Struck by the tedium of working in the quantification theory of propositions of *The Principles of Mathematics*, which we set out in an Appendix below, Russell explored different axioms for the quantificational theory of propositions in 1905. He published this quantification theory of proposition in a paper called "The Theory of Implication." The early Russellian language of quantification theory of propositions uses the horseshoe sign '⊃' in a way that is quite different from its use in modern logic. Its modern use it is flanked by *wffs* to form *wffs*. Russell's sign is flanked by terms to form a *wff*. Let us use the sign 'Ɔ' to mark the distinction.

The primitive signs of Russell's formal language of propositions are the individual variables, $x_1, \ldots, x_n$. But, for convenience, we can follow Russell and use any lower-case letter of the English alphabet as an individual variable. The language has as primitives the sign '**Ɔ**', the braces '(,)' and brackets '{ , and }'. The terms are defined thus: 1) individual variables are terms; (2) if A is a *wff*, then {A} is a term; (3) there are no other terms. The atomic *wffs* are of the form

x Ɔ *y*

where *x* and *y* are individual variables. The *wffs* are defined as the smallest set K such that all atomic *wffs* are in K and such that if α and β are terms in A is a *wff* containing *x* free, then α Ɔ β and (*x*)A are in K. Thus, where *x* and *y* are individual variables, *x* Ɔ *y* is a well-formed formula saying that *x* implies *y*.

The sign 'Ɔ' is a relation sign for the relation of implication. It is flanked by terms to form a formula. The modern horseshoe sign '⊃' is a statement connective which is flanked by formulas to form formulas. The modern horseshoe sign does not stand for a relation. Using the horseshoe in the modern way, '*x* ⊃ *y*' is ungrammatical. We read the modern expression 'A ⊃ B,' as 'if A then B.' In contrast, the expression '*x* Ɔ *y*,' where *x* and *y* are individual variables, is perfectly grammatical. It says that *x* implies *y*. The modern expression 'A ⊃ B' is emulated in language of substitution with '{A} Ɔ {B}.' One must transform (nominalize) a formula 'A' into a term '{A}' since the sign 'Ɔ' must be flanked by terms and never *wffs*. Consider, for example, the modern statement:

If Ansel is present, then Austin is absent.

This is translated into the language of propositions as:

Ansel's being present implies Austin's being absent.

Similarly, consider the modern statement:

Ansel is present, then if Austin is absent then Ansel is present.

This is translated into Russell's language of propositions as:

Ansel's being present implies Austin's being absent's implying Ansel's being present.

Of course, it is tedious to have to put

x Ɔ {*y* Ɔ *x*}

which is read as saying that x implies y's implying x. For convenience, Russell allows a subject position to mark a nominalizing transformation. Thus, we can write

$$x .\supset. y \supset x$$

instead of the more exacting expression '$x \supset \{y \supset x\}$.' Similarly,

$$A . \supset. B \supset A$$

is allowed as a convenience instead of the more exacting '$\{A\} \supset \{\{B\} \supset \{A\}\}$.' But we must never lose sight of the fact that terms, not *wffs*, flank the horseshoe sign of Russell's pre-*Principia* grammar for the language of logic.

Where A, B are schematic for a *wffs* and α, β, and δ are schematic for well-formed terms of the language, Russell develops various systems for quantification theory. One such system is the following:

Ax 1 $\alpha .\supset. \beta \supset \alpha$

Ax 2 $\alpha .\supset. \beta \supset \delta :\supset: \beta .\supset. \alpha \supset \delta$

Ax 3 $\alpha \supset \beta :\supset: \beta \supset \delta .\supset. \alpha \supset \delta$

Ax 4 $\alpha \supset \beta .\supset. \alpha :\supset: \alpha$

Ax 5 $(x)Ax .\supset. A\alpha$,

where α is free for free occurrences of x in A.

Ax 6 $(x)(\alpha \supset Ax) .\supset. \alpha \supset (x)Ax$,

where x is not free in α.

Modus Ponens From A and $\{A\} \supset \{B\}$, infer B

Universal Generalization From Ax infer $(x)Ax$.

The logical particles of the quantification theory of propositions are defined in terms of the relation of implication. Russell has:

$\alpha \vee \beta =df\ \alpha \supset \beta .\supset. \beta$

$\alpha \bullet \beta =df\ \sim(\alpha \supset \sim\beta)$

$\alpha \equiv \beta =df\ (\alpha \supset \beta) \bullet (\beta \supset \alpha)$.

$\sim\alpha =df\ \alpha \supset f$.
$f =df\ (x, y)(x \supset y)$
$(\exists x)Ax =df \sim (x) \sim Ax$.

An alternative Russell considers is to put

$\sim\alpha =df\ (x)(\alpha \supset x)$.

On this approach, $\sim\alpha$ says that α implies everything. On both of these approaches to defining Russell's tilde sign, the propositional part of the calculus is not separated from the quantificational theory. *The Principles of Mathematics* never separated a quantification-free subcomponent. Russell came to hold that such a separation requires him to take the tilde sign as primitive and add axioms accordingly. Axiom 4 is called ‘Reduction’ and it is particularly important for an axiomatization with tilde defined using quantification. If tilde is taken as a primitive sign, however, the Reduction axiom schema becomes unimportant. The quantification theory of propositions is consistent. Moreover, it is semantically complete with respect to analogs (in the language of propositions) of the logical truth quantification theory (Robbin 1969, p. 14).

