Extremal Problems, Reconstruction Formulas and Approximations of Gaussian Kernels

by

Felipe Ferreira Gonçalves, B.S., M.S.

DISSERTATION

Presented to the Faculty of

IMPA - Instituto Nacional de Matemática Pura e Aplicada

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

Instituto Nacional de Matemática Pura e Aplicada

Rio de Janeiro - Brasil

April 2016

To my mother Alice and my grandmother Maria.

Acknowledgments

I would like to express my deepest gratitude to my advisor Emanuel Carneiro for his guidance, support and motivation.

I would like thank the people with whom I worked in collaboration on some of the results presented in this thesis and from whom I learned so much: Professors Emanuel Carneiro, Michael Kelly, Friedrich Littmann and José Madrid. I am also deeply thankful to Professors Jeffrey Vaaler, William Beckner and Dimitar Dimitrov for their support during my Ph.D. studies.

I would like to acknowledge the financial support from CNPQ–Brazil (2012–2014) and FAPERJ–Brazil (2014–2016). I hereby also express my full appreciation for the Science Without Borders program (2015) which was crucial during my Ph.D. studies at The University of Texas at Austin.

I would like to also dedicate this thesis to my fellow Ph.D. students. I am only one among many excellent emerging mathematicians.

I owe a great deal of this journey to my dear childhood friends: Luís Gamma, Simão Neto, Thiago Sampaio e Vittorio Talone. For all the music, drinks and cigars that we enjoyed together.

I am deeply grateful to my lovely wife Kate. Her unconditional love and passionate caring during my Ph.D. adventures was fundamental for my success. The accomplishments of this thesis are meaningless without her in my life.

Finally, I would like to give a special thanks to my father João and my sister Fernanda for believing in my potential.

Extremal Problems, Reconstruction Formulas and Approximations of Gaussian Kernels

Felipe Ferreira Gonçalves, Ph.D.
IMPA, 2016

Supervisor: Emanuel Carneiro

In this Ph.D. thesis we discuss 3 different problems in analysis: (a) best approximations by functions with compactly supported Fourier transform for a variety of functions in one and several variables; (b) reconstruction of entire functions of exponential type via interpolation formulas involving derivatives; (c) a central limit theorem for operators, which characterizes the operators given by centered Gaussian kernels as the limiting family.

Nesta tese de Doutorado discutimos 3 problemas diferentes em análise: (a) melhores aproximações por funções com transformada de Fourier suportada em compactos para uma grande variedade de funções de uma e várias variáveis; (b) reconstrução de funções inteiras de tipo exponencial através de fórmulas de interpolação envolvendo derivadas; (c) um teorema central do limite para operadores, que caracteriza os operadores dados por núcleos Gaussianos centrados como a família limite.

Table of Contents

Chapter 1

Introduction

This Ph.D. thesis is composed by four independent chapters that are embedded in two central themes: *Approximation and Reconstruction*. This thesis compiles the developments of the following research articles:

[A1] Extremal Problems in de Branges Spaces: The Case of Truncated and Odd Functions (with Emanuel Carneiro), Mathematische Zeitschrift 280 (2015), 17–45.

[A2] Interpolation Formulas with Derivatives in de Branges Spaces, Trans. Amer. Math. Soc. (to appear).

[A3] One–Sided Band–Limited Approximations in Euclidean Spaces of Some Radial Functions (with Michael Kelly and José Madrid), Bull. Braz. Math. Soc. 46 (2015), no. 4, 563–599.

[A4] Interpolation Formulas with Derivatives in de Branges Spaces II (with Friedrich Littmann), Preprint (2015).

[A5] On Selberg's Box–Minorant Problem (with Jacob Carruth and Michael Kelly), Preprint (2015).

[A6] A Central Limit Theorem for Operators, Preprint (2015).

In Chapter 2 we find extremal one–sided approximations of exponential type for a class of truncated and odd functions with a certain exponential subordination. These approximations optimize the $L^1(\mathbb{R}, |E(x)|^{-2}\mathrm{d}x)$–error, where $E(z)$ is an arbitrary Hermite–Biehler entire function. This extends the work of Holt and Vaaler [47] for the signum function. We also provide periodic analogues of these results, finding optimal one–sided approximations by trigonometric polynomials of a given degree to a class of periodic functions with exponential subordination. These extremal trigonometric polynomials optimize the $L^1(\mathbb{R}/\mathbb{Z}, \mathrm{d}\vartheta)$–error, where $\mathrm{d}\vartheta$ is an arbitrary nontrivial measure on $\mathbb{R}/\mathbb{Z}$. The periodic results extend the work of Li and Vaaler [51], who considered this problem for the sawtooth function with respect to Jacobi measures. This chapter describes the results of article [A1] which is a joint work with Emanuel Carneiro (IMPA – Brazil).

In Chapter 3 we derive interpolation formulas involving derivatives for entire functions in reproducing kernel Hilbert spaces. We extend the interpolation formula derived by Vaaler in [73, Theorem 9] to de Branges spaces. We extensively use techniques from de Branges' theory of Hilbert spaces of entire functions, but a crucial passage involves the Hilbert–type inequalities as derived in [18]. Next, we extend our interpolation result to L^p de Branges spaces for which the structure function $E(z)$ of Hermite–Biehler class satisfies an additional hypothesis. Finally, we give three applications: (1) we derive an interpolation formula for entire functions of exponential type where the nodes of interpolation are the zeros of Bessel functions; (2) we prove a uniqueness result for extremal one–sided band–limited approximations of radial functions in Euclidean spaces; (3) we derive a condition for sampling and interpolation with derivatives in Paley–Wiener spaces in analogy with the work of Ortega–Cerdà and Seip [64]. This chapter compiles the results of articles [A2] and

[A4], the last one being a joint work with Friedrich Littmann (North Dakota State Univ. – USA).

Chapter 4 is separated in two independent, but correlated parts. In the *first part* we construct best approximations by band–limited functions in many variables for a class of functions that are subordinated to Gaussians. The majorants that we construct are shown to be extremal and our minorants are shown to be *asymptotically extremal* as the type becomes uniformly large. We then use our methods to derive periodic analogues of the main results. In the *second part* we study the problem of minorizing the indicator function of a box by a function with Fourier transform supported in the same box in such a way that the integral is maximized. This problem dates back to the work of Selberg, in which he developed a method to construct minorants of the box by using one–dimensional minorants and majorants of the same box. The drawback is that Selberg's minorant usually have *negative* integrals. First, we show that the interpolation strategy, which is very powerful in the one–dimensional case, fails to provide candidates in higher dimensions. Secondly, we prove that the integral of the best approximation decreases as the dimension N increases, and converges to zero as $N \to \infty$. Finally, we derive explicit *positive* lower bounds for dimension $N \leq 5$ and we estimate the critical N such that a quantity related to Selberg's construction vanishes. This chapter summarizes the results of articles [A3] and [A5]. The first one is a joint work with Michael Kelly (Univ. of Michigan – USA) and José Madrid (IMPA – Brazil), while the second one is a joint work with Jacob Carruth (The Univ. of Texas at Austin – USA) and Michael Kelly.

In Chapter 5 we prove an analogue of the Central Limit Theorem for operators. For every operator K defined on $\mathbb{C}[x]$ we construct a sequence of operators K_N defined on $\mathbb{C}[x_1, ..., x_N]$ and show that, under certain orthogonality conditions, this sequence converges in a weak sense to a unique operator. We show that the family of limiting operators $\mathcal{C}$ coincides with the family of operators given by centered Gaussian Kernels. Inspired in the approximation method used by Beckner in [4] to prove the sharp form of the Hausdorff–Young inequality, we show that Beckner's method is a special case of this general approximation method for operators. In particular, we characterize the Hermite semi–group as the family of limiting operators $\mathcal{C}$ associated with any multiplicative semi–group of operators. This chapter contains the results of article [A6].

Chapter 2

Extremal Functions of Exponential Type

2.1 Preliminaries

An entire function $F : \mathbb{C} \to \mathbb{C}$, not identically zero, is said to be of *exponential type* if

$$\tau(F) = \limsup_{|z|\to\infty} |z|^{-1} \log |F(z)| < \infty.$$

In this case, the non–negative number $\tau(F)$ is called the exponential type of $F(z)$. We say that $F(z)$ is *real entire* if $F(z)$ restricted to $\mathbb{R}$ is real valued. Given a function $f : \mathbb{R} \to \mathbb{R}$, a non–negative Borel measure $\mathrm{d}\sigma$ on $\mathbb{R}$, and a parameter $\delta > 0$, we address here the problem of finding a pair of real entire functions $L : \mathbb{C} \to \mathbb{C}$ and $M : \mathbb{C} \to \mathbb{C}$ of exponential type at most δ such that

$$L(x) \leq f(x) \leq M(x) \tag{2.1}$$

for all $x \in \mathbb{R}$, minimizing the integral

$$\int_{\mathbb{R}} \big\{M(x) - L(x)\big\} \,\mathrm{d}\sigma(x). \tag{2.2}$$

In the case of Lebesgue measure, this problem dates back to the work of Beurling in the 1930's (see [73]), where the function $f(x) = \mathrm{sgn}(x)$ was considered. Further developments of this theory provide the solution of this extremal problem for a wide class of functions $f(x)$ that includes, for instance, even, odd and truncated functions subject to a certain exponential or Gaussian subordination [14, 15, 18, 19, 20, 26, 40, 53, 55, 56, 73]. Several applications of these extremal functions arise in analytic number theory and analysis, for instance in connection to: large sieve inequalities [47, 60, 67, 68, 73], Erdös-Turán inequalities [19, 45, 51, 73], Hilbert–type inequalities [16, 18, 19, 40, 55, 62, 63, 73], Tauberian theorems [40], inequalities in signal processing [26], and bounds in the theory of the Riemann zeta–function [8, 10, 9, 22, 32, 34]. Similar approximation problems are treated, for instance, in [41, 33].

In the case of general measures $\mathrm{d}\sigma$, the problem (2.1) - (2.2) is still vastly open. In the remarkable paper [47], Holt and Vaaler considered the situation $f(x) = \mathrm{sgn}(x)$ and $\mathrm{d}\sigma(x) = |x|^{2\alpha+1}\,\mathrm{d}x$ with $\alpha > -1$. They solved this problem (in fact, for a more general class of measures) by establishing an interesting connection with the theory of de Branges spaces of entire functions [6]. This idea was further developed in [16] for a class of even functions $f(x)$ with exponential subordination and in [9, 56] for characteristic functions of intervals, both with respect to general de Branges measures. In particular, the optimal construction in [9] was used to improve the existing bounds for the pair correlation of zeros of the Riemann zeta–function, under the Riemann hypothesis, extending a classical result of Gallagher [32].

The purpose of this chapter is to complete the framework initiated in [16], where the case of even functions was treated. Here we develop an analogous extremal theory for a wide class of *truncated and odd functions* with exponential subordination, with respect to general de Branges measures (these are described below). In particular, this extends the work of Holt and Vaaler [47] for the signum function.

2.1.1 De Branges Spaces

In order to properly state our results, we need to briefly review the main concepts and terminology of the theory of Hilbert spaces of entire functions developed by de Branges [6]. More information about these spaces and their L^p version can be found in Appendix 6.1.

We briefly review the basics of de Branges' theory of Hilbert spaces of entire functions [6]. A function $F(z)$ analytic in the open upper half–plane

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \ \mathrm{Im}\,(z) > 0\}$$

has *bounded type* if it can be written as the quotient of two functions that are analytic and bounded in $\mathbb{C}^+$. If $F(z)$ has bounded type in $\mathbb{C}^+$ then, according to [6, Theorems 9 and 10], we have

$$\limsup_{y\to\infty} y^{-1} \log |F(iy)| = v(F) \in \mathbb{R}.$$

The number $v(F)$ is called the *mean type* of $F(z)$. We say that an entire function $F : \mathbb{C} \to \mathbb{C}$, not identically zero, has *exponential type* if

$$\limsup_{|z|\to\infty} |z|^{-1} \log |F(z)| = \tau(F) < \infty.$$

In this case, the non-negative number $\tau(F)$ is called the *exponential type* of $F(z)$. If $F : \mathbb{C} \to \mathbb{C}$ is entire we define $F^* : \mathbb{C} \to \mathbb{C}$ by $F^*(z) = \overline{F(\overline{z})}$ and if $F(z) = F^*(z)$ we say that it is *real entire*.

A *Hermite–Biehler function* $E : \mathbb{C} \to \mathbb{C}$ is an entire function that satisfies the inequality

$$|E(\overline{z})| < |E(z)|$$

for all $z \in \mathbb{C}^+$. We define the de Branges space[1] $\mathcal{H}^2(E)$ to be the space of entire functions $F : \mathbb{C} \to \mathbb{C}$ such that

$$\|F\|_E^2 := \int_{\mathbb{R}} |F(x)|^2 \, |E(x)|^{-2} \, \mathrm{d}x < \infty \,, \tag{2.3}$$

and such that F/E and F^*/E have bounded type and non–positive mean type in $\mathbb{C}^+$. This is a Hilbert space with respect to the inner product

$$\langle F, G \rangle_E := \int_{\mathbb{R}} F(x) \, \overline{G(x)} \, |E(x)|^{-2} \, \mathrm{d}x.$$

The Hilbert space $\mathcal{H}^2(E)$ has the special property that, for each $w \in \mathbb{C}$, the map $F \mapsto F(w)$ is a continuous linear functional on $\mathcal{H}^2(E)$. Therefore, there exists a function $z \mapsto K(w, z)$ in $\mathcal{H}^2(E)$ such that

$$F(w) = \langle F, K(w, \cdot) \rangle_E \,. \tag{2.4}$$

The function $K(w, z)$ is called the *reproducing kernel* of $\mathcal{H}^2(E)$. If we write

$$A(z) := \frac{1}{2}\big\{E(z) + E^*(z)\big\} \quad \text{and} \quad B(z) := \frac{i}{2}\big\{E(z) - E^*(z)\big\}, \tag{2.5}$$

then $A(z)$ and $B(z)$ are real entire functions with only real zeros and $E(z) = A(z) - iB(z)$. The reproducing kernel is then given by [6, Theorem 19]

$$K(w, z) = \frac{E(z)E^*(\overline{w}) - E^*(z)E(\overline{w})}{2\pi i(\overline{w} - z)} = \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z - \overline{w})}. \tag{2.6}$$

When $z = \overline{w}$ we have

$$\pi K(\overline{z}, z) = B'(z)A(z) - A'(z)B(z). \tag{2.7}$$

We may apply the Cauchy-Schwarz inequality in (2.4) to obtain that

$$|F(w)|^2 \leq \|F\|_E^2 K(w, w), \tag{2.8}$$

[1]In the next Chapter we will be dealing with L^p de Branges spaces which we denote by $\mathcal{H}^p(E)$. For this reason we use the notation $\mathcal{H}^2(E)$ instead of $\mathcal{H}(E)$, which is the usual notation for an de Branges space.

for every $F \in \mathcal{H}^2(E)$. By the reproducing kernel property (2.4) we have

$$K(w,z) = \langle K(w,\cdot), K(z,\cdot)\rangle_E.$$

Thus, $K(w,w) \geq 0$ and $|K(w,z)|^2 \leq K(w,w)K(z,z)$. Also, it is not hard to show that $K(w,w) = 0$ if and only if $w \in \mathbb{R}$ and $E(w) = 0$ (see [47, Lemma 11]).

We denote by $\varphi(z)$ a phase function associated to $E(z)$. This is an analytic function in a neighborhood of $\mathbb{R}$ defined by the condition $e^{i\varphi(x)}E(x) \in \mathbb{R}$ for all real x. Using (2.6) we obtain that

$$\varphi'(x) = \pi\frac{K(x,x)}{|E(x)|^2} > 0 \tag{2.9}$$

for all real x and thus $\varphi(x)$ is an increasing function of real x (see [6, Problem 48]). We also have that

$$e^{2i\varphi(x)} = \frac{A(x)^2}{|E(x)|^2} - \frac{B(x)^2}{|E(x)|^2} + 2i\frac{A(x)B(x)}{|E(x)|^2}$$

for all real x. As a consequence, the points $t \in \mathbb{R}$ such that $\varphi(t) \equiv 0 \mod \pi$ coincide with the real zeros of $B(z)/E(z)$ and the points $s \in \mathbb{R}$ such that $\varphi(s) \equiv \pi/2 \mod \pi$ coincide with the real zeros of $A(z)/E(z)$ and by (2.9), these zeros are simple. In other words, the function $B(z)/A(z)$ *has only simple real zeros and simple real poles that interlace.*

The zero set of the function $B(z)$ plays a special role in the theory of de Branges associated with a function $E(z)$ with no real zeros. In this case, the zeros of $B(z)$ coincide with the points $t \in \mathbb{R}$ such that $\varphi(t) \equiv 0 \mod \pi$. The following result can be found in [6, Theorem 22].

Theorem 2.1.1 (de Branges)**.** *Let $E(z)$ be a Hermite–Biehler function with no real zeros. Then the set of functions $\{B(z)/(z-t)\}_t$, where t varies among* *the real zeros of $B(z)$, is an orthogonal set in $\mathcal{H}^2(E)$ and for every $F \in \mathcal{H}^2(E)$ there is a constant $c(F) \geq 0$ such that*

$$\int_{\mathbb{R}} \left|\frac{F(x)}{E(x)}\right|^2 \mathrm{d}x = \sum_{B(t)=0} \frac{|F(t)|^2}{K(t,t)} + c(F). \tag{2.10}$$

Moreover, $c(F) = 0$ for every $F \in \mathcal{H}^2(E)$ if and only if $B \notin \mathcal{H}^2(E)$. In this case we have

$$F(z) = B(z) \sum_{B(t)=0} \frac{F(t)}{B'(t)(z-t)}$$

where the convergence is in the norm of the space.

Remark. An analogous result holds for the function $A(z)$, or more general, to any function of the form $e^{i\theta}E^*(z) - e^{-i\theta}E(z)$, $\theta \in \mathbb{R}$.

