradoxes impacting the Cantorian revolution in mathematics, e.g., the Russell paradoxes, Cantor's greatest cardinal, the Burali-Forti greatest ordinal. (Recall the Richard, the Berry, the König-Dixon, are dismissed as semantic pseudo-paradoxes based on equivocations.)

An outline of the axiom schema for a reconstruction of Russell's 1905 substitutional theory of propositional structure is outlined below. The term vs wff distinction is rigidly adhered to. The letters $\alpha$, $\beta$ , $\delta$, and so forth are for any terms, quantifier-free or otherwise. We have:

$^{1905}S_1$ $\alpha .\supset. \beta \supset \alpha$

$^{1905}S_2$ $\alpha .\supset. \beta \supset \delta :\supset: \beta .\supset. \alpha \supset \delta$

$^{1905}S_3$ $\alpha \supset \beta :\supset: \beta \supset \delta .\supset. \alpha \supset \delta$

$^{1905}S_4$ $\alpha \supset \beta .\supset. \alpha :\supset: \alpha$

$^{1905}S_5$ $(x)Ax .\supset. A[\alpha | x]$,

where $\alpha$ is free for free occurrences of *x* in A.

$^{1905}S_6$ $(x)(\alpha \supset Ax) .\supset. \alpha \supset (x)Ax$,

where *x* is not free in $\alpha$.

$^{1905}S_7$ $\alpha$ in $\{A\alpha\}$

$^{1905}S_8$ $\alpha$ in $\{A\beta_1,\ldots, \beta_n\}$ $.\supset.$ $\alpha = \{A\beta_1,\ldots, \beta_n\}$ $.\vee.$ $\alpha$ in $\beta_1$ $.\vee.,\ldots,.\vee.$ $\alpha$ in $\beta_n$,

where A is any *wff* all of whose distinct free terms are $\beta_1,\ldots, \beta_n$ .

$^{1905}S_9$ $(x, y)(x \text{ in } y .\bullet. y \text{ in } x :\supset: x = y)$

$^{1905}S_{10}$ $(x, y, z)(x \text{ in } y . \bullet. y \text{ in } z :\supset: x \text{ in } z)$

$^{1905}S_{11}$ $(p, a)(q)(x, y)( p \frac{x}{a} !q .\bullet. p \frac{y}{a} !q .\bullet. a \text{ in } p :\supset: x = y)$

$^{1905}S_{12}$ $(p, a)(z)(\forall q)( p \frac{z}{a} !q .\bullet. a \text{ in } p .\bullet. a \neq p :\supset: z \text{ in } q .\bullet. z \neq q)$

$^{1905}S_{13}$ $(x, y)( x \frac{y}{x} !y )$

$^{1905}S_{14}$ $(x, y)( x \frac{y}{y} !x )$

$^{1905}S_{15}$ $(p, a)(\forall x)(\exists q)( p \frac{z}{a} !q .\bullet. (r)( p \frac{z}{a} !r :\supset: q = r))$

$^{1905}S_{16}$ $(p)(\exists q)(q \text{ ex } p)$

$^{1905}S_{17}$ $(\exists u)( u \text{ out } \{Au|v\}) \supset (u)( \{A\alpha|v\} \frac{u}{a} !\{Au|v\}),$

where $a$ and $u$ are free for $v$ in A.

$^{1905}S_{18}$ $(\exists u_1,\ldots,u_n)( \alpha \text{ out } \{Au_1|v_1,\ldots, u_n|v_n\}) :\bullet: \alpha \neq \{Au_1|v_1,\ldots, u_n|v_n\}$

$.\bullet. \alpha \neq \beta_1 .\bullet. ,\ldots, .\bullet. \alpha \neq \beta_n .:\supset:.$

$(x)(\exists u_1,\ldots, u_n)( \{A\sigma_1|v_1,\ldots,\sigma_n|v_n\} \frac{x}{a} !\{Au|v\}/\forall;x!\{Au_1|v_1,\ldots, u_n|v_n\} :\bullet:$

$\sigma_1 \frac{x}{a} u_1 .\bullet. ,\ldots, .\bullet. \sigma_n \frac{x}{a} u_n )$

where each $u_i$ and $\sigma_i$ , $1 \leq i \leq N$, are free for their respective $v_i$ in A,

and $\beta_1,\ldots, \beta_n$ are all the terms occurring free in A.

