According to the interpretation offered by *Principia*^{L}, the formal language of *Principia Mathematica* is that of simple type theory. One may well ask, therefore, how it came to be regarded as a ramified type theory of attributes in intension. The answer is that primarily it is due to the influential interpretation of Alonzo Church. We call this *Principia*^{C}.

Church was clear in admonishing his readers that he was not attempting reconstruction of the historical *Principia.* Indeed, he was working from Russell's 1907 paper, "Mathematical Logic as Based on the Theory of Types" (*ML*) and developing a system of orders of propositions which, in his view "… is clearly demanded by the background and purpose of Russell's logic, and in spite of what seems to be an explicit denial by Whitehead and Russell in *Principia*" (Church 1976, p. 291, *fn*4). In *ML,* the language of a ramified-type theory of attributes in intension (with bindable predicate variables adorned with *order* and type indices) is adopted, but *only* for notational convenience. A translation procedure is given so that bindable predicate variables adorned with order and type indices can be eliminated in favor of a language of proposition variables adorned with order indices. In order to understand this the theory of *ML*, we shall have to discuss the history of Russell's substitutional theory of propositional structure. This substitutional theory propositions was once imagined as the philosophical correct theory, and Whitehead and Russell had planned to put it in an appendix to the first edition of *Principia Mathematica*. But let us postpone these details for the time being and present Church's theory of ramified types (*r*-types).

In *Principia Mathematica*, Whitehead and Russell intended a nominalistic semantics for the predicate variables of the work. In this way, they hoped for the variables to get "internal limitations" grounded in their "significance conditions." The only genuine variables, that isthose given an objectual semantics, are the individual variables of lowest type 0. Whitehead and Russell's nominalistic semantics offers a recursive definition of truth and of falsehood for the *wffs* of *Principia Mathematica* and the base case of that recursion (the theory of "truth" for atomic *wffs*) is Russell's multiple-relation theory of judgment. Unfortunately, the nominalistic semantics Whitehead and Russell intended does not make valid all instances of *12.*n*. Consider this

*12.1 $(\exists f^{(o)})(f^{(o)}(x^o) \equiv_{x^o} \varphi(x^o)$,

where $f^{(o)}$ is not free in the *wff* $\varphi(x^o)$. If the truth-conditions render a meaning of "truth_n for the *wff* $\varphi(x^o)$", then that must be reflected in an *order* index on the predicate variable $f^{(o)}$, otherwise the nominalistic semantics cannot validate the instance. Thus, for example, the nominalistic semantics validates only this:

$(\exists\, {}^2f^{(o)})(x^o)(\, {}^2f^{(o)}\,(x^o) \equiv_{x^o} (\exists^1\psi^{(o)})\ {}^1\psi^{(o)}(x^o)\,)$.

This motivated Church to reconstruct a system on behalf of Russell which allows *non-predicative* predicate variables such as ${}^2f^{(o)}$ and ${}^3f^{(o)}$ and so on, whose *order* can be above the order of its simple type. Church thereby invents comprehension axioms for *predicative* types—that is, those whose order is the order of the simple type symbol and for non-predicative types. (These would be made valid by the informal nominalistic semantics set out in the Introduction to *Principia Mathematica's* first edition.) And he invents comprehension axioms for non-predicative types. Finally, Church then introduces the *ad hoc* axioms Reducibility such as the following:

$$({}^{n}f^{(o)})\,(\exists^{1}f^{(o)})({}^{1}f^{(o)}(x^{o})\ \equiv_{x^{o}}\ {}^{n}f^{(o)}\,(x^{o})).$$

Thus was born Church's system of *r*-types (ramified types). Let's call it '*Principia*^{C}.'

In short, Church invented a formal language and formal system for *Principia* which codes a ramified-type structure into the syntax of the predicate variables. The primitive signs of the language of *Principia*^{C} are as before: ∨, ~, (,), ' (prime), but let us now add ∀for the universal quantifier since *r*-type symbols can be daunting. Predicate variables and individual variables come with order\type symbols. The individual variables are x_1^{o}, x_2^{o},…, x_n^{o} (informally x^{o}, y^{o}, z^{o}), and the predicate variables are $x_1^{t_1}$, …, $x_n^{t_n}$, informally φ^{t}, ψ^{t} f^{t}, g^{t}. Church's system of *r*-types sets out a recursive definition of *r*-types as follows:

(i) There is an r-type *o* to which all and only individuals belong, and whose order is 0.

(ii) If m ∈ ω - {0}, and t_1,…, t_n are given *r*-types, then there is an *r*-type (t_1,…, t_n)/*m* to which belong all and only *n*-ary attributes of level *m* and with arguments of *r*-types t_1,…, $t_n$respectively.

(iii) The *order* of such an attribute of r-type (t_1,…, t_n)/*m* is M+*m*, where M is the greatest of the *orders* corresponding to the *r*-types t_1,…, t_n (and M = 0 if *m* = 0). An attribute of *r*-type (t_1,…, t_n)/*m* is *predicative iff m* = 1 .

As we noted, in stark contrast with *Principia*^{L} we can see that Church's *Prinicpia*^{C} accepts *non-predicative* as well as *predicative* predicate variables and permits cumulation. That is, an argument to a predicate variable may have any *order* less than that of the predicate variable to which it is an argument.