2.1.2 Main Results

For our purposes we let $E(z)$ be a Hermite–Biehler function of bounded type in $\mathbb{C}^+$. In this case, a classical result of Krein (see [50] or [47, Lemma 9]) guarantees that E has exponential type and $\tau(E) = v(E)$. Moreover, an entire function $F(z)$ belongs to $\mathcal{H}^2(E)$ if and only if it has exponential type at most $\tau(E)$ and satisfies (2.3) (see [47, Lemma 12]).

Let $\mathrm{d}\mu$ be a (locally finite) signed Borel measure on $\mathbb{R}$ and denote by $\mu(x) = \mathrm{d}\mu((-\infty, x]))$ its distribution function. Assume that it satisfies the following properties:

(H1) The measure $\mathrm{d}\mu$ has support bounded by below (or equivalently, $\mu(x) = 0$ for all $x \leq C$ for some $C \in \mathbb{R}$).

(H2) The function $\mu(x)$ verifies

$$0 \leq \mu(x) \leq 1 \tag{2.11}$$

for all real x.

In some instances we require a third property:

(H3) The average value of the distribution function $\mu(x)$ is 1, that is

$$\lim_{y\to\infty} \frac{1}{y} \int_{-\infty}^{y} \mu(x)\,\mathrm{d}x = 1. \tag{2.12}$$

Remark. We remark that the constant 1 appearing on the right-hand sides of (2.11) and (2.12) could be replaced by any constant $C > 0$. For simplicity, we normalize the measure (by dilating) to work with $C = 1$. Observe that any probability measure $\mathrm{d}\mu$ on $\mathbb{R}$ satisfying (H1) automatically satisfies (H2) and (H3). Measures like $\mathrm{d}\mu(\lambda) = \chi_{(0,\infty)}(\lambda)\,\sin a\lambda\,\mathrm{d}\lambda$, for $a > 0$, which were considered by Littmann and Spanier in [57] (giving the truncated and odd Poisson kernels in the construction below), satisfy (H1) - (H2) but not (H3).

Let $\mathrm{d}\mu$ be a signed Borel measure on $\mathbb{R}$ satisfying (H1) - (H2). We define the function $f_\mu(z)$, the truncated Laplace transform of this measure, by

$$f_\mu(z) = \begin{cases} \displaystyle\int_{\mathbb{R}} e^{-\lambda z}\,\mathrm{d}\mu(\lambda), & \text{if } \mathrm{Re}\,(z) > 0; \\ 0, & \text{if } \mathrm{Re}\,(z) \le 0. \end{cases} \tag{2.13}$$

Observe that f_μ is a well–defined analytic function in $\mathrm{Re}\,(z) > 0$ since

$$f_\mu(z) = \int_{\mathbb{R}} e^{-\lambda z}\,\mathrm{d}\mu(\lambda) = \int_{\mathbb{R}} z e^{-\lambda z}\mu(\lambda)\,\mathrm{d}\lambda, \tag{2.14}$$

where we have used integration by parts. If we write

$$\mu^{(-1)}(y) := \int_{-\infty}^{y} \mu(x)\,\mathrm{d}x,$$

under the additional condition (H3) we find that (below we let $\operatorname{supp}(\mathrm{d}\mu) \subset (a,\infty)$)

$$\begin{aligned} f_\mu(0^+) &= \lim_{x\to 0^+} f_\mu(x) = \lim_{x\to 0^+} \int_a^\infty x e^{-\lambda x}\mu(\lambda)\,\mathrm{d}\lambda = \lim_{x\to 0^+} \int_a^\infty x^2 e^{-\lambda x}\mu^{(-1)}(\lambda)\,\mathrm{d}\lambda \\ &= \lim_{x\to 0^+} \int_{ax}^\infty x e^{-t}\mu^{(-1)}(t/x)\,\mathrm{d}t = \int_0^\infty t e^{-t}\,\mathrm{d}t \\ &= 1, \end{aligned} \tag{2.15}$$

by dominated convergence. Our first result is the following.

Theorem 2.1.2. *Let $E(z)$ be a Hermite–Biehler function of bounded type in $\mathbb{C}^+$ such that $E(0) \neq 0$. Let $\mathrm{d}\mu$ be a signed Borel measure on $\mathbb{R}$ satisfying* (H1) - (H2) - (H3). *Assume that* $\operatorname{supp}(\mathrm{d}\mu) \subset [-2\tau(E),\infty)$ *and let f_μ be defined by* (2.13). *If $L : \mathbb{C} \to \mathbb{C}$ and $M : \mathbb{C} \to \mathbb{C}$ are real entire functions of exponential type at most $2\tau(E)$ such that*

$$L(x) \leq f_\mu(x) \leq M(x) \tag{2.16}$$

for all $x \in \mathbb{R}$, then

$$\int_{\mathbb{R}} \big\{M(x) - L(x)\big\}\,|E(x)|^{-2}\,\mathrm{d}x \geq \frac{1}{K(0,0)}. \tag{2.17}$$

Moreover, there is a unique pair of real entire functions $L_\mu : \mathbb{C} \to \mathbb{C}$ and $M_\mu : \mathbb{C} \to \mathbb{C}$ of exponential type at most $2\tau(E)$ satisfying (2.16) *for which the equality in* (2.17) *holds.*

Our second result is the analogous of Theorem 2.1.2 for the odd function

$$\widetilde{f_\mu}(z) := f_\mu(z) - f_\mu(-z). \tag{2.18}$$

Note that if $\mathrm{d}\mu$ is the Dirac delta measure we have $\widetilde{f_\mu}(x) = \operatorname{sgn}(x)$.

Theorem 2.1.3. *Let $E(z)$ be a Hermite–Biehler function of bounded type in $\mathbb{C}^+$ such that $E(0) \neq 0$. Let $\mathrm{d}\mu$ be a signed Borel measure on $\mathbb{R}$ satisfying* (H1) - (H2) - (H3). *Assume that* $\mathrm{supp}(\mathrm{d}\mu) \subset [-2\tau(E), \infty)$ *and let $\widetilde{f}_\mu$ be defined by* (2.18). *If $L : \mathbb{C} \to \mathbb{C}$ and $M : \mathbb{C} \to \mathbb{C}$ are real entire functions of exponential type at most $2\tau(E)$ such that*

$$L(x) \leq \widetilde{f}_\mu(x) \leq M(x) \tag{2.19}$$

for all $x \in \mathbb{R}$, then

$$\int_{\mathbb{R}} \{M(x) - L(x)\} \, |E(x)|^{-2} \, \mathrm{d}x \geq \frac{2}{K(0,0)}. \tag{2.20}$$

Moreover, there is a unique pair of real entire functions $\widetilde{L}_\mu : \mathbb{C} \to \mathbb{C}$ and $\widetilde{M}_\mu : \mathbb{C} \to \mathbb{C}$ of exponential type at most $2\tau(E)$ satisfying (2.19) *for which the equality in* (2.20) *holds.*

Remarks.

(1) There is no loss of generality in assuming $E(0) \neq 0$ and $\mathrm{supp}(\mathrm{d}\mu) \subset [-2\tau(E), \infty)$ in Theorems 2.1.2 and 2.1.3. In fact, since $f_\mu(x)$ and $\widetilde{f}_\mu(x)$ are discontinuous at $x = 0$, if $E(0) = 0$ the integrals on the left-hand sides of (2.17) and (2.20) always diverge. Given $\varepsilon > 0$, if the set $\{x \in \mathbb{R};\ \mu(x) > 0\} \cap (-\infty, -2\tau(E) - \varepsilon)$ has nonzero Lebesgue measure, we find by (2.14) that $f_\mu(x) \geq C_\varepsilon \, x \, e^{(2\tau(E)+\varepsilon)x}$ for $x > 0$, and there is no entire function $M(z)$ of exponential type at most $2\tau(E)$ satisfying (2.16).

(2) The minorant problem for $f_\mu(z)$ can be solved without the hypothesis (H3). We give the details in Corollary 2.3.1 below.

(3) Note that we are allowing the measure $\mathrm{d}\mu$ to have part of its support on the negative axis. In principle, our function $f_\mu(x)$ could increase exponentially as $x \to \infty$ and does not necessarily belong to $L^1(\mathbb{R}, |E(x)|^{-2}\,\mathrm{d}x)$ (the same holds for $L(z)$ and $M(z)$). When $f_\mu \in L^1(\mathbb{R}, |E(x)|^{-2}\,\mathrm{d}x)$ (resp. $\widetilde{f_\mu} \in L^1(\mathbb{R}, |E(x)|^{-2}\,\mathrm{d}x)$) it is possible to determine the corresponding optimal values of

$$\int_{\mathbb{R}} M(x)\,|E(x)|^{-2}\,\mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}} L(x)\,|E(x)|^{-2}\,\mathrm{d}x$$

separately. This is detailed in Corollaries 2.3.2 and 2.4.1.

Remark. We use two main tools in the proofs of Theorems 2.1.2 and 2.1.3. The first is a basic Cauchy-Schwarz inequality in the Hilbert space $\mathcal{H}^2(E)$ that shows that the optimal choice for $M(z) - L(z)$ must be the square of the reproducing kernel at the origin (divided by a constant). The second tool, used to show the existence of such optimal majorants and minorants, is the construction of suitable entire functions that interpolate $f_\mu(x)$ at the zeros of a given Laguerre-Pólya function. The latter is detailed in Section 2.2 and extends the construction of Holt and Vaaler [47, Section 2], that was tailored specifically for the signum function.

There is a variety of examples of de Branges spaces [6, Chapter 3] for which Theorems 2.1.2 and 2.1.3 can be directly applied. Another interesting family arises in the discussion of [47, Section 5]. In the terminology of de Branges [6, Section 50], these are examples of homogeneous spaces and their definition can be found in Appendix 6.2.

For a given $\alpha > -1$ there exists a Hermite–Biehler function $E_\alpha(z)$ satisfying the estimate

$$c|x|^{2\alpha+1} \leq |E_\alpha(x)|^{-2} \leq C|x|^{2\alpha+1} \quad \text{for all } |x| \geq 1, \tag{2.21}$$

for some $c, C > 0$ and such that for each $F \in \mathcal{H}^2(E_\alpha)$ we have the remarkable identity

$$\int_{\mathbb{R}} |F(x)|^2 \, |E_\alpha(x)|^{-2} \, \mathrm{d}x = c_\alpha \int_{\mathbb{R}} |F(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x \,, \tag{2.22}$$

with $c_\alpha = \pi \, 2^{-2\alpha-1} \, \Gamma(\alpha+1)^{-2}$. Also, by the discussion in Appendix 6.2 we have that $F \in \mathcal{H}^2(E_\alpha)$ if and only if $F(z)$ has exponential type at most 1 and either side of (2.22) is finite.

Identity (2.22) makes $\mathcal{H}^2(E_\alpha)$ the suitable de Branges space to treat the extremal problem (2.1) - (2.2) for the power measure $\mathrm{d}\sigma(x) = |x|^{2\alpha+1} \, \mathrm{d}x$. In order to do so, we define

$$\Delta_\alpha(\delta, \mu) = \inf \int_{\mathbb{R}} \big\{ M(x) - L(x) \big\} \, |x|^{2\alpha+1} \, \mathrm{d}x \,,$$

where the infimum is taken over all pairs of real entire functions $L : \mathbb{C} \to \mathbb{C}$ and $M : \mathbb{C} \to \mathbb{C}$ of exponential type at most δ such that $L(x) \leq f_\mu(x) \leq M(x)$ for all $x \in \mathbb{R}$. If there is no such a pair we set $\Delta_\alpha(\delta, \mu) = \infty$. Define $\widetilde{\Delta}_\alpha(\delta, \mu)$ considering the analogous extremal problem for the odd function $\widetilde{f}_\mu$. The following result follows from Theorems 2.1.2 and 2.1.3.

Theorem 2.1.4. *Let $\alpha > -1$ and $\delta > 0$. Let $\mathrm{d}\mu$ be a signed Borel measure on $\mathbb{R}$ satisfying* (H1) - (H2) - (H3)*, and let f_μ be defined by* (2.13) (*resp. $\widetilde{f}_\mu$ be defined by* (2.18)). We have

$$\Delta_\alpha(\delta, \mu) = \begin{cases} \Gamma(\alpha+1)\,\Gamma(\alpha+2)\left(\frac{4}{\delta}\right)^{2\alpha+2}, & \text{if } \operatorname{supp}(\mathrm{d}\mu) \subset [-\delta, \infty); \\ \infty, & \text{otherwise;} \end{cases} \tag{2.23}$$

and

$$\widetilde{\Delta}_\alpha(\delta, \mu) = \begin{cases} 2\,\Gamma(\alpha+1)\,\Gamma(\alpha+2)\left(\frac{4}{\delta}\right)^{2\alpha+2}, & \text{if } \operatorname{supp}(\mathrm{d}\mu) \subset [-\delta, \infty); \\ \infty, & \text{otherwise.} \end{cases} \tag{2.24}$$

If $\Delta_\alpha(\delta, \mu)$ (resp. $\widetilde{\Delta}_\alpha(\delta, \mu)$) is finite, there exists a unique pair of corresponding extremal functions.

Proof. To see why Theorem 2.1.4 is indeed a consequence of Theorems 2.1.2 and 2.1.3 we proceed as follows. For $\kappa > 0$, we consider the measure $\mathrm{d}\mu_\kappa$ defined by $\mathrm{d}\mu_\kappa(\Omega) = \mathrm{d}\mu(k\Omega)$, where Ω is any Borel measurable set and $\kappa\Omega = \{\kappa\lambda;\ \lambda \in \Omega\}$. A simple dilation argument shows that

$$\Delta_\alpha(\delta, \mu) = \kappa^{2\alpha+2}\, \Delta_\alpha\big(\kappa\delta, \mu_{\kappa^{-1}}\big) \quad \text{and} \quad \widetilde{\Delta}_\alpha(\delta, \mu) = \kappa^{2\alpha+2}\, \widetilde{\Delta}_\alpha\big(\kappa\delta, \mu_{\kappa^{-1}}\big),$$

and we can reduce matters to the case $\delta = 2$. Now let $L(z)$ and $M(z)$ be a pair of real entire functions of exponential type at most 2 such that $L(x) \leq f_\mu(x) \leq M(x)$ for all $x \in \mathbb{R}$, and such that $(M - L) \in L^1(\mathbb{R}, |x|^{2\alpha+1}\, \mathrm{d}x)$. By (2.21) we have that $(M - L) \in L^1(\mathbb{R}, |E_\alpha(x)|^{-2}\, \mathrm{d}x)$. Since $(M(x) - L(x))$ is non–negative on $\mathbb{R}$, according to [47, Theorem 15] (see also [16, Lemma 14]) we can write $M(z) - L(z) = U(z)U^*(z)$ with $U \in \mathcal{H}^2(E_\alpha)$. Therefore, by identity (2.22) and Theorem 2.1.2, we have

$$\begin{aligned}
\int_{\mathbb{R}} \big\{M(x) - L(x)\big\}\, |x|^{2\alpha+1}\, \mathrm{d}x &= \int_{\mathbb{R}} |U(x)|^2\, |x|^{2\alpha+1}\, \mathrm{d}x \\
&= c_\alpha^{-1} \int_{\mathbb{R}} |U(x)|^2\, |E_\alpha(x)|^{-2}\, \mathrm{d}x \\
&= c_\alpha^{-1} \int_{\mathbb{R}} \big\{M(x) - L(x)\big\}\, |E_\alpha(x)|^{-2}\, \mathrm{d}x \\
&\geq c_\alpha^{-1}\, K_\alpha(0,0)^{-1},
\end{aligned}$$

where $c_\alpha = \pi\, 2^{-2\alpha-1}\, \Gamma(\alpha+1)^{-2}$ and

$$K_\alpha(0,0) = \frac{B_\alpha'(0) A_\alpha(0)}{\pi} = \frac{1}{2\pi(\alpha+1)}.$$

This establishes (2.23). A similar argument using Theorem 2.1.3 gives (2.24). $\square$

As illustrated in the argument above, in order to use the general machinery of Theorems 2.1.2 and 2.1.3 to solve the extremal problem (2.1) - (2.2) for a given measure $\mathrm{d}\sigma$, one has to first construct an appropriate de Branges space $\mathcal{H}^2(E)$ that is isometrically contained in $L^2(\mathbb{R}, \mathrm{d}\sigma)$. In particular, this construction was carried out in [9] for the measure

$$\mathrm{d}\sigma(x) = \left\{1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right\} \mathrm{d}x,$$

that appears in connection to Montgomery's formula and the pair correlation of zeros of the Riemann zeta–function (see [61]), and in [57, 59] for the measure

$$\mathrm{d}\sigma(x) = (x^2 + a^2)\, \mathrm{d}x,$$

where $a \geq 0$, that appears in connection to extremal problems with prescribed vanishing conditions.

2.2 Interpolation Tools

2.2.1 Laplace Transforms and Laguerre-Pólya Functions

In this subsection we review some basic facts concerning Laguerre-Pólya functions and the representation of their inverses as Laplace transforms as in [46, Chapters II to V]. The selected material we need is already well organized in [16, Section 2] and we follow closely their notation.

We say that an entire function $F : \mathbb{C} \to \mathbb{C}$ belongs to the *Laguerre-Pólya class* if it has only real zeros and its Hadamard factorization is given by

$$F(z) = \frac{F^{(r)}(0)}{r!}\, z^r\, e^{-az^2+bz} \prod_{j=1}^{\infty} \left(1 - \frac{z}{x_j}\right) e^{z/x_j}, \tag{2.25}$$

where $r \in \mathbb{Z}^+$, $a, b, x_j \in \mathbb{R}$, with $a \geq 0$, $x_j \neq 0$ and $\sum_{j=1}^{\infty} x_j^{-2} < \infty$ (with the appropriate change of notation in case of a finite number of zeros). Such functions are the uniform limits (in compact sets) of polynomials with only real zeros. We say that a Laguerre-Pólya function $F(z)$ represented by (2.25) has finite degree $\mathcal{N} = \mathcal{N}(F)$ when $a = 0$ and $F(z)$ has exactly $\mathcal{N}$ zeros counted with multiplicity. Otherwise we set $\mathcal{N}(F) = \infty$.

