10. Appendix B: The 1925 Experiment of *Principia* ^{W}

Principia ‘s analysis of mathematics rests on its comprehension axiom.*****12.*n* . Yet the semantics Russell intended for *Principia* was nominalistic and cannot validate *12.*n*. The order indices on predicate variables are philosophically explained in terms of the recursively defined hierarchy of meanings of “truth” and “falsehood,” with Russell’s multiple-relation theory handling the truth-conditions for the base (atomic) *wffs* for the recursion. Russell came to see that this philosophically grounds comprehension principles far weaker than *12.*n*.

The publication of a second-edition of *Principia* offered Russell an opportunity to investigate an idea he attributes to Wittgenstein. Reducibility would not be needed in a system in which there has been a complete analysis and re-conceptualization of all non-extensional contexts. Russell never endorsed this idea, and indeed he found it inadequate to mathematics because it fails to preserve Cantor’s work and Analysis. But he did investigate its merits in the second-edition of *Principia*. We shall call it *Principia* $^{\mathrm{W}}$

All too often, interpretations present Russell’s new introduction as abandoning the system of the first edition of *Principia* and endorsing the system of the second edition. This is quite salient in Monk’s excellently written book Wittgenstein, where he writes that in the new edition Russell was effectively “… abandoning Volumes II and III of the original work” (Monk, 2001, p. 46). Happily, this is not historically accurate. Russell expressed himself unequivocally in writing that he did not endorse the Wittgensteinian experiments of the new introduction and appendices B and C of the second edition. Quite to the contrary, his purpose was to evaluate them and he concluded that the Tractarian ideas, so interpreted, are a failure.

The grammar for the experiment that was *Principia* $^{\mathrm{W}}$ is quite different from that of *Principia’s* first edition. The new grammar introduces non-predicate variables ${}^{a}\varphi^{t}$, where *a* is any number greater than the *order* of the simple type symbol *t*. Moreover, and rather shockingly the grammar allows

$${}^{b}\psi^{(t)}\,(\,{}^{a}\varphi^{t})$$

a *wff* when $a > b$. As we can see, the grammar is radically different from that of *Principia* ^{L} and *Principia* ^{C}. One can have cases such as the following:

$${}^{2}\psi^{(t)}\,(\,{}^{5}\varphi^{t})$$

Unfortunately, the revisions to the formal grammar that Russell envisioned for the reconstruction of Wittgenstein’s ideas have been largely missed and the experimental system of the 1925 second edition have remined largely misunderstood. Admitting that he could not discern whether the system of the new edition was riddled “slipshod notations” or with outright errors, Myhill admitted that he simply found it unintelligible (Myhill, 1974, p. *xiv*). In contrast, we shall see below that Gödel (1944) must have found it intelligible since he found an error in its proof of theorem of *89.16 set out in its appendix B. Fortunately, a clearer picture of Russell’s new experimental grammar (as explained above) has recently come to the fore.

The system *Principia*^{W} introduces a schema of extensionality. Russell is clear in his endorsement of such a schema of extensionality even though it remains rather strikingly strong a thesis to adopt (*PM*, p. *xxxix*). There is controversy here, but perhaps the following represents what Russell had in mind. Where ${}^{a}P^{(t)}$ and ${}^{b}Q^{(t)}$ are any predicate terms of the system, and ${}^{m}y^{t}$ is used as a genuine object-language variable of the system, we have:

(EXT)

$(\forall {}^{m}y^{t})({}^{a}P^{(t)}({}^{m}y^{t}) \equiv {}^{b}Q^{(t)}({}^{m}y^{t})) .\supset. A \supset A^{*}$,

where A^{*} results from replacing some (not necessarily all) free occurrences of ${}^{a}P^{(t)}$ by free occurrences of ${}^{b}Q^{(t)}$ in A, and neither ${}^{a}P^{(t)})$ nor ${}^{b}Q^{(t)}$ are logical properties. (With all the type and order indices, the above adopting the convenience of using the sign ∀ for the universal quantifier.)

With the adoption of (EXT), there is no longer any need of the *Principia'*s no-classes emulation of class and relation-in-extension expressions—a construction that generates extensional context from intensional contexts. Given the extensionality of the system, Russell regards class and relation-in-extension abstracts as *themselves* complex predicate terms (*PM*, p. xxxix). Russell therefore uses the notation $\hat{x}\varphi x$ as a complex predicate term which avoids the formal pitfalls of using $\varphi\hat{x}$. It is more convenient to use lambda notation $[\lambda x\ Ax]$ to form a complex predicate terms instead of the notation $\hat{x}Ax$. But Russell clearly means to have

$a \in \hat{x}Ax =df\ [\lambda x\ Ax](a)$.

And in general (for relations) he would have:

$< a_1,\ldots, a_n> \in \hat{z}_1,\ldots, \hat{z}_n\ A(z_1,\ldots, z_n)\ =df\ [\lambda z_1,\ldots, z_n\ A(z_1,\ldots, z_n)](a_1,\ldots, a_n)$

In any case, the general point is that instead of comprehension axiom schema, the system allows

$\hat{z}_1,\ldots, \hat{z}_n\ A(z_1,\ldots\ z_n)$

as a *predicate* term with

$<a_1,\ldots, a_n> \in\ \hat{z}_1,\ldots, \hat{z}_n\ A(z_1,\ldots, z_n) \equiv A(a,\ldots, a)$

as an axiom schema. In the new grammar, an argument to a predicate variable of simple- type (*t*) must be of simple-type *t*, but its order can even be greater than the order of the predicate variable to which it is argument! This is a striking new feature.