Church's notion of level keeps track of the *order* of the simple-type symbol. Indeed, if the level is kept always at 1, Church's notation is just a variant of the notation of simple-types where the order is always the order of the simple type symbol. To illustrate, consider Church's $\varphi^{(o)/1}$. Its order is 1 and this is computed by taking the order of the *r*-type symbol *o*, namely 0, plus the level 1. In the language of *Principia*^{L} this is simply the order of the simple-type symbol (*o*). Church's predicate variable $\varphi^{((o)/1)/1}$ is for an order 2 attribute of order 1 attributes of individuals. The order is 2 and this is computed by taking the order of the *r*-type (*o*)/1, namely 1 plus the level 1. In the terminology of *Principia*^{L} this is the order of the simple-type symbol ((*o*)), namely 2. Similarly, Church takes $\varphi^{(((o)/1)/1)/1}$ to be a predicate variable for an attribute of attributes of *r*-type ((*o*)/1) /1. The order here is 3**.** In the terminology of *Principia*^{L} this is the order of the simple type symbol (((*o*))), namely 3. These are monadic (one-place) attributes. Church uses $\varphi^{(o,o)/1}$ for a dyadic relation of individuals. A predicate variable $\varphi^{((o,(o)/1)/1}$is for a dyadic heterogeneous relation between an individual and an attribute of *r*-type (*o*)/1. Its order is 2. This is computed by taking the order of its highest order of argument, namely that of (*o)*/1, plus the level. Once again, in the terminology of *Principia*^{L} this is the order of the simple type symbol (*o*, (*o*)), namely 2. All these attributes are predicative attributes of predicative attributes (or individuals).

Church's grammar allows *non-predicative* variables such as these: $\varphi^{(o)/2}$, and $\varphi^{((o)/1)/2}$ and even $\varphi^{((o)/2)/1}$. These have no analogs in *Principia*^{L}, because they do not occur in the historical *Principia Mathematica.* As we noted, the historical *Principia Mathematica* demands that all and only variables be predicative. Because of such variables, Church offers a grammar for his *Principia*^{C} that is cumulative. That is, he allows *wffs* such as:

$\psi^{((o)/2)/1}(\varphi^{(o)/1})$

$\psi^{((o)/2)/1}(\varphi^{(o)/2})$

Church recognizes that such cumulative *r*-types nowhere to be found in the historical *Prinicpia.* Indeed, he regards $\psi^{((o)/2)/1}$ as predicative and as we shall see, it is accommodated in his Axiom of Reducibility which is independent of his comprehension principles for non-predicative *r*-type.

Church's atomic formulas are of the form,

$\varphi^{(t_1,\ldots,t_n)/m}\,(x_1^{\beta_1},\ldots,x_n^{\beta_n})$

where the type of β_i is equal to that of t_i, but the order may be less than or equal to the order of t_i. The *wffs* are the smallest set K containing all atomic *wffs* such that if A, B, C are *wffs* in K and $x^{t/n}$ is an individual variable free in C, then so are ~(A), A v B, and $(\forall x^{t/n})$C.

Together with *1.2-*1.6 of the sentential calculus, Church has the following axioms governing quantificaiton with *r*-types:

Church(*10.1)

$$(\forall x^{t/n})\text{A} \supset \text{A}(y^{\beta/m}),$$

where $y^{\beta/m}$ is free for $x^{t/n}$ in A and the *order* of the *r-type* of β/m is not greater than that of *t/n.*

Church(*10.12)

$$(\forall x^{t/n}))(\text{B} \vee \text{A})\ .\supset.\ \text{B} \vee (\forall x^{t/n})\text{A}.$$

where $x^{t/n}$ does not occur free in the *wff* B.

Church(*12.1*n non-Predicative* axiom schema)

$$(\exists\varphi^{(t_1,\ldots,t_n)/m})(\forall x_1^{\beta_1}),\ldots,(\forall x_n^{\beta_n})(\varphi^{(t_1,\ldots,t_n)/m})(x_1^{\beta_1},\ldots,x_n^{\beta_n}) \equiv \text{A}),$$

where $\varphi^{(t_1,\ldots,t_n)/m}$ is not free in the wff A and the *r*-types $\beta_1,\ldots,\beta_n$ are not greater in order than than the order of the r-types $t_1,\ldots,t_n$.

Church (*Reducibility*)

$(\forall\theta^{(t_1,\ldots,t_n)/m})\ (\exists\varphi^{(t_1,\ldots,t_n)/1})\ (\forall x_1^{\beta_1}),\ldots,(\forall x_n^{\beta_n})($

$\theta^{(t_1,\ldots,t_n)/m})(x_1^{\beta_1},\ldots,x_n^{\beta_n}) \equiv \varphi^{(t_1,\ldots,t_n)/1})(x_1^{\beta_1},\ldots,x_n^{\beta_n})\ ),$

As we see, Church's *Reducibility* axiom is quite separate from his comprehensive axiom schemata, which introduce non-predicative attributes in intension. Church's cumulative grammar plays a central role in his characterization of this axiom. Neither is to be found in the historical *Principia Mathematica.*

As we noted, Church's interpretation of *Principia* was based largely on Russell's 1907 paper *ML* where propositions were still a part of the theory. Church's interpretation has become the orthodoxy in spite of his admonishing readers that his intent was not to faithfully represent what is in the historical work. To properly understand the differences, we have to turn to Russell's substitutional logic of propositional structure. But before we can take up the formal logic of Russell's substitutional theory, we must first discuss the basic logic of quantification theory when couched in Russell's ontology of propositions.