If $F(z)$ is a Laguerre-Pólya function with $\mathcal{N}(F) \geq 2$, and $c \in \mathbb{R}$ is such that $F(c) \neq 0$, we henceforth denote by $g_c(t)$ the frequency function given by

$$g_c(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{ts}}{F(s)} \, \mathrm{d}s. \tag{2.26}$$

Observe that the integral in (2.26) is absolutely convergent since the condition $\mathcal{N}(F) \geq 2$ implies that $1/|F(c+iy)| = O(|y|^{-2})$ as $|y| \to \infty$. If $(\tau_1, \tau_2) \subset \mathbb{R}$ is the largest open interval containing no zeros of $F(z)$ such that $c \in (\tau_1, \tau_2)$, the residue theorem implies that $g_c(t) = g_d(t)$ for any $d \in (\tau_1, \tau_2)$. Moreover, the Laplace transform representation

$$\frac{1}{F(z)} = \int_{\mathbb{R}} g_c(t)\, e^{-tz} \, \mathrm{d}t \tag{2.27}$$

holds in the strip $\tau_1 < \mathrm{Re}\,(z) < \tau_2$ (the integral in (2.27) is in fact absolutely convergent due to Lemma 2.2.1 below). If $\mathcal{N}(F) = 0$ or 1, we can still represent $F(z)^{-1}$ as a Laplace transform on vertical strips. In fact, if $\mathcal{N}(F) = 1$, we let τ be the zero of $F(z)$, written in the form (2.25). If $\tau = 0$ then (2.27) holds with

$$g_c(t) = \begin{cases} F'(0)^{-1}\, \chi_{(b,\infty)}(t), & \text{for } c > 0; \\ -F'(0)^{-1}\, \chi_{(-\infty,b)}(t), & \text{for } c < 0. \end{cases}$$

If $\tau \neq 0$ then (2.27) holds with

$$g_c(t) = \begin{cases} -\tau\, F(0)^{-1}\, e^{\tau(t-b)-1} \chi_{(b+\tau^{-1},\infty)}(t), & \text{for } c > \tau; \\ \tau\, F(0)^{-1}\, e^{\tau(t-b)-1} \chi_{(-\infty,b+\tau^{-1})}(t), & \text{for } c < \tau. \end{cases}$$

If $\mathcal{N}(F) = 0$ then (2.27) holds with

$$g_c(t) = F(0)^{-1}\, \delta(t-b),$$

for any $c \in \mathbb{R}$, where δ denotes the Dirac delta measure.

The fundamental tool for the development of our interpolation theory in this section is the precise qualitative knowledge of the frequency functions $g_c(t)$. This is extensively discussed in [46, Chapters II to V] and we collect the relevant facts for our purposes in the next lemma.

Lemma 2.2.1. *Let $F(z)$ be a Laguerre–Pólya function of degree $\mathcal{N} \geq 2$ and let g_c be defined by (2.26), where $c \in \mathbb{R}$ and $F(c) \neq 0$. The following propositions hold:*

(i) The function $g_c \in C^{\mathcal{N}-2}(\mathbb{R})$ and is real valued.

(ii) The function $g_c(t)$ is of one sign, and its sign equals the sign of $F(c)$.

(iii) If $(\tau_1, \tau_2) \subset \mathbb{R}$ is the largest open interval containing no zeros of $F(z)$ such that $c \in (\tau_1, \tau_2)$, then for any $\tau \in (\tau_1, \tau_2)$ we have the following estimate

$$\left| g_c^{(n)}(t) \right| \ll_{\tau,n} e^{\tau t} \quad \forall\, t \in \mathbb{R}, \tag{2.28}$$

where $0 \leq n \leq \mathcal{N} - 2$.

Proof. Parts (i) and (ii) follow from [46, Chapter IV, Theorems 5.1 and 5.3]. Part (iii) follows from [46, Chapter II, Theorem 8.2 and Chapter V, Theorem 2.1]. □

2.2.2 Interpolation at the Zeros of Laguerre-Pólya Functions

In this subsection we construct suitable entire functions that interpolate our $f_\mu(x)$ at the zeros of a given Laguerre-Pólya function. In order to accomplish this, we make use of the representation in (2.27) and Lemma 2.2.1. The material in this subsection extends the classical work of Graham and Vaaler in [40, Section 3], where this construction was achieved for the particular function $F(x) = (\sin \pi x)^2$ of Laguerre-Pólya class.

If $F(z)$ is a Laguerre-Pólya function, we henceforth denote by α_F the smallest positive zero of $F(z)$ (if no such zero exists, we set $\alpha_F = \infty$). Let $g(t) = g_{\alpha_F/2}(t)$ (if $\alpha_F = \infty$ take $g(t) = g_1(t)$). If $\mathrm{d}\mu$ is a signed Borel measure on $\mathbb{R}$ satisfying (H1) - (H2), it is clear that the function

$$g * \mathrm{d}\mu(t) = \int_{\mathbb{R}} g(t-\lambda)\,\mathrm{d}\mu(\lambda) = \int_{\mathbb{R}} g'(t-\lambda)\,\mu(\lambda)\,\mathrm{d}\lambda = g' * \mu(t) \tag{2.29}$$

satisfies the same growth conditions as in (2.28) for $\tau \in (0, \alpha_F)$, for $0 \leq n \leq \mathcal{N} - 3$, with the implied constants now depending also on $\mathrm{d}\mu$. We are now in position to define the building blocks of our interpolation.

Proposition 2.2.2. *Let $F(z)$ be a Laguerre-Pólya function with $\mathcal{N}(F) \geq 2$. Let $g(t) = g_{\alpha_F/2}(t)$ and assume that $F(\alpha_F/2) > 0$ (in case $\alpha_F = +\infty$, let $g(t) = g_1(t)$ and assume $F(1) > 0$). Let $\mathrm{d}\mu$ be a signed Borel measure on $\mathbb{R}$ satisfying* (H1) - (H2), *and let $f_\mu(z)$ be defined by* (2.13). *Define*

$$\mathcal{A}_1(F, \mu, z) = F(z) \int_{-\infty}^{0} g * \mathrm{d}\mu(t)\, e^{-tz}\,\mathrm{d}t \quad \text{for } \operatorname{Re}(z) < \alpha_F, \tag{2.30}$$

$$\mathcal{A}_2(F, \mu, z) = f_\mu(z) - F(z) \int_{0}^{\infty} g * \mathrm{d}\mu(t)\, e^{-tz}\,\mathrm{d}t \quad \text{for } \operatorname{Re}(z) > 0. \tag{2.31}$$

Then $z \mapsto \mathcal{A}_1(F, \mu, z)$ is analytic in $\operatorname{Re}(z) < \alpha_F$, $z \mapsto \mathcal{A}_2(F, \mu, z)$ is analytic in $\operatorname{Re}(z) > 0$, and these functions are restrictions of an entire function, which we will denote by $\mathcal{A}(F, \mu, z)$. Moreover, if supp$(\mathrm{d}\mu) \subset [-\tau, \infty)$, there exists $c > 0$ so that

$$|\mathcal{A}(F, \mu, z)| \leq c\left(|z|\, e^{\tau x}\, \chi_{(0,\infty)}(x) + |F(z)|\right) \tag{2.32}$$

for all $z = x + iy \in \mathbb{C}$, and

$$\mathcal{A}(F, \mu, \xi) = f_\mu(\xi) \tag{2.33}$$

for all $\xi \in \mathbb{R}$ *with* $F(\xi) = 0$.

Proof. We have already noted in (2.14) that $z \mapsto f_\mu(z)$ is analytic in $\mathrm{Re}\,(z) > 0$ when $\mathrm{d}\mu$ satisfies (H1) - (H2). If $\mathcal{N}(F) \geq 3$, from (2.29) and Lemma 2.2.1 (iii) we see that the integrals on the right-hand sides of (2.30) and (2.31) converge absolutely and define analytic functions in the stated half–planes. If $\mathcal{N}(F) = 2$, it can be verified directly that $g(t)$ is continuous and C^1 by parts, and that the function $g'(t)$ thus obtained has at most one discontinuity and still satisfies the growth condition (2.28). Therefore (2.29) holds and, as before, this suffices to establish the absolute convergence and analiticity of (2.30) and (2.31) in the stated half–planes.

Now let $0 < x < \alpha_F$. Using (2.29), (2.27) and (2.14) we get

$$\begin{aligned}
\mathcal{A}_1(F,\mu,x) - \mathcal{A}_2(F,\mu,x) &= -f_\mu(x) + F(x)\int_{\mathbb{R}} g' * \mu(t)\, e^{-tx}\,\mathrm{d}t \\
&= -f_\mu(x) + F(x)\int_{\mathbb{R}}\int_{\mathbb{R}} g'(t-\lambda)\,\mu(\lambda)\, e^{-tx}\,\mathrm{d}\lambda\,\mathrm{d}t \\
&= -f_\mu(x) + F(x)\int_{\mathbb{R}}\left(\int_{\mathbb{R}} g'(t-\lambda)\, e^{-tx}\,\mathrm{d}t\right)\mu(\lambda)\,\mathrm{d}\lambda \\
&= -f_\mu(x) + F(x)\int_{\mathbb{R}}\left(\int_{\mathbb{R}} g'(s)\, e^{-sx}\,\mathrm{d}s\right) e^{-\lambda x}\,\mu(\lambda)\,\mathrm{d}\lambda \\
&= -f_\mu(x) + \int_{\mathbb{R}} x\, e^{-\lambda x}\,\mu(\lambda)\,\mathrm{d}\lambda \\
&= 0.
\end{aligned}$$

This implies that $\mathcal{A}_1(F,\mu,z) = \mathcal{A}_2(F,\mu,z)$ in the strip $0 < \mathrm{Re}\,(z) < \alpha_F$. Hence, $z \mapsto \mathcal{A}_1(F,\mu,z)$ and $z \mapsto \mathcal{A}_2(F,\mu,z)$ are analytic continuations of each other and this defines the entire function $z \mapsto \mathcal{A}(F,\lambda,z)$. The integral representations for $\mathcal{A}$ and (2.29) imply, for $\mathrm{Re}\,(z) \le \alpha_F/2$, that

$$\begin{aligned} |\mathcal{A}(F,\mu,z)| &\le |F(z)| \int_{-\infty}^{0} |g'| * \mu(t)\, e^{-t\,\mathrm{Re}\,(z)}\, \mathrm{d}t \\ &\le |F(z)| \int_{-\infty}^{0} |g'| * \mu(t)\, e^{-t\,\alpha_F/2}\, \mathrm{d}t\,, \end{aligned} \tag{2.34}$$

while for $\mathrm{Re}\,(z) \ge \alpha_F/2$ we have

$$|\mathcal{A}(F,\mu,z)| \le |f_\mu(z)| + |F(z)| \int_{0}^{\infty} |g'| * \mu(t)\, e^{-t\,\alpha_F/2}\, \mathrm{d}t. \tag{2.35}$$

Since $\mathrm{supp}(\mathrm{d}\mu) \subset [-\tau, \infty)$ we use (2.14) and (H2) to obtain, for $\mathrm{Re}\,(z) \ge \alpha_F/2$,

$$\begin{aligned} |f_\mu(z)| &= \left| \int_{-\tau}^{\infty} z\, e^{-\lambda z}\, \mu(\lambda)\, \mathrm{d}\lambda \right| \le |z| \left| \int_{-\tau}^{\infty} e^{-\lambda\,\mathrm{Re}\,(z)}\, \mathrm{d}\lambda \right| \\ &= \frac{|z|}{\mathrm{Re}\,(z)}\, e^{\tau\,\mathrm{Re}\,(z)} \le \frac{2|z|}{\alpha_F}\, e^{\tau\,\mathrm{Re}\,(z)}. \end{aligned} \tag{2.36}$$

Estimates (2.34), (2.35) and (2.36) plainly verify (2.32). The remaining identity (2.33) follows from the definition of $\mathcal{A}$. □

Proposition 2.2.3. *Let $F(z)$ be a Laguerre-Pólya function that has a double zero at the origin. Let $g(t) = g_{\alpha_F/2}(t)$ and assume that $F(\alpha_F/2) > 0$ (in case $\alpha_F = +\infty$, let $g(t) = g_1(t)$ and assume $F(1) > 0$). Let $\mathrm{d}\mu$ be a signed Borel measure on $\mathbb{R}$ satisfying* (H1) - (H2)*, and let f_μ be defined by* (2.13)*. With $z \mapsto \mathcal{A}(F,\mu,z)$ defined by Proposition 2.2.2, consider the entire functions $z \mapsto L(F,\mu,z)$ and $z \mapsto M(F,\mu,z)$ defined by*

$$L(F,\mu,z) = \mathcal{A}(F,\mu,z) + g * \mathrm{d}\mu(0) \frac{F(z)}{z} \tag{2.37}$$

and

$$M(F,\mu,z) = L(F,\mu,z) + \frac{2F(z)}{F''(0)z^2}. \tag{2.38}$$

The following propositions hold:

(i) We have

$$F(x)\big\{f_\mu(x) - L(F,\mu,x)\big\} \geq 0 \tag{2.39}$$

for all $x \in \mathbb{R}$ *and*

$$L(F,\mu,\xi) = f_\mu(\xi) \tag{2.40}$$

for all $\xi \in \mathbb{R}$ *with* $F(\xi) = 0$.

(ii) We have

$$F(x)\big\{M(F,\mu,x) - f_\mu(x)\big\} \geq 0 \tag{2.41}$$

for all $x \in \mathbb{R}$ *and*

$$M(F,\mu,\xi) = f_\mu(\xi) \tag{2.42}$$

for all $\xi \in \mathbb{R} \setminus \{0\}$ *with* $F(\xi) = 0$. *At* $\xi = 0$ *we have*

$$M(F,\mu,0) = 1.$$

(iii) The equality

$$\big|M(F,\mu,x) - f_\mu(x)\big| + \big|f_\mu(x) - L(F,\mu,x)\big| = \frac{2|F(x)|}{x^2 F''(0)} \tag{2.43}$$

holds for all $x \in \mathbb{R}$.

Proof. Part (i). For $x < 0$, using (2.29), (2.30) and (2.37) we get

$$f_\mu(x) - L(F,\mu,x) = -F(x) \int_{-\infty}^{0} \big\{g' * \mu(t) - g' * \mu(0)\big\}\, e^{-tx}\, \mathrm{d}t, \tag{2.44}$$

and, for $x > 0$, using (2.29), (2.31) and (2.37) we get

$$f_\mu(x) - L(F,\mu,x) = F(x) \int_{0}^{\infty} \big\{g' * \mu(t) - g' * \mu(0)\big\}\, e^{-tx}\, \mathrm{d}t. \tag{2.45}$$

If $\mathcal{N}(F) \geq 4$, integration by parts in (2.27) shows that the Laplace transforms of $g'(t)$ and $g''(t)$ in the strip $0 < \mathrm{Re}\,(z) < \alpha_F$ are $z/F(z)$ and $z^2/F(z)$, respectively (here we use Lemma 2.2.1 (*iii*) to eliminate the boundary terms). Since $F(\alpha_F/2) > 0$, we conclude by Lemma 2.2.1 (*ii*) that $g'(t)$ and $g''(t)$ are non–negative on $\mathbb{R}$. In particular, $g'(t)$ is also non–decreasing on $\mathbb{R}$. If $\mathcal{N}(F) = 2$ or 3, it can be verified directly that $g'(t)$ is non–decreasing on $\mathbb{R}$. In either case, this implies that $g' * \mu(t)$ is non–decreasing, and (2.39) and (2.40) (for $\xi \neq 0$) then follow from (2.44) and (2.45). For $\xi = 0$ we see directly from (2.33) and (2.37) that $L(F, \mu, 0) = 0$.

Part (*ii*). For $x < 0$, using (2.29), (2.30) and (2.38) we get

$$M(F,\mu,x) - f_\mu(x) = F(x) \int_{-\infty}^{0} \left\{ g' * \mu(t) - g' * \mu(0) - \frac{2t}{F''(0)} \right\} e^{-tx}\, \mathrm{d}t, \tag{2.46}$$

and, for $x > 0$, using (2.29), (2.31) and (2.38) we get

$$M(F,\mu,x) - f_\mu(x) = -F(x) \int_{0}^{\infty} \left\{ g' * \mu(t) - g' * \mu(0) - \frac{2t}{F''(0)} \right\} e^{-tx}\, \mathrm{d}t. \tag{2.47}$$

In order to prove (2.41) it suffices to verify that

$$\left| g' * \mu(t) - g' * \mu(0) \right| \leq \frac{2|t|}{F''(0)} \tag{2.48}$$

for all $t \in \mathbb{R}$.

If $\mathcal{N}(F) \geq 4$, we have already noted that the Laplace transform of $g''(t)$ in the strip $0 < \mathrm{Re}\,(z) < \alpha_F$ is $z^2/F(z)$. Since $F(z)/z^2$ does not vanish at the origin, we see from Lemma 2.2.1 that $g''(t)$ is non–negative and decays exponentially as $|t| \to \infty$. By a direct verification, the same holds for $\mathcal{N}(F) = 3$, where $g''(t)$ might have one discontinuity. Thus $g''(t)$ is integrable on $\mathbb{R}$ and by (2.27) we find

$$\int_{\mathbb{R}} g''(t)\, \mathrm{d}t = 2F''(0)^{-1}. \tag{2.49}$$

We are now in position to prove (2.48) for $\mathcal{N}(F) \geq 3$. We have already noted in part (i) that $g'(t)$ is a non–decreasing function. Therefore, for $t > 0$, we use (H2) and (2.49) to get

$$\begin{aligned} g' * \mu(t) - g' * \mu(0) &= \int_{\mathbb{R}} \left\{ g'(t-\lambda) - g'(-\lambda) \right\} \mu(\lambda)\, \mathrm{d}\lambda \\ &\leq \int_{\mathbb{R}} \int_0^t g''(s-\lambda)\, \mathrm{d}s\, \mathrm{d}\lambda \\ &= 2\, t\, F''(0)^{-1}. \end{aligned}$$

An analogous argument holds for $t < 0$. If $\mathcal{N}(F) = 2$, we have $F(z) = \frac{1}{2} F''(0)\, e^{bz}\, z^2$ and $g(t) = \frac{2}{F''(0)}(t-b)\, \chi_{(b,\infty)}(t)$, and (2.48) can be verified directly.