$^{1905}S_{19}$ $\{(u)Au\} = \{(v)Av|u\}$, where $v$ is free for $u$ in A.

$^{1905}S_{20}$ $\{(u)Au\} = \{(u)Bu\} .\supset. (u)(\{Au\} = \{Bu\})$

$^{1905}S_{21}$ $\{(u)Au\} \neq \{\alpha \supset \beta\}$

$^{1905}S_{22}$ $\{\alpha \supset \beta\} \neq \supset .\bullet. \{(u)Au\} \neq \supset$

$^{1905}S_{23}$ $\alpha \textit{ in } \{(\forall u)Au\} .\bullet. \alpha \neq \{(u)Au\} .\supset. (u)( \alpha \textit{ in } \{Au\}),$

where $\alpha$ is not the individual variable $u$ and $u$ is not free in $\alpha$.

*Modus Ponens*

From A and $\{A\} \supset \{B\}$, infer B

*Universal Generalization*

From A$x$ infer $(\forall x)Ax$.

Definition of logical particles are as before. But now Russell includes the following:

$\sigma = \beta =df (p, a, q, r)( p \frac{\sigma}{a} !q \bullet p \frac{\beta}{a} !r :\supset: q \supset r).$

$\alpha \text{ out} \beta =df (x)( \beta \frac{x}{a} !\beta).$

$\alpha \text{ in } \beta =df \sim (\alpha \text{ out } \beta).$

$\alpha \text{ ind } \beta =df \alpha \text{ out } \beta .\bullet. \beta \text{ out } \alpha.$

$\alpha \text{ ex } \beta =df \sim (\exists x)( x \text{ in } \alpha .\bullet. x \text{ in } \beta).$

This reconstructs the system that Russell set out in a manuscript of 22 December 1905. The axiom schemata S19, S20 and S21 are adapted from Church (1984).

Russell's substitutional theory endeavors to extend the type-free quantificational calculus of propositions to emulate a simple type theory of universals (and thereby classes). As we can see, the discovery of the theory of definite descriptions was central to the theory. In the June 1905 manuscript "On Fundamentals" most of the details of the new symbolism and theory of definite descriptions, including the matter of scope, are already worked out. In the manuscript (already mentioned) called "On Substitution" dated 22 December 1905, we find Russell using the formalism for the theory of definite descriptions. The theory of definite descriptions is employed to form definite descriptions of propositions. The expression $p \frac{x}{a} !q$ says that (proposition) $q$ is structurally exactly like $p$ except containing $x$ where $p$ contains $a$. Russell then has:

$$p \frac{x}{a} =df\, (\iota q)( p \frac{x}{a} !q).$$

The contextual definitions for definite descriptions apply. To emulate simple-type indexed predicate variables $\varphi^{(o)}$ Russell uses two individual variables $p$ and $a$. Consider this pair:

$$(\varphi^{(o)})( x^{o} ) \sim\varphi^{(o)}(x^{o})$$

$$(p)(a)(x)\sim( p \frac{x}{a} ).$$

The first is in the simple type regimented language of predicate variables. The second is its translation into the language of substitution. Similarly, consider this:

$$(\varphi^{(o)})( \varphi^{(o)}(x^{o}) \supset_{x^{o}} \varphi^{(o)}(x^{o}))$$

$$(p)(a)( p \frac{x}{a} \supset_{x} p \frac{x}{a} ).$$

When $\varphi^{(o)}(x^{o})$ is embedded, we can employ a definite description of a proposition. But when it is not, we have to write it out as $(\exists q)( p \frac{x}{a} !r \equiv_{r} r = q \,.\bullet.\, q)$. Of course, since the substitutional theory has axiom $^{1905}S_{15}$ assuring the uniqueness of the outcome of a substitution, we can simply write $(\exists q)( p \frac{x}{a} !q \,.\bullet.\, q)$.