In his paper “The Theory of Implication,” and also in a letter Russell wrote to Frege on December 12th,1904 (Frege 1980, p. 169), Russell also imagines a definition of his tilde as follows:

$$\sim\alpha =df\ (r)\ r\ .$$

Unfortunately, ‘$(r)\ r$’ should be regarded as grammatically ill-formed unless it is to be an abbreviation of one or another of the following:

$$(r)r\ =\ df\ (x)(x \supset \mathrm{r})$$

$$(r)r\ =\ df\ (x)(x \supset x\ .\supset.\ r \supset x\ ).$$

As we shall see in our Appendix, this issue turns out to be important when it comes to understanding how certain errors crept into quantificational logic of propositions set out in *The Principles of Mathematics*.

Of course, in Russell’s quantification theory for the logic of propositions, only a well-formed formula can appear on a line of proof. No individual variable or complex term obtained by nominalizing a *wff* may occur on a line of proof. Confusion on this matter has led some to read

$$\alpha \bullet \beta$$

as if it were α is true and β is true, but this is misguided. There is no predicate ‘true’ in the formal logic of propositions. The expression ‘$\alpha \bullet \beta$’ says that

α’s implying *f’s* implying β implies *f.*

This is quite different from the ordinary modern notion of conjunction. Indeed, Russell allows the expression ‘$x \bullet y$.’ The expression ‘x & y’ with the modern conjunction sign ‘&’ is nonsense. Russell’s derived rule of *simplification* cannot be the incoherent rule:

From $\alpha \bullet \beta$ infer α.

This would yield the following unintelligible instance:

From $x \bullet y$ infer x.

In Russell’s language of propositions, the derived rule of *simplification* is this:

From $\{A\} \bullet \beta$ infer A.

As we can see, with a little care, we can avoid conflating Russell’s logical particles the modern logical connectives and understand Russell’s system quantification theory of propositions anew.

One intriguing feature of Russell's quantification theory of propositions is that it embraces a perfectly consistent form of what logical (non-psychological) aboutness that enables ontological *self-reference*. Russell's general propositions have logical aboutness. This notion, however, has nothing to do with intentionality characteristic of mind. It is simply that the law of universal generalization assures that general propositions quantify over everything, including themselves. Every instance of the following logical axiom of universal instantiation is accepted:

$(x)\mathrm{A}x \;.\supset.\; \mathrm{A}\{(x)\mathrm{A}x\}.$

For example, consider the following:

$(x)(x = x) \;.\supset.\; \{(x)(x = x)\} = \{(x)(x = x)\}.$

This form of ontological *self-reference* involves a sort of circularity, but it is *not* a source of contradiction or paradox. Confusions concerning this point abound in the literature on Russell. For example, Goldfarb writes (Goldfarb 1989, p. 29):

> The proposition, e.g., that no proposition having property is true can be expressed as
> (1) $(p)(\varphi p \supset \sim p)$
> …The Epimenides paradox can then be generated with no use of semantic notions. All that is required is a value for the propositional function φ in (1) that is uniquely satisfied by the proposition expressed in (1). Such a propositional function, it seems, would not be hard to imagine. Thus, it appears, a Ramseyan solution to the paradox is simply not available.

A similar claim can be found in Hilton (1990), Potter (2000) and Stevens (2005). For a great many years this mistaken interpretation has carried the day, and it has impeded progress toward understanding the historical development of Russell's philosophy of logic.

Russell's quantificational logic of propositions allows an ontological *self-reference* which is not problematic in any way. There is no *wff* A of the language of the quantificational theory of Russell's propositional logic that satisfies the following:

$\mathrm{A}p \;.\equiv_p.\; p = \{(s)(\mathrm{A}s \supset \sim s)\}.$

The following propositional liar *cannot* be formulated in Russell's quantificational theory:

$m \text{ believes } p \;.\equiv_p.\; p = \{(s)(m \text{ believes } s \supset \sim s)\}.$

Indeed, in a manuscript dated June 1905 called "On Fundamentals" Russell explicitly dismissed any worry that a propositional Liar paradox might arise in his quantification theory of propositions. Russell recognized it is *not* the business of logic to address it. There is no relation of logical aboutness (or logical assertion) at all, and the language of the quantificational logic of propositions does not embrace expressions for contingent psychological relations such as that *belief* (or psychological *assertion* or and so forth) One has a great deal of conceptual room to investigate the vexing question of what contingent psychological theory might be the best account of psychological propositional attitudes and contingent psychological paradoxes (if any)

Sadly, this simple point is often missed by historians who do not stop to distinguish questions concerning the logically necessary existing propositions assured by Russell's theory and claims, which have no bearing on Russell's logic of propositions, that this or that contingently existing proposition exists. Historians often note that in *Principles*, Russell engaged with the question of whether there is a logical (non-psychological) notion of *assertion*—so that a logically asserted proposition '$x \supset y$' differs ontologically from what occurs unasserted occurring, for example, in the proposition '$\{x \supset y\} \supset y$.' But in *Principles* (*PoM*, p. 504), Russell also expressed doubt about the cogency of his concern that *logical assertion* is an ontological feature that must be respected in a formal quantification theory of propositions. And he soon completely dropped the notion. He never introduced a relation of 'logical assertion' which might give rise to a paradox. The confusion persists to this day, and with it comes the mistaken view that Russellian propositions are somehow involved paradoxes such as the Liar. No propositional Liar can be formulated in spite of the rampant form of ontological "self-reference" allowed in Russell's early quantificational theory of propositions.