For $\xi \neq 0$, the interpolation property (2.42) follows directly from (2.46) and (2.47). At $\xi = 0$, since $L(F, \mu, 0) = 0$, it follows from (2.38) that $M(F, \mu, 0) = 1$.

Part (iii). Identity (2.43) follows easily from (2.37), (2.38), (2.39) and (2.41). □

2.3 Proof of Theorem 2.1.2

Recall from (2.15) that under (H3) we have

$$f_\mu(0^+) = \lim_{x \to 0^+} f_\mu(x) = 1.$$

Optimality. Let $L(z)$ and M(z) be real entire functions of exponential type at most $2\tau(E)$ such that $L(x) \leq f_\mu(x) \leq M(x)$ for all $x \in \mathbb{R}$ and

$$\int_{\mathbb{R}} \left\{ M(x) - L(x) \right\} |E(x)|^{-2}\, \mathrm{d}x < \infty.$$

Since $(M(z) - L(z))$ is non–negative on $\mathbb{R}$, by [47, Theorem 15] (or alternatively [16, Lemma 14]) we may write

$$M(z) - L(z) = U(z) U^*(z)$$

with $U \in \mathcal{H}^2(E)$. Since $f_\mu(0^-) = 0$ and $f_\mu(0^+) = 1$, we find that $|U(0)|^2 = M(0) - L(0) \geq 1$. From the reproducing kernel identity and the Cauchy-Schwarz inequality, it follows that

$$1 \leq |U(0)|^2 = \left|\langle U, K(0,\cdot)\rangle_E\right|^2 \leq \|U\|_E^2 \, \|K(0,\cdot)\|_E^2 = \|U\|_E^2 \, K(0,0), \tag{2.50}$$

and therefore

$$\begin{aligned} \int_{\mathbb{R}} \big\{M(x) - L(x)\big\} \, |E(x)|^{-2} \, \mathrm{d}x &= \int_{\mathbb{R}} |U(x)|^2 \, |E(x)|^{-2} \, \mathrm{d}x \\ &= \|U\|_E^2 \geq \frac{1}{K(0,0)}. \end{aligned} \tag{2.51}$$

This establishes (2.17). Moreover, equality in (2.50) (and thus in (2.51)) happens if and only if $U(z) = c\,K(0,z)$ with $|c| = K(0,0)^{-1}$. This implies that we must have

$$M(z) - L(z) = \frac{K(0,z)^2}{K(0,0)^2}. \tag{2.52}$$

Existence. By multiplying $E(z)$ by a complex constant of absolute value 1, we may assume without loss of generality that $E(0) \in \mathbb{R}$. Since $E(z)$ is a Hermite–Biehler function of bounded type, we see that $E^*(z)$ also has bounded type. The companion function $B(z)$ defined by (2.5) is then a real entire function of bounded type with only real zeros. By [6, Problem 34] (see [49] for a generalization) we conclude that $B(z)$ belongs to the Laguerre-Pólya class. The function $B(z)$ has exponential type and it is clear that $\tau(B) \leq \tau(E)$. Note also that $B(z)$ has a simple zero at $z = 0$ (since $E(0) \neq 0$ we have $K(0,0) > 0$ and, by (2.7), $z = 0$ cannot be a double zero of $B(z)$).

Applying Proposition 2.2.3 to the function $B^2(z)$, we construct the entire functions

$$L_\mu(z) = L(B^2, \mu, z) \tag{2.53}$$

and

$$M_\mu(z) = M(B^2, \mu, z). \tag{2.54}$$

It follows from (2.39) and (2.41) that

$$L_\mu(x) \le f_\mu(x) \le M_\mu(x)$$

for all $x \in \mathbb{R}$. From (2.32), (2.37) and (2.38) if follows that $L_\mu(z)$ and $M_\mu(z)$ have exponential type at most $2\tau(E)$. Finally, from (2.6), (2.7), (2.37) and (2.38) we have that

$$M_\mu(z) - L_\mu(z) = \frac{B^2(z)}{B'(0)^2\, z^2} = \frac{K(0,z)^2}{K(0,0)^2},$$

and as we have seen in (2.52), this is the condition for equality in (2.17).

Uniqueness. From the equality condition (2.52) and the existence of an optimal pair $\{L_\mu(z), M_\mu(z)\}$ we conclude that this pair must be unique.

2.3.1 Further Results

Without assuming (H3) it is possible to solve the minorant problem for $f_\mu(x)$. However, we do have to assume that the companion function that generates the nodes of interpolation does not belong to the space $\mathcal{H}^2(E)$.

Corollary 2.3.1. *Let $E(z)$ be a Hermite–Biehler function of bounded type in $\mathbb{C}^+$ such that $E(0) > 0$. Let $\mathrm{d}\mu$ be a signed Borel measure on $\mathbb{R}$ satisfying* (H1) - (H2). *Assume that* $\mathrm{supp}(\mathrm{d}\mu) \subset [-2\tau(E), \infty)$ *and let $f_\mu(x)$ be defined by* (2.13). *Assume that $B \notin \mathcal{H}^2(E)$. Let $L_\mu(z)$ be the real entire function of exponential type at most $2\tau(E)$ defined by* (2.53). *If $L : \mathbb{C} \to \mathbb{C}$ is a real entire function of exponential type at most $2\tau(E)$ such that*

$$L(x) \le f_\mu(x)$$

for all $x \in \mathbb{R}$, then

$$\int_{\mathbb{R}} \big\{ f_\mu(x) - L(x) \big\} \, |E(x)|^{-2} \, \mathrm{d}x \ge \int_{\mathbb{R}} \big\{ f_\mu(x) - L_\mu(x) \big\} \, |E(x)|^{-2} \, \mathrm{d}x. \quad (2.55)$$

Proof. From (2.43) and (2.53) we observe first that the right-hand side of (2.55) is indeed finite. If the left-hand side of (2.55) is $+\infty$ there is nothing to prove. Assume then that $(f_\mu - L) \in L^1(\mathbb{R}, |E(x)|^{-2}\,\mathrm{d}x)$. We use the fact that there exists a majorant $M_\mu(z)$ (not necessarily extremal anymore) defined by (2.54), and from (2.43) we see that $(M_\mu - f_\mu) \in L^1(\mathbb{R}, |E(x)|^{-2}\,\mathrm{d}x)$. By the triangle inequality we get $(M_\mu - L_\mu) \in L^1(\mathbb{R}, |E(x)|^{-2}\,\mathrm{d}x)$ and $(M_\mu - L) \in L^1(\mathbb{R}, |E(x)|^{-2}\,\mathrm{d}x)$. Since the last two functions are non–negative on $\mathbb{R}$, from [47, Theorem 15] (or alternatively [16, Lemma 14]) we can write

$$M_\mu(z) - L(z) = U(z)U^*(z)$$

and

$$M_\mu(z) - L_\mu(z) = V(z)V^*(z),$$

with $U, V \in \mathcal{H}^2(E)$. This gives us

$$L_\mu(z) - L(z) = U(z)U^*(z) - V(z)V^*(z).$$

Since $B \notin \mathcal{H}^2(E)$, from [6, Theorem 22] the set $\{z \mapsto \overline{E}(\xi)^{-1}K(\xi, z);\, B(\xi) = 0\}$ is an orthogonal basis for $\mathcal{H}^2(E)$ (note here that if $E(\xi) = 0$, the function $\overline{E}(\xi)^{-1}K(\xi, z)$ has to be interpreted as the appropriate limit). We now use Parseval's identity and the the fact that $L_\mu(z)$ interpolates $f_\mu(x)$ at the zeros of $B(z)$ to get

$$\begin{aligned}\int_{\mathbb{R}} \big\{L_\mu(x) - L(x)\big\}\, |E(x)|^{-2}\,\mathrm{d}x &= \int_{\mathbb{R}} \big\{|U(x)|^2 - |V(x)|^2\big\}\, |E(x)|^{-2}\,\mathrm{d}x \\ &= \sum_{B(\xi)=0} \frac{\big\{|U(\xi)|^2 - |V(\xi)|^2\big\}}{K(\xi,\xi)} = \sum_{B(\xi)=0} \frac{\big\{L_\mu(\xi) - L(\xi)\big\}}{K(\xi,\xi)} \\ &= \sum_{B(\xi)=0} \frac{\big\{f_\mu(\xi) - L(\xi)\big\}}{K(\xi,\xi)} \geq 0.\end{aligned}$$

This concludes the proof of the corollary. □

When $f_\mu \in L^1(\mathbb{R}, |E(x)|^{-2}\,\mathrm{d}x)$ it is possible to determine the precise values of the optimal integrals in our extremal problem separately.

Corollary 2.3.2. *Let $E(z)$ be a Hermite–Biehler function of bounded type in $\mathbb{C}^+$ such that $E(0) > 0$. Let $\mathrm{d}\mu$ be a signed Borel measure on $\mathbb{R}$ satisfying* (H1) - (H2). *Assume that* $\mathrm{supp}(\mathrm{d}\mu) \subset [-2\tau(E), \infty)$ *and let $f_\mu(x)$ be defined by* (2.13). *Assume that*

$$\int_{\mathbb{R}} |f_\mu(x)|\,|E(x)|^{-2}\,\mathrm{d}x < \infty \tag{2.56}$$

and that $B \notin \mathcal{H}^2(E)$.

(*i*) *Let $L_\mu(z)$ be the extremal minorant of exponential type at most $2\tau(E)$ defined by* (2.53). *We have*

$$\int_{\mathbb{R}} L_\mu(x)\,|E(x)|^{-2}\,\mathrm{d}x = \sum_{\substack{\xi>0\\ B(\xi)=0}} \frac{f_\mu(\xi)}{K(\xi,\xi)}. \tag{2.57}$$

(*ii*) *Assuming* (H3), *let $M_\mu(z)$ be the extremal majorant of exponential type at most $2\tau(E)$ defined by* (2.54). *We have*

$$\int_{\mathbb{R}} M_\mu(x)\,|E(x)|^{-2}\,\mathrm{d}x = \frac{1}{K(0,0)} + \sum_{\substack{\xi>0\\ B(\xi)=0}} \frac{f_\mu(\xi)}{K(\xi,\xi)}. \tag{2.58}$$

Proof. We first prove (*ii*). The function $M_\mu(z)$ is non–negative on $\mathbb{R}$ and belongs to $L^1(\mathbb{R}, |E(x)|^{-2}\,\mathrm{d}x)$ from (2.56) (observe in particular that E cannot have non–negative zeros in this situation). From [47, Theorem 15] (or alternatively [16, Lemma 14]) we can write

$$M_\mu(z) = U(z)U^*(z) \tag{2.59}$$

with $U \in \mathcal{H}^2(E)$. We use again the fact that the set $\{z \mapsto \frac{K(\xi,z)}{\overline{E}(\xi)}; B(\xi) = 0\}$ is an orthogonal basis for $\mathcal{H}^2(E)$ since $B \notin \mathcal{H}^2(E)$ [6, Theorem 22]. From

Parseval's identity and the the fact that $M_\mu(z)$ interpolates $f_\mu(x)$ at the zeros of $B(z)$ (with $M_\mu(0) = 1$) we arrive at

$$\begin{aligned}\int_{\mathbb{R}} M_\mu(x)\,|E(x)|^{-2}\,\mathrm{d}x &= \int_{\mathbb{R}} |U(x)|^2\,|E(x)|^{-2}\,\mathrm{d}x = \sum_{B(\xi)=0} \frac{|U(\xi)|^2}{K(\xi,\xi)} \\ &= \sum_{B(\xi)=0} \frac{M_\mu(\xi)}{K(\xi,\xi)} = \frac{1}{K(0,0)} + \sum_{\substack{\xi>0 \\ B(\xi)=0}} \frac{f_\mu(\xi)}{K(\xi,\xi)}.\end{aligned}$$

This establishes (2.58).

We now prove (i). In this case, we still have a majorant $M_\mu(z)$ (not necessarily extremal anymore) and the factorization (2.59) still holds. From (2.43) we see that $(M_\mu - L_\mu) \in L^1(\mathbb{R}, |E(x)|^{-2}\,\mathrm{d}x)$ and we can write again $M_\mu(z) - L_\mu(z) = V(z)V^*(z)$, with $V \in \mathcal{H}^2(E)$. This gives us

$$L_\mu(z) = U(z)U^*(z) - V(z)V^*(z). \tag{2.60}$$

Using Parseval's identity again, and the fact that $L_\mu(z)$ interpolates $f_\mu(x)$ at the zeros of B, we arrive at

$$\begin{aligned}\int_{\mathbb{R}} L_\mu(x)\,|E(x)|^{-2}\,\mathrm{d}x &= \int_{\mathbb{R}} \left\{|U(x)|^2 - |V(x)|^2\right\}|E(x)|^{-2}\,\mathrm{d}x \\ &= \sum_{B(\xi)=0} \frac{|U(\xi)|^2 - |V(\xi)|^2}{K(\xi,\xi)} = \sum_{B(\xi)=0} \frac{L_\mu(\xi)}{K(\xi,\xi)} \\ &= \sum_{\substack{\xi>0 \\ B(\xi)=0}} \frac{f_\mu(\xi)}{K(\xi,\xi)}.\end{aligned}$$

This establishes (2.57) and completes the proof. □

2.4 Proof of Theorem 2.1.3

Optimality. This follows as in the optimality part of Theorem 2.1.2, just observing that

$$\widetilde{f}_\mu(0^-) = -1 \quad \text{and} \quad \widetilde{f}_\mu(0^+) = 1.$$

Existence. We use Proposition 2.2.3 with the Laguerre-Pólya functions $B^2(z)$ and its reflection $B^2(-z)$ to define

$$\widetilde{L}_\mu(z) = L(B^2(z), \mu, z) - M(B^2(-z), \mu, -z) \tag{2.61}$$

and

$$\widetilde{M}_\mu(z) = M(B^2(z), \mu, z) - L(B^2(-z), \mu, -z). \tag{2.62}$$

These are real entire functions of exponential type at most $2\tau(E)$ that satisfy

$$\widetilde{L}_\mu(x) \le \widetilde{f}_\mu(x) \le \widetilde{M}_\mu(x)$$

for all $x \in \mathbb{R}$. As before, from (2.6), (2.7), (2.37) and (2.38) we find that

$$\widetilde{M}_\mu(z) - \widetilde{L}_\mu(z) = \frac{2B^2(z)}{B'(0)^2\, z^2} = \frac{2K(0,z)^2}{K(0,0)^2},$$

and this is the condition for equality in (2.20).

Uniqueness. It follows as in the proof of Theorem 2.1.2.

2.4.1 Further Results

Corollary 2.4.1. *Let $E(z)$ be a Hermite–Biehler function of bounded type in $\mathbb{C}^+$ such that $E(0) > 0$. Let $\mathrm{d}\mu$ be a signed Borel measure on $\mathbb{R}$ satisfying* (H1) - (H2) - (H3). *Assume that* $\mathrm{supp}(\mathrm{d}\mu) \subset [-2\tau(E), \infty)$ *and let $\widetilde{f}_\mu$ be defined by* (2.18). *Assume that*

$$\int_{\mathbb{R}} |\widetilde{f}_\mu(x)|\, |E(x)|^{-2}\, \mathrm{d}x < \infty \tag{2.63}$$

and that $B \notin \mathcal{H}^2(E)$. Let $\widetilde{L}_\mu(z)$ and $\widetilde{M}_\mu(z)$ be the extremal functions of exponential type at most $2\tau(E)$ defined by (2.61) *and* (2.62)*, respectively. We have*

$$\int_{\mathbb{R}} \widetilde{L}_\mu(x)\, |E(x)|^{-2}\, \mathrm{d}x = -\frac{1}{K(0,0)} + \sum_{\substack{\xi \neq 0 \\ B(\xi)=0}} \frac{\widetilde{f}_\mu(\xi)}{K(\xi,\xi)}$$

and

$$\int_{\mathbb{R}} \widetilde{M}_\mu(x)\,|E(x)|^{-2}\,\mathrm{d}x = \frac{1}{K(0,0)} + \sum_{\substack{\xi\neq 0\\ B(\xi)=0}} \frac{\widetilde{f}_\mu(\xi)}{K(\xi,\xi)}.$$

Proof. From the integrability condition (2.63) we see that $E(z)$ cannot have real zeros and we may use (2.61), (2.62), (2.59) and (2.60) to write

$$\widetilde{L}_\mu(z) = \big(U_1(z)U_1^*(z) - V_1(z)V_1^*(z)\big) - U_2(z)U_2^*(z)$$

and

$$\widetilde{M}_\mu(z) = U_3(z)U_3^*(z) - \big(U_4(z)U_4^*(z) - V_4(z)V_4^*(z)\big),$$

where $U_i, V_j \in \mathcal{H}^2(E)$. Once we have completed this passage from L^1 to L^2, the remaining steps are analogous to the proof of Corollary 2.3.2. $\square$

2.5 Periodic Analogues

In this section we consider the periodic version of this extremal problem. Throughout this section we write $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$. A trigonometric polynomial of degree at most N is an entire function of the form

$$\mathcal{W}(z) = \sum_{k=-N}^{N} a_k\, e(kz),$$

where $a_k \in \mathbb{C}$. We say that $\mathcal{W}(z)$ is a *real trigonometric polynomial* if $\mathcal{W}(z)$ is real for z real. Given a periodic function $\mathcal{F} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, a probability measure $\mathrm{d}\vartheta$ on $\mathbb{R}/\mathbb{Z}$ and a degree $N \in \mathbb{Z}^+$, we address the problem of finding a pair of real trigonometric polynomials $\mathcal{L} : \mathbb{C} \to \mathbb{C}$ and $\mathcal{M} : \mathbb{C} \to \mathbb{C}$ of degree at most N such that

$$\mathcal{L}(x) \le \mathcal{F}(x) \le \mathcal{M}(x) \tag{2.64}$$

for all $x \in \mathbb{R}/\mathbb{Z}$, minimizing the integral

$$\int_{\mathbb{R}/\mathbb{Z}} \big\{\mathcal{M}(x) - \mathcal{L}(x)\big\}\, \mathrm{d}\vartheta(x). \tag{2.65}$$

When $\mathrm{d}\vartheta$ is the Lebesgue measure, this problem was considered, for instance, in [7, 19, 51, 73] in connection to discrepancy inequalities of Erdös-Turán type. For general even measures $\mathrm{d}\vartheta$, the case of even periodic functions with exponential subordination was considered in [5, 16]. In [51], Li and Vaaler solved this extremal problem for the sawtooth function

$$\psi(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & , \text{ if } x \notin \mathbb{Z}; \\ 0 & , \text{ if } x \in \mathbb{Z}; \end{cases}$$

with respect to the Jacobi measures.