Now the number of bound individual variables increases as we ascend logical types. The expression $s \frac{p,a}{t,w} !q$ says that $q$ is structurally exactly like s except containing $p$ wherever $s$ contains $t$ and containing $a$ wherever $s$ contains $w$. This is defined by a careful succession of substitutions. The complicated definition needn't detain us. Russell has

$$s \frac{p,a}{t,w} =df\; (\iota q)( s \frac{p,a}{t,w} !q ).$$

The substitutional theory emulates the binding a predicate variable of type $\psi^{((o))}$ by using three individual variables $s$, $t$ and $w$. For example, consider the following transcription pair:

$$(\forall\psi^{((o))})( \psi^{((o))}(\varphi^{(o)}) \supset_{\varphi^{(o)}} \psi^{((o))}(\varphi^{(o)}) )$$

$(s, t, w)( s\,\frac{p,a}{t,w} \supset_{p,a} s\,\frac{p,a}{t,w} )$.

As we see, the language of an impredicative simple-type regimented theory of attributes is emulated by the substitutional theory.

Every instance of the comprehension axiom schema for a simple-type theory of attributes is emulated within the substitutional theory. For example, the relation of identity is captured in a simple type theory of attributes by the following instance of comprehension. We have:

$$(\exists\varphi^{(o)})( \varphi^{(o)}(x^o, y^o ) \equiv_{x^o,y^o} x^o = y^o),$$

$$(\exists\, p ,a) ( p\,\frac{x,y}{a,b} \equiv_{x,y} \{x = y\}).$$

To emulate the comprehension of attributes of attributes of individuals, consider the following attribute of attributes of entities:

$$(\exists\psi^{((o))})( \psi^{((o))}(\varphi^{(o)} ) \equiv_{\varphi^o} (\forall x^o)\sim \varphi^{(o)}( x^o) )$$

$$(\exists\, s, t ,w) ( s\,\frac{p,a}{t,w} \equiv_{p,a} \{(x)( \sim(p\,\frac{x}{a} )\}).$$

Observe that the number of substitutions does not fully track types of attributes. The way the substitution is used is important as well. But the substitutional theory shows why Russell says his theory of definite descriptions "…was the first step toward overcoming the difficulties which has baffled me for so long" (*Auto*, p. 229).

The substitutional theory offers a genuine syntactic *solution* of the paradoxes that arise in naïve theories that assume comprehension principles for universals or classes. The paradoxical constructions are ungrammatical. Observe that the following expression is well-formed:

$$p\,\frac{p\frac{x}{a}}{a}\,!t\;.$$

We have only to recall that $p\,\frac{x}{a}$ abbreviates the definite description $(\iota q)(p\,\frac{x}{a}\,!q)$. Hence, by applying a version of Russell's contextual definition *14.01 we have:

$$p\,\frac{p\frac{x}{a}}{a}\,!t =df\ (\exists q)( p\,\frac{x}{a}\,!r \equiv_r r = q \,.\bullet.\, p\,\frac{q}{a}\,!t).$$

In stark contrast, the expression φ(φ) which violates simple-type theory cannot be emulated in the syntax of the substitutional theory. It would require the ungrammatical expression

$$p\,\frac{x,y}{a}\,!q\;.$$

It is meaningless to speak of substituting two entities *x* and *y* for one entity *a* in entity *p*. At the same time every formula of the simple impredicative type theory of universals in intension can be emulated in the substitutional theory.