Another striking feature of the new grammar of *Principia*^{W} is that predicate variables (and predicate terms) with *different* order regimentation an meaningfully flank the identity sign. This will at first appear to be outrageous. In a realist semantics for the system it suggests that universals ramified by different orders may nonetheless be "identical." On a realist semantics, univesals with different orders must surely be distinct. The explanation for both of these striking features this is that Russell intended a new nominalistic semantics for the predicate terms and variables of *Principia*^{W}. And he imagined that "identity" in the system interpreted as indiscernibility by the lights of the ideography of the formal system. The identity sign in *Principia*^{W} is defined just as in the 1910 *Principia*. (See *PM*, p. *xxxvii*.) Thus we put:

$$ {}^{a}P^{(t)} = {}^{b}Q^{(t)} =df \ (\forall {}^{c}\varphi^{((t))})({}^{c}\varphi^{((t))} ({}^{a}P^{(t)}) \equiv {}^{c}\varphi^{((t))} ({}^{b}Q^{(t)})). $$

where c = max(a,b) + 1. This definition of identity will allow for full substitutivity. Thus, the extensionality axiom assures that identity is sufficient for full substitutivity. This result will not, however, apply to individual variables. Presumably Russell also intends the following axiom, though he did not state it:

$$ {}^{o}x^{o} = {}^{o}y^{o} \ .\supset. \ A \supset A^{*}, $$

where A^* is just like A except that free ${}^{o}y^{o}$ replaces some (not necessarily all) free occurrences of ${}^{o}x^{o}$ in A.

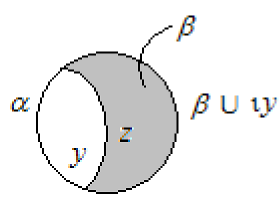
In Appendix *B* of the second edition, Russell attempted a proof of mathematical induction without assuming a reducibility principle. Gödel found a flaw in one of the lemmas, namely, (*89.16), and wondered if it could be avoided (Gödel, 1944, p. 145). The lemma is this:

$$ {}^{*}89.16. \quad \alpha \notin \text{Cls induct}_3 \ \& \ \gamma \in \text{Cls induct}_3 \ .\supset. \ \exists!\alpha - \gamma. $$

Gödel noticed that the proof employed the erroneous line

$$ \exists!\alpha - \beta \ \& \ \alpha \subseteq \beta \cup \iota y \ .\supset. \ \alpha = \beta \cup \iota y. $$

The error can be understood from the following diagram:



$$ \beta = (\beta \cup \iota y) \cap \breve{\alpha} $$

The class β is the shaded area. The antecedent conditions are met. There is something, namely y, that is in α and not in β, and α (which is the area in white) is a subset of $\beta \cup \iota y$ (which is the area inside the circle). Nonetheless, α is not equal to $\beta \cup \iota y$ as long as there is something, namely z, in β and not in α. Happily, a new proof can be found for *89.16 and the system of Appendix B restored. (See Landini, 1996).

Interestingly, if we do not restrict (EXT) we can prove a form of Reducibility in the system. In lowest type, the theorem is this:

1925Reduc

$$ ({}^{m}\psi^{(o)})(\exists\ \varphi^{(o)} ((\forall y^{o})({}^{1}\varphi^{(o)}(y^{o}) \equiv {}^{m}\psi^{(o)} (y^{o})). $$

It is important to recall, however, that the grammar of *Principia*^{W} has changed markedly. Thus, 1925 Reduc is certainly *not* Church's (*12.*n Reducibility*). One might be concerned that the provability of

1925Reduc shows that this strong principle (EXT) was not what Russell intended. (See Hazen 2004.) To be sure, with 1925Reduc, we can recover Cantor's power-class theorem and Analysis in *Principia*^{W}. Russell's assessment of the power of the system of the second edition would be, on this account, mistake. But this may well have been *precisely* the sort of result Russell had hoped for. (See Landini 2013).

In sum, Myhill showed that mathematical induction cannot be recovered if we work in what amounts to Church's system of *Principia*^{C} and simply drop Reducibility and append extensionality principles. But Myhill was not working with the new system, and the radically new grammar, set out in *Principia's* second edition. Russell's rectification of mathematical induction was a positive result on behalf of Wittgenstein that Russell was eager to herald in the new edition of *Principia*. But as we saw, all things considered Russell's assessment of Wittgenstein's idea was negative. *Principia's* second edition does not endorse the new system *Principia*^{W} and finds it inadequate for *Analysis* and the recovery of Cantor's work. Russell concluded that Cantor's power-class theorem and Analysis is not recoverable. There is no reason whatever to think that Russell embraced the system *Principia*^{W} as a replacement for the logic of *Principia.* The only replacement Russell endorses in the second edition is the elimination of free variables effected by Russell's directive that readers replace section *9 of the first edition with section *8 of the section edition.