The purpose of Section 2.5 is to extend the work [51], solving this problem for a general class of functions with exponential subordination (which are the periodizations of our functions $f_\mu(x)$ and $\widetilde{f}_\mu(x)$, including the sawtooth function as a particular case) with respect to arbitrary nontrivial probability measures $\mathrm{d}\vartheta$ (we say that $\mathrm{d}\vartheta$ is trivial if it has support on a finite number of points). The main tools we use here are the theory of reproducing kernel Hilbert spaces of polynomials and the theory of orthogonal polynomials in the unit circle, and we start by reviewing the terminology and the basic facts of these two well–established subjects. In doing so, we follow the notation of [16, 52, 69] to facilitate some of the references.

2.5.1 Reproducing Kernel Hilbert Spaces of Polynomials

We write $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ for the open unit disc and $\partial\mathbb{D}$ for the unit circle. Let $n \in \mathbb{Z}^+$ and let $\mathcal{P}_n$ be the set of polynomials of degree at most n with complex coefficients. If $Q \in \mathcal{P}_n$ we define the conjugate polynomial $Q^{*,n}(z)$ by

$$Q^{*,n}(z) = z^n\, \overline{Q\big(\bar{z}^{\,-1}\big)}. \tag{2.66}$$

If $Q(z)$ has exact degree n, we sometimes omit the superscript n and write $Q^*(z)$ for simplicity.

Let $P(z)$ be a polynomial of exact degree $n+1$ with no zeros on $\partial\mathbb{D}$ such that

$$|P^*(z)| < |P(z)| \tag{2.67}$$

for all $z \in \mathbb{D}$. We consider the Hilbert space $\mathcal{H}_n(P)$ consisting of the elements in $\mathcal{P}_n$ with scalar product

$$\langle Q, R\rangle_{\mathcal{H}_n(P)} = \int_{\mathbb{R}/\mathbb{Z}} Q(e(x))\,\overline{R(e(x))}\,|P(e(x))|^{-2}\,\mathrm{d}x.$$

From Cauchy's integral formula, it follows easily that the reproducing kernel for this finite–dimensional Hilbert space is given by

$$\mathcal{K}(w,z) = \frac{P(z)\overline{P(w)} - P^*(z)\overline{P^*(w)}}{1-\bar{w}z}\,,$$

that is, for every $w \in \mathbb{C}$ we have the identity

$$\langle Q, \mathcal{K}(w,\cdot)\rangle_{\mathcal{H}_n(P)} = Q(w).$$

As before, we define the companion polynomials

$$\mathcal{A}(z) := \frac{1}{2}\big\{P(z) + P^*(z)\big\} \quad \text{and} \quad \mathcal{B}(z) := \frac{i}{2}\big\{P(z) - P^*(z)\big\}\,,$$

and we find that $\mathcal{A}(z) = \mathcal{A}^*(z)$, $\mathcal{B} = \mathcal{B}^*(z)$ and $P(z) = \mathcal{A}(z) - i\mathcal{B}(z)$. Since the coefficients of z^0 and z^{n+1} of $P(z)$ do not have the same absolute value (this would contradict (2.67) at $z=0$) the polynomials $\mathcal{A}(z)$ and $\mathcal{B}(z)$ have exact degree $n+1$. From (2.67) we also see that $\mathcal{A}(z)$ and $\mathcal{B}(z)$ have all of their zeros in $\partial\mathbb{D}$.

The reproducing kernel has the alternative representation

$$\mathcal{K}(w,z) = \frac{2}{i}\left(\frac{\mathcal{B}(z)\overline{\mathcal{A}(w)} - \mathcal{A}(z)\overline{\mathcal{B}(w)}}{1-\bar{w}z}\right). \tag{2.68}$$

Observe that

$$\mathcal{K}(w,w) = \langle \mathcal{K}(w,\cdot), \mathcal{K}(w,\cdot)\rangle_{\mathcal{H}_n(P)} \geq 0$$

for all $w \in \mathbb{C}$. If there is $w \in \mathbb{C}$ such that $\mathcal{K}(w,w) = 0$, then $\mathcal{K}(w,\cdot) \equiv 0$ and $Q(w) = 0$ for every $Q \in \mathcal{P}_n$, a contradiction. Therefore $\mathcal{K}(w,w) > 0$ for all $w \in \mathbb{C}$. From the representation (2.68) it follows that $\mathcal{A}(z)$ and $\mathcal{B}(z)$ have only simple zeros and their zeros never agree.

From (2.68) we see that the sets $\{z \mapsto \mathcal{K}(\zeta, z);\ \mathcal{A}(\zeta) = 0\}$ and $\{z \mapsto \mathcal{K}(\zeta, z);\ \mathcal{B}(\zeta) = 0\}$ are orthogonal bases for $\mathcal{H}_n(P)$ and, in particular, we arrive at Parseval's formula (see [52, Theorem 2])

$$||Q||^2_{\mathcal{H}_n(P)} = \sum_{\mathcal{A}(\zeta)=0} \frac{|Q(\zeta)|^2}{\mathcal{K}(\zeta,\zeta)} = \sum_{\mathcal{B}(\zeta)=0} \frac{|Q(\zeta)|^2}{\mathcal{K}(\zeta,\zeta)}. \tag{2.69}$$

2.5.2 Orthogonal Polynomials in the Unit Circle

The map $x \mapsto e(x)$ allows us to identify measures on $\mathbb{R}/\mathbb{Z}$ with measures on the unit circle $\partial\mathbb{D}$. Let $\mathrm{d}\vartheta$ be a nontrivial probability measure on $\mathbb{R}/\mathbb{Z} \sim \partial\mathbb{D}$ (recall that $\mathrm{d}\vartheta$ is trivial if it has support on a finite number of points) and consider the space $L^2(\partial\mathbb{D}, \mathrm{d}\vartheta)$ with inner product given by

$$\langle f, g\rangle_{L^2(\partial\mathbb{D},\mathrm{d}\vartheta)} = \int_{\partial\mathbb{D}} f(z)\,\overline{g(z)}\,\mathrm{d}\vartheta(z) = \int_{\mathbb{R}/\mathbb{Z}} f(e(x))\,\overline{g(e(x))}\,\mathrm{d}\vartheta(x).$$

We define the *monic orthogonal polynomials* $\Phi_n(z) = \Phi_n(z; \mathrm{d}\vartheta)$ by the conditions

$$\Phi_n(z) = z^n + \text{lower order terms}\,; \qquad \langle \Phi_n, z^j\rangle_{L^2(\partial\mathbb{D},\mathrm{d}\vartheta)} = 0 \qquad (0 \leq j < n);$$

and we define the *orthonormal polynomials* by $\varphi_n(z) = c_n\Phi_n(z)/||\Phi_n||_2$, where c_n is a complex number of absolute value one such that $\varphi_n(1) \in \mathbb{R}$ (this normalization will be used later). Observe that

$$\langle Q^{*,n}, R^{*,n}\rangle_{L^2(\partial\mathbb{D},\mathrm{d}\vartheta)} = \langle R, Q\rangle_{L^2(\partial\mathbb{D},\mathrm{d}\vartheta)}$$

for all polynomials $Q, R \in \mathcal{P}_n$, where the conjugation map $*$ was defined in (2.66). The next lemma collects the relevant facts for our purposes from Simon's survey article [69].

Lemma 2.5.1. *Let* $\mathrm{d}\vartheta$ *be a nontrivial probability measure on* $\mathbb{R}/\mathbb{Z}$.

(*i*) $\varphi_n(z)$ *has all its zeros in* $\mathbb{D}$ *and* $\varphi_n^*(z)$ *has all its zeros in* $\mathbb{C}\setminus\overline{\mathbb{D}}$.

(*ii*) *Define a new measure* $\mathrm{d}\vartheta_n$ *on* $\mathbb{R}/\mathbb{Z}$ *by*

$$\mathrm{d}\vartheta_n(x) = \frac{\mathrm{d}x}{\left|\varphi_n(e(x); \mathrm{d}\vartheta)\right|^2},$$

Then $\mathrm{d}\vartheta_n$ *is a probability measure on* $\mathbb{R}/\mathbb{Z}$, $\varphi_j(z; \mathrm{d}\vartheta) = \varphi_j(z; \mathrm{d}\vartheta_n)$ *for* $j = 0, 1, \ldots, n$ *and for all* $Q, R \in \mathcal{P}_n$ *we have*

$$\langle Q, R\rangle_{L^2(\partial\mathbb{D}, \mathrm{d}\vartheta)} = \langle Q, R\rangle_{L^2(\partial\mathbb{D}, \mathrm{d}\vartheta_n)}. \tag{2.70}$$

Proof. (*i*) This is [69, Theorem 4.1].

(*ii*) This follows from [69, Theorem 2.4, Proposition 4.2 and Theorem 4.3]. □

Let $n \geq 0$ and $\varphi_{n+1}(z) = \varphi_{n+1}(z; \mathrm{d}\vartheta)$. By Lemma 2.5.1 (*i*) and the maximum principle we have

$$|\varphi_{n+1}(z)| < |\varphi_{n+1}^*(z)|$$

for all $z \in \mathbb{D}$. By Lemma 2.5.1 (*ii*) we note (Christoffel–Darboux formula) that $\mathcal{P}_n$ with the scalar product $\langle\cdot,\cdot\rangle_{L^2(\partial\mathbb{D}, \mathrm{d}\vartheta)}$ is a reproducing kernel Hilbert space with reproducing kernel given by

$$\mathcal{K}_n(w, z) = \frac{\varphi_{n+1}^*(z)\,\overline{\varphi_{n+1}^*(w)} - \varphi_{n+1}(z)\,\overline{\varphi_{n+1}(w)}}{1 - \bar{w}z}. \tag{2.71}$$

Observe that $\varphi^*_{n+1}(z)$ plays the role of $P(z)$ in Section 2.5.1. As before, we define the two companion polynomials (here we use the subscript according to the degree of the polynomial)

$$\mathcal{A}_{n+1}(z) = \frac{1}{2}\big\{\varphi^*_{n+1}(z) + \varphi_{n+1}(z)\big\} \quad \text{and} \quad \mathcal{B}_{n+1}(z) = \frac{i}{2}\big\{\varphi^*_{n+1}(z) - \varphi_{n+1}(z)\big\}, \tag{2.72}$$

and we note that (2.69) holds.

We now derive the quadrature formula that is suitable for our purposes. This result appears in [16, Corollary 26] and we present a short proof here for convenience.

Proposition 2.5.2. *Let* $\mathrm{d}\vartheta$ *be a nontrivial probability measure on* $\mathbb{R}/\mathbb{Z}$ *and let* $\mathcal{W} : \mathbb{C} \to \mathbb{C}$ *be a trigonometric polynomial of degree at most* N*. Let* $\varphi_{N+1}(z) = \varphi_{N+1}(z; \mathrm{d}\vartheta)$ *be the* $(N+1)$*-th orthonormal polynomial in the unit circle with respect to this measure and consider* $\mathcal{K}_N(w, z)$*,* $\mathcal{A}_{N+1}(z)$ *and* $\mathcal{B}_{N+1}(z)$ *as defined in* (2.71) *and* (2.72)*. Then we have*

$$\int_{\mathbb{R}/\mathbb{Z}} \mathcal{W}(x)\,\mathrm{d}\vartheta(x) = \sum_{\substack{\xi\in\mathbb{R}/\mathbb{Z} \\ \mathcal{A}_{N+1}(e(\xi))=0}} \frac{\mathcal{W}(\xi)}{\mathcal{K}_N(e(\xi), e(\xi))} = \sum_{\substack{\xi\in\mathbb{R}/\mathbb{Z} \\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{\mathcal{W}(\xi)}{\mathcal{K}_N(e(\xi), e(\xi))}.$$

Proof. Write

$$\mathcal{W}(z) = \sum_{k=-N}^{N} a_k\, e(kz)$$

and assume first that $\mathcal{W}(z)$ is real valued on $\mathbb{R}$, that is, $a_k = \overline{a_{-k}}$. Let $\tau = \min_{x\in\mathbb{R}} \mathcal{W}(x)$. Then $z \mapsto \mathcal{W}(z) - \tau$ is a real trigonometric polynomial of degree at most N that is non–negative on $\mathbb{R}$. By the Riesz-Féjer theorem there exists a polynomial $Q \in \mathcal{P}_N$ such that

$$\mathcal{W}(z) - \tau = Q(e(z))\,\overline{Q(e(\overline{z}))}$$

for all $z \in \mathbb{C}$. Writing $\tau = |\tau_1|^2 - |\tau_2|^2$, and using (2.70) and (2.69), we obtain

$$\begin{aligned}\int_{\mathbb{R}/\mathbb{Z}} \mathcal{W}(x)\,\mathrm{d}\vartheta(x) &= \int_{\mathbb{R}/\mathbb{Z}} \left\{|Q(e(x))|^2 + |\tau_1|^2 - |\tau_2|^2\right\} \mathrm{d}\vartheta(x) \\ &= \int_{\mathbb{R}/\mathbb{Z}} \frac{|Q(e(x))|^2 + |\tau_1|^2 - |\tau_2|^2}{\left|\varphi_{N+1}(e(x))\right|^2}\,\mathrm{d}x \\ &= \sum_{\substack{\xi\in\mathbb{R}/\mathbb{Z}\\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{|Q(e(\xi))|^2 + |\tau_1|^2 - |\tau_2|^2}{\mathcal{K}_N(e(\xi), e(\xi))} \\ &= \sum_{\substack{\xi\in\mathbb{R}/\mathbb{Z}\\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{\mathcal{W}(\xi)}{\mathcal{K}_N(e(\xi), e(\xi))},\end{aligned}$$

and analogously at the nodes given by the roots of $\mathcal{A}_{N+1}(z)$. The general case follows by writing $\mathcal{W}(z) = \mathcal{W}_1(z) - i\mathcal{W}_2(z)$, with $\mathcal{W}_1(z) = \sum_{k=-N}^{N} b_k\, e(kz)$ and $\mathcal{W}_2(z) = \sum_{k=-N}^{N} c_k\, e(kz)$, where $b_k = \frac{1}{2}(a_k + \overline{a_{-k}})$ and $c_k = \frac{i}{2}(a_k - \overline{a_{-k}})$. $\square$

2.5.3 Main Results

We now present the solution of the extremal problem (2.64) - (2.65) for a class of periodic functions with a certain exponential subordination. As described below, this class comes from the periodization of the functions $f_\mu(x)$ and $\widetilde{f}_\mu(x)$ defined in (2.13) and (2.18).

Throughout this section we let $\mathrm{d}\mu$ be a (locally finite) signed Borel measure on $\mathbb{R}$ satisfying conditions (H1') - (H2). The condition (H1') is simply a restriction of our current (H1), namely:

(H1') The measure $\mathrm{d}\mu$ has support on $[0, \infty)$.

When convenient, we may require additional properties on $\mathrm{d}\mu$. The first one is our usual (H3), and we now introduce the following summability condition:

(H4) The distribution function $\mu(x) := \mu((-\infty, x])$ verifies

$$\int_0^\infty \frac{1}{\lambda^2}\,\mu(\lambda)\,\mathrm{d}\lambda < \infty.$$

For $\lambda > 0$ we consider the following truncated function that appears on the right-hand side of (2.14):

$$v(\lambda, x) = \begin{cases} xe^{-\lambda x} & \text{if } x > 0; \\ 0, & \text{if } x \leq 0, \end{cases}$$

and define the 1–periodic function

$$\begin{aligned} h(\lambda, x) &:= \sum_{n\in\mathbb{Z}} v(\lambda, x+n) \\ &= \frac{e^{-\lambda(x-\lfloor x\rfloor-\frac{1}{2})}\big\{2\sinh(\lambda/2)(x-\lfloor x\rfloor-\frac{1}{2})+\cosh(\lambda/2)\big\}}{4\sinh(\lambda/2)^2}. \end{aligned}$$

If $\mathrm{d}\mu$ is a signed Borel measure satisfying (H1') - (H2) - (H4) we define the 1-periodic function

$$\mathcal{F}_\mu(x) := \int_0^\infty h(\lambda, x)\,\mu(\lambda)\,\mathrm{d}\lambda = \sum_{n\in\mathbb{Z}} f_\mu(x+n), \tag{2.73}$$

where the last equality follows from (2.14) and Fubini's theorem. We observe that $\mathcal{F}_\mu(x)$ is differentiable for $x \notin \mathbb{Z}$ and that

$$\mathcal{F}_\mu(0^-) = \mathcal{F}_\mu(0).$$

For $0 \leq x \leq 1$ we have

$$h(\lambda, x) = xe^{-\lambda x} + \frac{e^{-\lambda}}{\left(1-e^{-\lambda}\right)^2}\left(xe^{-\lambda x}\left(1-e^{-\lambda}\right)+e^{-\lambda x}\right),$$

and we see from dominated convergence and the computation in (2.15) that

$$\limsup_{x\to 0^+} \mathcal{F}_\mu(x) \leq \mathcal{F}_\mu(0) + 1,$$

and *under the additional condition* (H3) we have

$$\mathcal{F}_\mu(0^+) = \mathcal{F}_\mu(0^-) + 1 = \mathcal{F}_\mu(0) + 1. \tag{2.74}$$

We now define the odd counterpart. First we let, for $\lambda > 0$,

$$\widetilde{v}(\lambda, x) := v(\lambda, x) - v(\lambda, -x)$$

and consider the 1-periodic function

$$\begin{aligned}\widetilde{h}(\lambda, x) &:= \sum_{n\in\mathbb{Z}} \widetilde{v}(\lambda, x+n) \\ &= \frac{-\frac{1}{2}\cosh(\lambda/2)\sinh(\lambda\psi(x)) + \psi(x)\sinh(\lambda/2)\cosh(\lambda\psi(x))}{\sinh(\lambda/2)^2}.\end{aligned}$$

where

$$\psi(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & , \text{ if } x \notin \mathbb{Z}; \\ 0 & , \text{ if } x \in \mathbb{Z}; \end{cases}$$

is the sawtooth function. If $\mathrm{d}\mu$ is a signed Borel measure satisfying (H1') - (H2) we define the odd 1-periodic function

$$\widetilde{\mathcal{F}}_\mu(x) := \int_0^\infty \widetilde{h}(\lambda, x)\, \mu(\lambda)\, \mathrm{d}\lambda. \tag{2.75}$$

Note that *we do not have to assume* (H4) in order to define $\widetilde{\mathcal{F}}_\mu(x)$ in (2.75) since, for all $x \in \mathbb{R}$, the function $\lambda \mapsto \widetilde{h}(\lambda, x)$ is $O(\lambda)$ as $\lambda \to 0$. If, however, we have (H4), the function $\mathcal{F}_\mu(x)$ is well-defined and we have

$$\widetilde{\mathcal{F}}_\mu(x) = \mathcal{F}_\mu(x) - \mathcal{F}_\mu(-x) = \sum_{n\in\mathbb{Z}} \widetilde{f}_\mu(x+n),$$

verifying that $\widetilde{\mathcal{F}}_\mu(x)$ is in fact the periodization of $\widetilde{f}_\mu(x)$. We note that $\widetilde{\mathcal{F}}_\mu(x)$ is differentiable for $x \notin \mathbb{Z}$. For $0 \le x \le 1$ we may write alternatively

$$\begin{aligned}\widetilde{h}(\lambda, x) &= h(\lambda, x) - h(\lambda, 1-x) \\ &= xe^{-\lambda x} - (1-x)e^{-\lambda(1-x)} + \frac{e^{-\lambda}}{\left(1-e^{-\lambda}\right)^2}\Big(xe^{-\lambda x}\left(1-e^{-\lambda}\right) \\ &+ e^{-\lambda x} - (1-x)e^{-\lambda(1-x)}\left(1-e^{-\lambda}\right) - e^{-\lambda(1-x)}\Big),\end{aligned} \tag{2.76}$$

and we may use dominated convergence in (2.75) together with the computation in (2.15) to conclude that, under (H1') - (H2) - (H3), we have

$$\widetilde{\mathcal{F}}_{\mu}(0^{\pm}) = \pm 1.$$

We highlight the fact that when $\mathrm{d}\mu$ is the Dirac delta measure, we recover the sawtooth function (multiplied by -2) in (2.75). In fact, observing that for $x \notin \mathbb{Z}$ we have

$$\widetilde{h}(\lambda, x) = -\frac{\partial}{\partial \lambda}\left(\frac{\sinh\big(-\lambda(x - \lfloor x \rfloor - \frac{1}{2})\big)}{\sinh(\lambda/2)}\right),$$

we find, for $x \notin \mathbb{Z}$,

$$\widetilde{\mathcal{F}}_{\mu}(x) = \int_0^{\infty} \widetilde{h}(\lambda, x)\,\mathrm{d}\lambda = -2(x - \lfloor x \rfloor - \tfrac{1}{2}).$$

This is expected since the corresponding $\widetilde{f}_{\mu}(x)$ is the signum function. In particular, the results we present below extend the work of Li and Vaaler [51] on the sawtooth function.

The following two results provide a complete solution of the extremal problem (2.64) - (2.65) for the periodic functions $\mathcal{F}_{\mu}(x)$ and $\widetilde{\mathcal{F}}_{\mu}(x)$ defined in (2.73) and (2.75), with respect to arbitrary nontrivial probability measures $\mathrm{d}\vartheta$. This completes the framework initiated in [16], where this extremal problem was solved for an analogous class of even periodic functions with exponential subordination. In what follows we let $\varphi_{N+1}(z) = \varphi_{N+1}(z; \mathrm{d}\vartheta)$ be the $(N+1)$-th orthonormal polynomial in the unit circle with respect to this measure and consider $\mathcal{K}_N(w,z), \mathcal{A}_{N+1}(z), \mathcal{B}_{N+1}(\mathrm{z})$ as defined in (2.71) and (2.72).

Theorem 2.5.3. *Let* $\mathrm{d}\mu$ *be a signed Borel measure on* $\mathbb{R}$ *satisfying* (H1') - (H2) - (H4)*, and let* $\mathcal{F}_{\mu}(x)$ *be defined by* (2.73)*. Let* $\mathrm{d}\vartheta$ *be a nontrivial probability measure on* $\mathbb{R}/\mathbb{Z}$ *and* $N \in \mathbb{Z}^{+}$*.*

(*i*) *If* $\mathcal{L} : \mathbb{C} \to \mathbb{C}$ *is a real trigonometric polynomial of degree at most* N *such that*

$$\mathcal{L}(x) \leq \mathcal{F}_\mu(x) \tag{2.77}$$

for all $x \in \mathbb{R}/\mathbb{Z}$*, then*

$$\int_{\mathbb{R}/\mathbb{Z}} \mathcal{L}(x)\, \mathrm{d}\vartheta(x) \leq \frac{\mathcal{F}_\mu(0)}{\mathcal{K}_N(1,1)} + \sum_{\substack{\xi\in\mathbb{R}/\mathbb{Z}\,;\,\xi\neq 0\\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{\mathcal{F}_\mu(\xi)}{\mathcal{K}_N(e(\xi), e(\xi))}. \tag{2.78}$$

Moreover, there is a unique real trigonometric polynomial $\mathcal{L}_\mu : \mathbb{C} \to \mathbb{C}$ *of degree at most* N *satisfying* (2.77) *for which the equality in* (2.78) *holds.*

(*ii*) *Assume that* $\mathrm{d}\mu$ *also satisfies* (H3)*. If* $\mathcal{M} : \mathbb{C} \to \mathbb{C}$ *is a real trigonometric polynomial of degree at most* N *such that*

$$\mathcal{F}_\mu(x) \leq \mathcal{M}(x) \tag{2.79}$$

for all $x \in \mathbb{R}/\mathbb{Z}$*, then*

$$\int_{\mathbb{R}/\mathbb{Z}} \mathcal{M}(x)\, \mathrm{d}\vartheta(x) \geq \frac{\mathcal{F}_\mu(0^+)}{\mathcal{K}_N(1,1)} + \sum_{\substack{\xi\in\mathbb{R}/\mathbb{Z}\,;\,\xi\neq 0\\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{\mathcal{F}_\mu(\xi)}{\mathcal{K}_N(e(\xi), e(\xi))}. \tag{2.80}$$

Moreover, there is a unique real trigonometric polynomial $\mathcal{M}_\mu : \mathbb{C} \to \mathbb{C}$ *of degree at most* N *satisfying* (2.79) *for which the equality in* (2.80) *holds.*

Theorem 2.5.4. *Let* $\mathrm{d}\mu$ *be a signed Borel measure on* $\mathbb{R}$ *satisfying* (H1') - (H2) - (H3)*, and let* $\widetilde{\mathcal{F}}_\mu(x)$ *be defined by* (2.75)*. Let* $\mathrm{d}\vartheta$ *be a nontrivial probability measure on* $\mathbb{R}/\mathbb{Z}$ *and* $N \in \mathbb{Z}^+$*.*

(*i*) *If* $\mathcal{L} : \mathbb{C} \to \mathbb{C}$ *is a real trigonometric polynomial of degree at most* N *such that*

$$\mathcal{L}(x) \leq \widetilde{\mathcal{F}}_\mu(x) \tag{2.81}$$

for all $x \in \mathbb{R}/\mathbb{Z}$*, then*

$$\int_{\mathbb{R}/\mathbb{Z}} \mathcal{L}(x)\,\mathrm{d}\vartheta(x) \leq -\frac{1}{\mathcal{K}_N(1,1)} + \sum_{\substack{\xi\in\mathbb{R}/\mathbb{Z}\,;\,\xi\neq 0\\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{\widetilde{\mathcal{F}}_\mu(\xi)}{\mathcal{K}_N(e(\xi),e(\xi))}. \tag{2.82}$$

Moreover, there is a unique real trigonometric polynomial $\widetilde{\mathcal{L}}_\mu : \mathbb{C} \to \mathbb{C}$ *of degree at most* N *satisfying* (2.81) *for which the equality in* (2.82) *holds.*

(*ii*) *If* $\mathcal{M} : \mathbb{C} \to \mathbb{C}$ *is a real trigonometric polynomial of degree at most* N *such that*

$$\mathcal{F}_\mu(x) \leq \mathcal{M}(x) \tag{2.83}$$

for all $x \in \mathbb{R}/\mathbb{Z}$*, then*

$$\int_{\mathbb{R}/\mathbb{Z}} \mathcal{M}(x)\,\mathrm{d}\vartheta(x) \geq \frac{1}{\mathcal{K}_N(1,1)} + \sum_{\substack{\xi\in\mathbb{R}/\mathbb{Z}\,;\,\xi\neq 0\\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{\widetilde{\mathcal{F}}_\mu(\xi)}{\mathcal{K}_N(e(\xi),e(\xi))}. \tag{2.84}$$

Moreover, there is a unique real trigonometric polynomial $\widetilde{\mathcal{M}}_\mu : \mathbb{C} \to \mathbb{C}$ *of degree at most* N *satisfying* (2.83) *for which the equality in* (2.84) *holds.*

2.5.4 Periodic Interpolation

Before we proceed to the proofs of Theorems 2.5.3 and 2.5.4 we state and prove the periodic version of Proposition 2.2.3. Below we keep the notation already used in Section 2.2.

Proposition 2.5.5. *Let* $F(z)$ *be a* 1*-periodic Laguerre-Pólya function of exponential type* $\tau(F)$*. Assume that* $F(z)$ *has a double zero at the origin and that* $F(\alpha_F/2) > 0$*. Let* $\mathrm{d}\mu$ *be a signed Borel measure on* $\mathbb{R}$ *satisfying* (H1') - (H2) - (H4)*, and let* $\mathcal{F}_\mu(x)$ *be defined by* (2.73)*.*

(i) *The functions* $x \mapsto L(F,\mu,x)$ *and* $x \mapsto M(F,\mu,x)$ *belong to* $L^1(\mathbb{R})$.

(ii) *Define the trigonometric polynomials*

$$\mathcal{L}(F,\mu,z) = \sum_{|k|<\frac{\tau(F)}{2\pi}} \widehat{L}(F,\mu,k)\, e(kz) \tag{2.85}$$

and

$$\mathcal{M}(F,\mu,z) = \sum_{|k|<\frac{\tau(F)}{2\pi}} \widehat{M}(F,\mu,k)\, e(kz). \tag{2.86}$$

Then we have

$$F(x)\,\mathcal{L}(F,\mu,x) \leq F(x)\,\mathcal{F}_\mu(x) \leq F(x)\,\mathcal{M}(F,\mu,x) \tag{2.87}$$

for all $x \in \mathbb{R}$.

(iii) *Moreover,*

$$\mathcal{L}(F,\mu,\xi) = \mathcal{F}_\mu(\xi) = \mathcal{M}(F,\mu,\xi) \tag{2.88}$$

for all $\xi \in \mathbb{R} \setminus \mathbb{Z}$ *with* $F(\xi) = 0$. *At* $\xi \in \mathbb{Z}$ *we have*

$$\mathcal{L}(F,\mu,\xi) = \mathcal{F}_\mu(0) \quad \text{and} \quad \mathcal{M}(F,\mu,\xi) = \mathcal{F}_\mu(0) + 1. \tag{2.89}$$

Proof. We have already noted, from (2.32), that $z \mapsto L(F,\mu,z)$ and $z \mapsto M(F,\mu,z)$ are entire functions of exponential type at most $\tau(F)$. From (2.43) we find that

$$|L(F,\mu,x)| + |M(F,\mu,x)| \ll f_\mu(x) + \frac{1+|F(x)|}{1+x^2}$$

for $x \in \mathbb{R}$. Since $F(z)$ is 1-periodic, it is bounded on the real line. Hence, in order to prove *(i)*, it suffices to verify that $f_\mu \in L^1(\mathbb{R})$. This is a simple application of Fubini's theorem and conditions (H1') - (H2) - (H4). In fact,

$$\int_0^\infty f_\mu(x)\,\mathrm{d}x = \int_0^\infty \int_0^\infty x e^{-\lambda x}\,\mu(\lambda)\,\mathrm{d}\lambda\,\mathrm{d}x = \int_0^\infty \frac{1}{\lambda^2}\,\mu(\lambda)\,\mathrm{d}\lambda < \infty.$$

This establishes (i). The Paley–Wiener theorem implies that the Fourier transforms

$$\widehat{L}(F,\mu,t) = \int_{\mathbb{R}} L(F,\mu,x)\, e(-tx)\, \mathrm{d}x$$

and

$$\widehat{M}(F,\mu,t) = \int_{\mathbb{R}} M(F,\mu,x)\, e(-tx)\, \mathrm{d}x$$

are continuous functions supported in the compact interval $[-\frac{\tau(F)}{2\pi}, \frac{\tau(F)}{2\pi}]$. By a classical result of Plancherel and Pólya [65], the functions $z \mapsto L'(F,\mu,z)$ and $z \mapsto M'(F,\mu,z)$ also have exponential type at most $\tau(F)$ and belong to $L^1(\mathbb{R})$. Therefore, the Poisson summation formula holds as a pointwise identity and we have

$$\mathcal{L}(F,\mu,x) = \sum_{|k|<\frac{\tau(F)}{2\pi}} \widehat{L}(F,\mu,k)\, e(kx) = \sum_{n\in\mathbb{Z}} L(F,\mu,x+n) \tag{2.90}$$

and

$$\mathcal{M}(F,\mu,x) = \sum_{|k|<\frac{\tau(F)}{2\pi}} \widehat{M}(F,\mu,k)\, e(kx) = \sum_{n\in\mathbb{Z}} M(F,\mu,x+n). \tag{2.91}$$

Using the fact that

$$\mathcal{F}_\mu(x) = \sum_{n\in\mathbb{Z}} f_\mu(x+n)$$

for all $x \in \mathbb{R}$, (2.87), (2.88) and (2.89) now follow from (2.90), (2.91) and Proposition 2.2.3, since $F(z)$ is 1-periodic . This establishes (ii) and (iii). □

2.5.5 Proof of Theorem 2.5.3

Recall that we have normalized our orthonormal polynomials $\varphi_{N+1}(z)$ in order to have $\varphi_{N+1}(1) \in \mathbb{R}$. This implies that $\mathcal{B}_{N+1}(1) = 0$.

Optimality. If $\mathcal{L} : \mathbb{C} \to \mathbb{C}$ is a real trigonometric polynomial of degree at most N such that

$$\mathcal{L}(x) \leq \mathcal{F}_\mu(x)$$

for all $x \in \mathbb{R}/\mathbb{Z}$, from Proposition 2.5.2 we find that

$$\begin{aligned} \int_{\mathbb{R}/\mathbb{Z}} \mathcal{L}(x)\, \mathrm{d}\vartheta(x) &= \sum_{\substack{\xi \in \mathbb{R}/\mathbb{Z} \\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{\mathcal{L}(\xi)}{\mathcal{K}_N(e(\xi), e(\xi))} \\ &\leq \frac{\mathcal{F}_\mu(0)}{\mathcal{K}_N(1,1)} + \sum_{\substack{\xi \in \mathbb{R}/\mathbb{Z}\,;\, \xi \neq 0 \\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{\mathcal{F}_\mu(\xi)}{\mathcal{K}_N(e(\xi), e(\xi))}. \end{aligned} \tag{2.92}$$

This establishes (2.78). Under (H3) recall that we have

$$\mathcal{F}_\mu(0^+) = \mathcal{F}_\mu(0^-) + 1 = \mathcal{F}_\mu(0) + 1.$$

In an analogous way, using Proposition 2.5.2, it follows that if $\mathcal{M} : \mathbb{C} \to \mathbb{C}$ is a real trigonometric polynomial of degree at most N such that

$$\mathcal{F}_\mu(x) \leq \mathcal{M}(x)$$

for all $x \in \mathbb{R}/\mathbb{Z}$ then

$$\begin{aligned} \int_{\mathbb{R}/\mathbb{Z}} \mathcal{M}(x)\, \mathrm{d}\vartheta(x) &= \sum_{\substack{\xi \in \mathbb{R}/\mathbb{Z} \\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{\mathcal{M}(\xi)}{\mathcal{K}_N(e(\xi), e(\xi))} \\ &\geq \frac{\mathcal{F}_\mu(0^+)}{\mathcal{K}_N(1,1)} + \sum_{\substack{\xi \in \mathbb{R}/\mathbb{Z}\,;\, \xi \neq 0 \\ \mathcal{B}_{N+1}(e(\xi))=0}} \frac{\mathcal{F}_\mu(\xi)}{\mathcal{K}_N(e(\xi), e(\xi))}. \end{aligned}$$

This establishes (2.80).

Existence. Define the trigonometric polynomial

$$\mathfrak{B}_{N+1}(z) = \mathcal{B}_{N+1}(e(z))\, \overline{\mathcal{B}_{N+1}(e(\overline{z}))}. \tag{2.93}$$

Since the polynomial $\mathcal{B}_{N+1}(z)$ has degree $N+1$ and has only simple zeros in the unit circle, we conclude that the trigonometric polynomial $\mathfrak{B}_{N+1}(z)$ has degree $N+1$, is non–negative on $\mathbb{R}$ and has only double real zeros. Since every trigonometric polynomial is of bounded type in the upper half–plane $\mathbb{C}^+$, it follows by [6, Problem 34] that $\mathfrak{B}_{N+1}(z)$ is a Laguerre-Pólya function.

We now use Proposition 2.5.5 to construct the functions

$$\mathcal{L}_\mu(z) := \mathcal{L}(\mathfrak{B}_{N+1}, \mu, z);$$
$$\mathcal{M}_\mu(z) := \mathcal{M}(\mathfrak{B}_{N+1}, \mu, z).$$

Since $\mathfrak{B}_{N+1}(z)$ has exponential type $2\pi(N+1)$ we see from (2.85) and (2.86) that $\mathcal{L}_\mu(z)$ and $\mathcal{M}_\mu(z)$ are trigonometric polynomials of degree at most N. Since $\mathfrak{B}_{N+1}(z)$ is non–negative on $\mathbb{R}$ we conclude from (2.87) that

$$\mathcal{L}_\mu(x) \leq \mathcal{F}_\mu(x) \leq \mathcal{M}_\mu(x)$$

for all $x \in \mathbb{R}/\mathbb{Z}$. Moreover, from (2.88), (2.89) and the quadrature formula given by Proposition 2.5.2, we conclude that the equality in (2.78) holds. Under the additional condition (H3), we use (2.74) to see that the equality in (2.80) also holds.

Uniqueness. If $\mathcal{L} : \mathbb{C} \to \mathbb{C}$ is a real trigonometric polynomial of degree at most N satisfying (2.77) for which the equality in (2.78) holds, from (2.92) we must have

$$\mathcal{L}(\xi) = \mathcal{F}_\mu(\xi) = \mathcal{L}_\mu(\xi)$$

for all $\xi \in \mathbb{R}/\mathbb{Z}$ such that $\mathcal{B}_{N+1}(e(\xi)) = 0$. Since $\mathcal{F}_\mu(x)$ is differentiable at $\mathbb{R}/\mathbb{Z} - \{0\}$, from (2.77) we must also have

$$\mathcal{L}'(\xi) = \mathcal{F}'_\mu(\xi) = \mathcal{L}'_\mu(\xi)$$

for all $\xi \in \mathbb{R}/\mathbb{Z} - \{0\}$ such that $\mathcal{B}_{N+1}(e(\xi)) = 0$. These $2N+1$ conditions completely determine a trigonometric polynomial of degree at most N, hence $\mathcal{L}(z) = \mathcal{L}_\mu(z)$. The proof for the majorant is analogous.

2.5.6 Proof of Theorem 2.5.4

Optimality and Uniqueness. These follow exactly as in the proof of Theorem 2.5.3 using the fact that

$$\widetilde{\mathcal{F}}_\mu(0^\pm) = \pm 1.$$

Existence. We proceed with the construction of the extremal trigonometric polynomials in two steps:

Step 1. Suppose that $\mu(\lambda)$ satisfies (H4).

In this case we know that

$$\widetilde{\mathcal{F}}_\mu(x) = \mathcal{F}_\mu(x) - \mathcal{F}_\mu(-x) \tag{2.94}$$

for all $x \in \mathbb{R}$. With the notation of Proposition 2.5.5 and $\mathfrak{B}_{N+1}(z)$ given by (2.93), we define

$$\widetilde{\mathcal{L}}_\mu(z) = \mathcal{L}(\mathfrak{B}_{N+1}(z), \mu, z) - \mathcal{M}(\mathfrak{B}_{N+1}(-z), \mu, -z) \tag{2.95}$$

and

$$\widetilde{\mathcal{M}}_\mu(z) = \mathcal{M}(\mathfrak{B}_{N+1}(z), \mu, z) - \mathcal{L}(\mathfrak{B}_{N+1}(-z), \mu, -z). \tag{2.96}$$

It is clear from (2.94) and (2.87) that

$$\widetilde{\mathcal{L}}_\mu(x) \leq \widetilde{\mathcal{F}}_\mu(x) \leq \widetilde{\mathcal{M}}_\mu(x) \tag{2.97}$$

for all $x \in \mathbb{R}/\mathbb{Z}$. Moreover, from (2.88) and (2.89) we find that

$$\widetilde{\mathcal{L}}_\mu(\xi) = \widetilde{\mathcal{F}}_\mu(\xi) = \widetilde{\mathcal{M}}_\mu(\xi) \tag{2.98}$$

for all $\xi \in \mathbb{R}/\mathbb{Z} - \{0\}$ such that $\mathcal{B}_{N+1}(e(\xi)) = 0$ and

$$\widetilde{\mathcal{L}}_\mu(0) = -1 \quad \text{and} \quad \widetilde{\mathcal{M}}_\mu(0) = 1. \tag{2.99}$$

Using the quadrature formula given by Proposition 2.5.2, we see that equality holds in (2.82) and (2.84).

Step 2. The case of general $\mathrm{d}\mu$.

For every $n \in \mathbb{N}$ we define a measure $\mathrm{d}\mu_n$ given by

$$\mathrm{d}\mu_n(\Omega) := \mathrm{d}\mu\big(\Omega - \tfrac{1}{n}\big),$$

where $\Omega \subset \mathbb{R}$ is a Borel set. Note that $\mathrm{d}\mu_n$ satisfies (H1') - (H2) - (H3) - (H4). Let $\widetilde{\mathcal{F}}_n(x) := \widetilde{\mathcal{F}}_{\mu_n}(x)$, and $\widetilde{\mathcal{L}}_n(z) := \widetilde{\mathcal{L}}_{\mu_n}(z)$ and $\widetilde{\mathcal{M}}_n(z) := \widetilde{\mathcal{M}}_{\mu_n}(z)$ as in (2.95) and (2.96). Since properties (2.97), (2.98) and (2.99) hold for each $n \in \mathbb{N}$, in order to conclude, it suffices to prove that $\widetilde{\mathcal{F}}_n(x)$ converges pointwise to $\widetilde{\mathcal{F}}_\mu(x)$ and that $\widetilde{\mathcal{L}}_n(z)$ and $\widetilde{\mathcal{M}}_n(z)$ converge pointwise (passing to a subsequence, if necessary) to trigonometric polynomials $\widetilde{\mathcal{L}}_\mu(z)$ and $\widetilde{\mathcal{M}}_\mu(z)$.

Observe first that

$$\widetilde{\mathcal{F}}_n(x) = \int_0^\infty \widetilde{h}\big(\lambda + \tfrac{1}{n}, x\big)\, \mu(\lambda)\, \mathrm{d}\lambda \tag{2.100}$$

for all $x \in \mathbb{R}$. From (2.76) we see that, for $0 \leq x \leq 1$,

$$\Big|\widetilde{h}\big(\lambda + \tfrac{1}{n}, x\big)\Big| \leq x e^{-\lambda x} + (1-x) e^{-\lambda(1-x)} + r(\lambda), \tag{2.101}$$

where $r(\lambda)$ is $O(1)$ for $\lambda < 1$ and $O(e^{-\lambda})$ for $\lambda \geq 1$, uniformly in $x \in [0,1]$ and $n \in \mathbb{N}$. For any $x \in [0,1)$ the right-hand side of (2.101) belongs to $L^1(\mathbb{R}^+, \mu(\lambda)\,\mathrm{d}\lambda)$, and therefore we may use dominated convergence in (2.100) to conclude that $\widetilde{\mathcal{F}}_n(x) \to \widetilde{\mathcal{F}}_\mu(x)$ as $n \to \infty$.

From (2.43), (2.90) and (2.91) we find that

$$\widetilde{\mathcal{M}}_n(x) - \widetilde{\mathcal{L}}_n(x) = \frac{4\,\mathfrak{B}_{N+1}(x)}{\mathfrak{B}''_{N+1}(0)} \sum_{k \in \mathbb{Z}} \frac{1}{(x+k)^2}. \tag{2.102}$$

for all $x \in \mathbb{R}$. Note that the right-hand side of (2.102) is bounded since $\mathfrak{B}_{N+1}(z)$ is a trigonometric polynomial with a double zero at the integers. Therefore, we arrive at

$$-\frac{4\,\mathfrak{B}_{N+1}(x)}{\mathfrak{B}''_{N+1}(0)}\sum_{k\in\mathbb{Z}}\frac{1}{(x+k)^2}+\widetilde{\mathfrak{F}}_n(x)\le\widetilde{\mathfrak{L}}_n(x)\le\widetilde{\mathfrak{F}}_n(x)$$

and

$$\widetilde{\mathfrak{F}}_n(x)\le\widetilde{\mathfrak{M}}_n(x)\le\widetilde{\mathfrak{F}}_n(x)+\frac{4\,\mathfrak{B}_{N+1}(x)}{\mathfrak{B}''_{N+1}(0)}\sum_{k\in\mathbb{Z}}\frac{1}{(x+k)^2}.$$

From (2.100) and (2.101) we see that

$$\begin{aligned}\big|\widetilde{\mathfrak{F}}_n(x)\big| &\le \int_0^\infty xe^{-\lambda x}\,\mu(\lambda)\,\mathrm{d}\lambda+\int_0^\infty(1-x)e^{-\lambda(1-x)}\,\mu(\lambda)\,\mathrm{d}\lambda+\int_0^\infty r(\lambda)\,\mu(\lambda)\,\mathrm{d}\lambda\\ &\le C\end{aligned}$$

for all $x \in [0,1]$ and $n \in \mathbb{N}$, since each of the first two integrals is a continuous function of $x \in (0,1)$, with finite side limits as $x \to 0$ and $x \to 1$, due to condition (H3) and the computation in (2.15). This implies that $\widetilde{\mathfrak{L}}_n(z)$ and $\widetilde{\mathfrak{M}}_n(z)$ are uniformly bounded on $\mathbb{R}$. The $2N+1$ Fourier coefficients of $\widetilde{\mathfrak{L}}_n(z)$ and $\widetilde{\mathfrak{M}}_n(z)$ are then uniformly bounded on $\mathbb{R}$ and we can extract a subsequence $\{n_k\}$ such that $\widetilde{\mathfrak{L}}_{n_k}(z) \to \widetilde{\mathfrak{L}}_\mu(z)$ and $\widetilde{\mathfrak{M}}_{n_k}(z) \to \widetilde{\mathfrak{M}}_\mu(z)$ uniformly in compact sets, where $\widetilde{\mathfrak{L}}_\mu(z)$ and $\widetilde{\mathfrak{M}}_\mu(z)$ are trigonometric polynomials of degree at most N. This completes the proof.

Chapter 3

Reconstruction Formulas in de Branges Spaces

3.1 Preliminaries

One of the classical problems in complex analysis is to reconstruct an entire function from a countable set of data. For example, the Weierstrass factorization reconstructs a given entire function $F(z)$ using its set of zeros.

In this chapter we study another type of reconstruction based on *interpolation*. We consider the problem of reconstructing an entire function $F(z)$ from its values and the values of its derivatives up to a specified order at a discrete set of points on the real line. To accomplish this we use an interpolation formula. Some assumptions about the growth of $F(z)$ at infinity will be required in order to achieve complete characterizations.

Given two real numbers $p \in [1, \infty)$ and $\tau > 0$ the Paley–Wiener space $PW^p(\tau)$ is defined as the space of entire functions of exponential type at most τ such that their restriction to the real axis belongs to $L^p(\mathbb{R})$. The space $PW^p(\tau)$ is a Banach space, and it is a Hilbert space for $p = 2$. These are special spaces with a reproducing kernel structure. The reproducing kernel of $PW^p(\tau)$ is given by

$$K(w, z) = \frac{\sin \tau(z - \overline{w})}{\pi(z - \overline{w})}$$

and

$$F(w) = \int_{\mathbb{R}} F(x)\overline{K(w, x)}\mathrm{d}x$$

for every $F \in PW^p(\tau)$ and $w \in \mathbb{C}$.

A basic result of the theory of Paley–Wiener spaces is that for all $F \in PW^p(\tau)$ we have

$$F(z) = \frac{\sin(\tau z)}{\tau} \sum_{n\in\mathbb{Z}} (-1)^n \frac{F(\pi n/\tau)}{(z - \pi n/\tau)}, \tag{3.1}$$

where the sum converges uniformly in compact sets of $\mathbb{C}$ and also in the norm of the space if $p = 2$ (this is sometimes called the Shannon-Whittaker interpolation formula). The reproducing kernel structure is intrinsically related with the above formula by the fact that, when $p = 2$, the functions $\{K(\pi n/\tau, z)\}_{n\in\mathbb{Z}}$ form an orthogonal basis of the space and formula (3.1) is a simple representation of the function in terms of this basis.

The existence of interpolation formulas using derivatives is also known in Paley–Wiener spaces. In [73, Theorem 9] Vaaler proved that

$$F(z) = \left(\frac{\sin \tau z}{\tau}\right)^2 \sum_{n\in\mathbb{Z}} \left(\frac{F(\pi n/\tau)}{(z - \pi n/\tau)^2} + \frac{F'(\pi n/\tau)}{(z - \pi n/\tau)} \right) \tag{3.2}$$

for every $F \in PW^p(2\tau)$. We highlight that Vaaler's proof of (3.2) is based on Fourier Analysis, and his method does not generalize to the function spaces that we consider in this chapter. We aim to generalize this classical interpolation formula for $PW^p(\tau)$ to the setting of *de Branges spaces*.

The Paley–Wiener spaces are a special family of a wider class of spaces of entire functions $\mathcal{H}^p(E)$ called de Branges spaces. Intuitively, a de Branges space can be seen as a weighted Paley–Wiener space. Given a Hermite–Biehler function $E(z)$ (see the definition in Section 2.1.1) and a number $p \in [1, \infty)$, the space $\mathcal{H}^p(E)$ is defined as the space of entire functions $F(z)$ that satisfy a certain growth condition relatively to $E(z)$ and such that F/E belongs to $L^p(\mathbb{R})$. These spaces are uniquely determined by the structure function $E(z)$ of Hermite–Biehler class and they contain the Paley–Wiener space $PW^p(\tau)$ as a special case that can be recovered by using the function $E(z) = e^{-i\tau z}$.

Formula (3.2) is useful in many applications to approximation theory. In [40], Graham and Vaaler used this formula to construct extremal one–sided approximations of exponential type to a given real–valued function $g(x)$. Under certain restrictions on $g(x)$, they characterized the pair of entire functions $M(z)$ and $L(z)$ of exponential type at most 2π that satisfy $L(x) \leq g(x) \leq M(x)$ for all real x and minimize the quantities

$$\int_{\mathbb{R}} \{M(x) - g(x)\}\,\mathrm{d}x \quad \text{and} \quad \int_{\mathbb{R}} \{g(x) - L(x)\}\,\mathrm{d}x.$$

In [18], Carneiro, Littmann and Vaaler applied the same methods to produce extremal one–sided band–limited approximations for functions $g(x)$ that are in some sense subordinated to the Gaussian function. Other important works that apply such interpolation formulas are [14, 19, 73]. If, instead of the $L^1(\mathbb{R}, \mathrm{d}x)$-norm, one decides to minimize a weighted norm $L^1(\mathbb{R}, \mathrm{d}\mu(x))$, where $\mathrm{d}\mu$ is a non-negative Borel measure on the real line, the Fourier transform tools are no longer available. The alternative theory to approach these new extremal problems is the theory of de Branges spaces. Several works have been done in this direction, see [9, 13, 16, 17, 47, 54, 57]. The methods used in these later works were very different than the previous ones, since generalizations of the formula (3.2) to de Branges spaces were not known at the time. As already mentioned in Section 2.1, these special functions $M(z)$ and $L(z)$ have been used in a variety of interesting applications in number theory and analysis, for instance in connection to: large sieve inequalities [47, 67, 68, 73], Erdös-Turán inequalities [19, 73], Hilbert–type inequalities [16, 18, 19, 40, 55, 62, 63, 73], Tauberian theorems [40] and bounds in the theory of the Riemann zeta–function and L–functions [8, 9, 10, 11, 12, 22, 32, 34].

3.1.1 Problem Formulation

This work deals with *reproducing kernel Hilbert spaces* $\mathcal{H}^2$ of entire functions $F(z)$ with reproducing kernel $K(w,z)$. We require that the space $\mathcal{H}^2$ is closed under differentiation, that is, $F' \in \mathcal{H}^2$ whenever $F \in \mathcal{H}^2$. This last assumption will imply that the function

$$\partial_{\overline{w}}^j K(w,z)$$

is the reproducing kernel for the differential operator ∂_z^j, that is,

$$F^{(j)}(w) = \langle F(\cdot), \partial_{\overline{w}}^j K(w,\cdot)\rangle_{\mathcal{H}^2}$$

for all $F \in \mathcal{H}^2$.

We recall that a system $\{\varphi_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}^2$ is called a *weighted frame* if there exists $C > 0$ and $\lambda_n > 0$ such that

$$C^{-1}\|F\|_{\mathcal{H}^2}^2 \leq \sum_{n\in\mathbb{Z}} \lambda_n |\langle F, \varphi_n\rangle_{\mathcal{H}^2}|^2 \leq C\|F\|_{\mathcal{H}^2}^2 \tag{3.3}$$

for all $F \in \mathcal{H}^2$, and the frame is called *exact* if (3.3) fails to hold for any $C > 0$ if one of the terms in the series is removed.

In this terminology, for a given integer $\nu > 0$ we seek to find a discrete set of points $\mathcal{T}_\nu \subseteq \mathbb{R}$ such that the collection $\mathcal{D}_\nu$ of functions $D_{\nu,j}(z,t)$ defined by

$$D_{\nu,j}(z,t) = \partial_{\overline{w}}^j K(w,z)\Big|_{w=t} \quad t \in \mathcal{T}_\nu, \quad j = 0,\dots,\nu-1 \tag{3.4}$$

forms an exact, weighted frame of $\mathcal{H}^2$. In order to obtain an interpolation series we also seek to find a *dual frame* $\mathcal{G}_\nu$ consisting of functions $z \mapsto G_{\nu,j}(z,t) \in \mathcal{H}^2$ such that an inequality of the form (3.3) holds for $\mathcal{G}_\nu$, and

$$F(z) = \sum_{t\in\mathcal{T}_\nu} \sum_{j=0}^{\nu-1} F^{(j)}(t) G_{\nu,j}(z,t)$$

for all $F \in \mathcal{H}^2$, where the convergence of the series to $F(z)$ is in the norm of $\mathcal{H}^2$.

3.1.2 Main Results

A Hermite-Biehler function $E : \mathbb{C} \to \mathbb{C}$ is an entire function which satisfies the fundamental inequality

$$|E(\overline{z})| < |E(z)|$$

for all $z \in \mathbb{C}^+$. For every Hermite-Biehler function $E(z)$ we can associate the de Branges space $\mathcal{H}^2(E)$, which is a reproducing kernel Hilbert space of entire functions. Recall the definition of $\mathcal{H}^2(E)$, with reproducing kernel $K(w,z)$ and companion function $A(z)$ and $B(z)$ defined in Section 2.1.1.

We say that $\mathcal{H}^2(E)$ is closed under differentiation if $F' \in \mathcal{H}^2(E)$ whenever $F \in \mathcal{H}^2(E)$. Inequality (2.8) together with the fact that $w \in \mathbb{C} \mapsto K(w,w)$ is a continuous function implies that convergence in the norm of $\mathcal{H}^2(E)$ implies uniform convergence in compact sets of $\mathbb{C}$. As a consequence, differentiation defines a closed linear operator on $\mathcal{H}^2(E)$ and therefore by the Closed Graph Theorem defines a bounded linear operator on $\mathcal{H}^2(E)$.

Let $\nu > 0$ be an integer and $E(z)$ be a Hermite–Biehler function with no real zeros (hence the zeros of $B(z)$ are simple). In order to obtain the desired interpolation series, we need to work in $\mathcal{H}^2(E^\nu)$, which has the proper structure. Denote by $A_\nu(z)$ and $B_\nu(z)$ the real entire functions that satisfy $E(z)^\nu = A_\nu(z) - iB_\nu(z)$ and by $K_\nu(w,z)$ the reproducing kernel associated with $\mathcal{H}^2(E^\nu)$. We define the collection $\mathcal{B}_\nu$ of functions $z \mapsto B_{\nu,j}(z,t)$ given by

$$B_{\nu,j}(z,t) = \frac{B(z)^\nu}{(z-t)^j}, \tag{3.5}$$

where t varies among the real zeros of $B(z)$ and $1 \le j \le \nu$. For an integer $\ell \ge 0$ we denote by $P_{\nu,\ell}(z,t)$ the Taylor polynomial of degree ℓ of $B_{\nu,\nu}(z,t)^{-1}$ expanded into a power series at $z = t$ as a function of z, that is

$$\frac{(z-t)^\nu}{B(z)^\nu} = P_{\nu,\ell}(z,t) + O(|z-t|^{\ell+1}).$$

Finally, we denote by $\mathcal{G}_\nu$ the collection of functions $z \mapsto G_{\nu,j}(z,t)$ defined by

$$G_{\nu,j}(z,t) = B_{\nu,\nu-j}(z,t)\frac{P_{\nu,\nu-j-1}(z,t)}{j!} \tag{3.6}$$

for $j = 0, \ldots, \nu - 1$ and $B(t) = 0$. We note that

$$G_{\nu,j}(z,t) = \frac{(z-t)^j}{j!} - \frac{B(z)^\nu}{j!}\sum_{n\geq\nu-j} a_{\nu,n}(t)(z-t)^{n+j-\nu}$$

where the quantity $a_{\nu,n}(t)$ is the coefficient of $(z-t)^n$ in the Taylor expansion of $1/B_{\nu,\nu}(z,t)$ about the point $z = t$. We easily see that these functions satisfy the following property

$$G^{(\ell)}_{\nu,j}(s,t) = \delta_0(s-t)\delta_0(\ell - j) \tag{3.7}$$

for any $\ell, j = 0, \ldots, \nu - 1$ and s, t zeros of $B(z)$.

The next result essentially says that $\mathcal{D}_\nu$ defined in (3.4) is an exact, weighted frame for $\mathcal{H}^2(E^\nu)$ with dual frame $\mathcal{G}_\nu$. As part of the proof we will also show that $\mathcal{B}_\nu$ is a frame (not weighted) for $\mathcal{H}^2(E^\nu)$. We emphasize that $K(w,z)$ is the reproducing kernel of $\mathcal{H}^2(E)$ while $K_\nu(w,z)$ is the reproducing kernel of $\mathcal{H}^2(E^\nu)$.

Theorem 3.1.1. *Let $E(z)$ be a Hermite–Biehler function with phase function $\varphi(z)$. Let $\nu \geq 2$ be an integer with the property that the space $\mathcal{H}^2(E^\nu)$ is closed under differentiation and denote by D the norm of the differentiation operator. Assume also that $B \notin \mathcal{H}^2(E)$ and that there exists $\delta > 0$ such that $\varphi'(t) \geq \delta$ whenever $B(t) = 0$. Then the following statements hold:*

(1) *For every $F \in \mathcal{H}^2(E^\nu)$*

$$F(z) = \sum_{B(t)=0}\sum_{j=0}^{\nu-1} F^{(j)}(t)G_{\nu,j}(z,t) \tag{3.8}$$

where the series converges to $F(z)$ in the norm of $\mathcal{H}^2(E^\nu)$.

(2) *There exists a positive constant* $C = C(\nu, D, \delta) > 0$ *such that*

$$C^{-1}\|F\|_{\mathcal{H}^2(E^\nu)} \leq \sum_{B(t)=0} \sum_{j=0}^{\nu-1} \frac{|F^{(j)}(t)|^2}{K_\nu(t,t)} \leq C\|F\|_{\mathcal{H}^2(E^\nu)} \tag{3.9}$$

and

$$C^{-1}\|F\|_{\mathcal{H}^2(E^\nu)} \leq \sum_{B(t)=0} \sum_{j=0}^{\nu-1} K_\nu(t,t)|\langle F, G_{\nu,j}(\cdot,t)\rangle_{\mathcal{H}^2(E^\nu)}|^2 \leq C\|F\|_{\mathcal{H}^2(E^\nu)} \tag{3.10}$$

for all $F \in \mathcal{H}^2(E^\nu)$.

(3) *If any of the terms of the series in* (3.9) (*respect.* (3.10)) *is removed, the modified inequality fails to hold for some* $F \in \mathcal{H}^2(E^\nu)$, *for any choice of* $C > 0$ (*that is, the frames are exact*).

Remarks.

(1) The requirement $B \notin \mathcal{H}^2(E)$ is necessary, otherwise $B^\nu \in \mathcal{H}^2(E^\nu)$ and estimate (3.9) would not be valid for $F(z) = B(z)^\nu$. Also, the proof is based on an induction argument in which this condition is necessary for the base case.

(2) For $\nu = 1$ the two frames agree, and (3.9) holds with $C = 1$ without any assumption on the phase and the differentiation operator (see [6, Theorem 22]).

(3) Conditions for the boundedness of the differentiation operator were given by Baranov in [1, 2]. It was also shown there that $E(z)$ has exponential type and no real zeros if $\mathcal{H}^2(E^\nu)$ is closed under differentiation.

(4) The fact that every zero of $B(z)$ is also a zero of $B_\nu(z)$ for every $\nu \geq 1$ is a crucial ingredient in the proof of the proposed theorem.

The next theorem extends the previous interpolation result for $\mathcal{H}^p(E^\nu)$ with $p \neq 2$. However, the convergence of the formula is only uniformly in compact sets of $\mathbb{C}$.

Given a function $E(z)$ of Hermite–Biehler class and $p \in [1, \infty)$ the de Branges space $\mathcal{H}^p(E)$ is defined as the space of entire functions $F(z)$ such that $F(z)/E(z)$ and $F^*(z)/E(z)$ are of bounded type with non–positive mean type and $F(x)/E(x) \in L^p(\mathbb{R})$ when restricted to the real axis (see Appendix 6.1 for a detailed discussion of these spaces and their connection with Hardy spaces).

In what follows we will need $E(z)$ to satisfy some special conditions, namely:

(C1) The mean type of $E^*(z)/E(z)$ is negative, that is,

$$v(E^*/E) = \limsup_{y\to\infty} \frac{\log|E(-iy)/E(iy)|}{y} < 0.$$

(C2) There exists some $h > 0$ such that all the zeros of $E(z)$ lie in the half–plane $\operatorname{Im} z \leq -h$.

Theorem 3.1.2. *Assume all the hypotheses of Theorem 3.1.1. Then for every $p \in [1, 2)$ and $F \in \mathcal{H}^p(E^\nu)$ we have*

$$F(z) = \sum_{B(t)=0} \sum_{j=0}^{\nu-1} F^{(j)}(t) G_{\nu,j}(z,t), \tag{3.11}$$

where the formula converges uniformly in compact sets of $\mathbb{C}$. Furthermore, formula (3.11) is also valid for $F \in \mathcal{H}^p(E^\nu)$ with $p \in (2, \infty)$ if we additionally assume that $E(z)$ satisfies condition (C1) or condition (C2).

Remarks.

(1) We note that closure under differentiation of the space $\mathcal{H}^p(E)$ does not imply, in general, closure under differentiation of the space $\mathcal{H}^q(E)$ with $q \neq p$. However, this implication will be true if $E(z)$ satisfies condition (C1) or (C2) (see Theorem 3.3.4). Moreover, the proof of the above theorem relies heavily on the case $p = 2$, in which a reduction argument is used, and for this reason we only need to assume closure under differentiation of the space $\mathcal{H}^2(E^\nu)$.

(2) The functions $E(z)$ that satisfy all the properties of the previous theorem in general do not have simple analytic expressions. A better way to construct such functions is via their Weierstrass factorization formula. A special subfamily of Hermite–Biehler functions with a manageable Weierstrass factorization formula is the Pólya class. This class is defined as those entire functions that can be arbitrarily approximated in any compact set of $\mathbb{C}$ by polynomials with no zeros in the upper half–plane (see [6, Section 7]). In fact, a function of Pólya class can be characterized by its Hadamard factorization formula. A function $E(z)$ with nonzero zeros $w_n = x_n - iy_n$, belongs to the Pólya class if and only if it has the following factorization

$$E(z) = E^{(r)}(0)(z^r/r!)e^{-az^2-ibz}\prod_n\left(1-\frac{z}{w_n}\right)e^{zh_n},$$

where $a \geq 0$, $\operatorname{Re} b \geq 0$, $h_n = \frac{x_n}{|w_n|^2}$, $y_n \geq 0$ and

$$\sum_n \frac{1+y_n}{|w_n|^2} < \infty.$$

In this situation, condition (C1) is equivalent to $\operatorname{Re} b > 0$.

Notation Remark. Given two non-negative quantities Q and Q' and N elements $r_1, \ldots, r_N$ of a set Ω we write $Q \ll_{r_1,\ldots,r_N} Q'$ when $Q \leq C(r_1, \ldots, r_N)Q'$, where $C : \Omega \to (0, \infty)$ is some function. We also write $Q \simeq_{r_1,\ldots,r_N} Q'$ when both $Q \ll_{r_1,\ldots,r_N} Q'$ and $Q' \ll_{r_1,\ldots,r_N} Q$ hold. Often, the quantities Q and Q' will depend on a function $F(z)$ and other quantities. We write $Q(F) \ll Q'(F)$ when there exists a constant $C > 0$, which does not depend on $F(z)$, such that $Q(F) \leq CQ'(F)$.

3.2 The L^2–case

The recipe for the proof of Theorem 3.1.1 is the following.

(1) We show that the span of the collection $\mathcal{G}_\nu$ is dense in $\mathcal{H}^2(E^\nu)$, which in turn by (3.7) implies that there exists a dense set of functions in $\mathcal{H}^2(E^\nu)$ for which (3.8) holds.

(2) We derive estimates involving the inner products of the collection $\mathcal{G}_\nu$ in order to prove Theorem 3.1.1 item (2) for a dense set of functions (and hence for the whole space).

First we prove density statements for the classes $\mathcal{B}_\nu$ and $\mathcal{G}_\nu$.

Lemma 3.2.1. *Let $E(z)$ be a Hermite–Biehler function with no real zeros and assume that $B \notin \mathcal{H}^2(E)$. Then the span of $\mathcal{B}_\nu$ and the span of $\mathcal{G}_\nu$ defined in* (3.5) *and* (3.6) *are both dense in $\mathcal{H}^2(E^\nu)$ for every integer $\nu \geq 1$.*

Proof. First we show via induction on ν that the span of the collection $\mathcal{B}_\nu$ is dense in $\mathcal{H}^2(E^\nu)$. Recall that in this scenario the zeros of $A(z)$ and $B(z)$ are simple and interlace.

It follows from Theorem 2.1.1 that the span of $\mathcal{B}_1$ is dense in $\mathcal{H}^2(E)$. Let $\nu > 0$ be an integer and assume that the span of $\mathcal{B}_\nu$ is dense in $\mathcal{H}^2(E^\nu)$.

By [1, Lemma 4.1], if $E_a(z)$ and $E_b(z)$ are two Hermite–Biehler functions then

$$\mathcal{H}^2(E_aE_b) = E_a^*\mathcal{H}^2(E_b) \oplus^\perp E_b\mathcal{H}^2(E_a),$$

where the sum is orthogonal. This implies that

$$\mathcal{H}^2(E^{\nu+1}) = A\mathcal{H}^2(E^\nu) \oplus B\mathcal{H}^2(E^\nu),$$

where the sum is direct. Therefore, the span of the collection $\mathcal{C} = A\mathcal{B}_\nu \cup B\mathcal{B}_\nu$ is dense in $\mathcal{H}^2(E^{\nu+1})$. Evidently $B\mathcal{B}_\nu$ is a subset of $\mathcal{B}_{\nu+1}$, and it remains to show that the collection $A\mathcal{B}_\nu$ can be arbitrarily approximated in the norm of $\mathcal{H}^2(E^{\nu+1})$ by elements of the span of $\mathcal{B}_{\nu+1}$.

Now note that if t is a zero of $B(z)$ then

$$A(z)B_{\nu,\nu}(z,t) = \frac{A(t)}{B'(t)}[B_{\nu+1,\nu+1}(z,t) - B(z)G(z,t)],$$

where

$$G(z,t) = B_{\nu-1,\nu-1}(z,t)\frac{A(t)B(z) - B'(t)(z-t)A(z)}{A(t)(z-t)^2}.$$

Evidently $G(z,t) \in \mathcal{H}^2(E^\nu)$ for each zero t of $B(z)$. Hence, for any given $\varepsilon > 0$ there exists $H(z)$ belonging to the span of $\mathcal{B}_\nu$ such that

$$\|G(z,t) - H(z)\|_{E^\nu} < \varepsilon B'(t)/A(t).$$

It follows that $[B_{\nu+1,\nu+1}(z,t) - B(z)H(z)]$ belongs to the span of $\mathcal{B}_{\nu+1}$ and

$$\begin{aligned}\left\|A(z)B_{\nu,\nu}(z,t) - \frac{A(t)}{B'(t)}[B_{\nu+1,\nu+1}(z,t) - B(z)H(z)]\right\|_{E^{\nu+1}} \\ = \frac{A(t)}{B'(t)}\|B(z)[G(z,t) - H(z)]\|_{E^{\nu+1}} \\ \leq \frac{A(t)}{B'(t)}\|G(z,t) - H(z)\|_{E^\nu} < \varepsilon.\end{aligned}$$

We conclude that $A(z)B_{\nu,\nu}(z,t)$ is an element of the closure of the span of $\mathcal{B}_{\nu+1}$. If $1 \le j < \nu$ then evidently $A(z)B_{\nu,j}(z,t) = B(z)H(z)$ for some $H \in \mathcal{H}^2(E^\nu)$ and the same argument above could be replicated. This proves the first part of the lemma.

Now, denote by $a_{\nu,j}(t)$ the coefficient of $(z-t)^j$ in the Taylor series representation of $1/B_{\nu,\nu}(z,t)$ as a function of z about $z=t$. Then the function $G_{\nu,k}(z,t)$ may be represented as

$$\begin{aligned} G_{\nu,k}(z,t) &= \frac{1}{k!}\sum_{j=0}^{\nu-k-1} a_{\nu,j}(t)B_{\nu,\nu-j-k}(z,t) \\ &= \frac{1}{k!}\sum_{m=1}^{\nu-k} a_{\nu,\nu-m-k}(t)B_{\nu,m}(z,t). \end{aligned}$$

Suppressing the arguments t and z, this is in matrix notation

$$(G_{\nu,k})_{0\le k\le \nu-1} = \begin{pmatrix} \frac{a_{\nu,\nu-1}}{0!} & \cdots & \frac{a_{\nu,1}}{0!} & \frac{a_{\nu,0}}{0!} \\ \frac{a_{\nu,\nu-2}}{1!} & \cdots & \frac{a_{\nu,0}}{1!} & 0 \\ \vdots & & 0 & 0 \\ \frac{a_{\nu,0}}{(\nu-1)!} & \cdots & 0 & 0 \end{pmatrix} (B_{\nu,m})_{1\le m\le \nu}.$$

Since $a_{\nu,0}(t) = 1/B'(t)^\nu \ne 0$, it follows that the above matrix is invertible and, in particular, any element of $\mathcal{B}_\nu$ is a linear combination of elements from $\mathcal{G}_\nu$ and vice versa. This concludes the lemma. □

For the proof of item (2) of Theorem 3.1.1 we will need to estimate the norm of a linear combination of elements from $\mathcal{B}_\nu$. The following four lemmas collect the necessary upper bounds for each term that will appear.

Lemma 3.2.2. *Let $\nu \ge 2$ be an integer and let $E(z)$ be a Hermite–Biehler function such that $\mathcal{H}^2(E^\nu)$ is closed under differentiation and denote by D the norm of the differentiation operator on $\mathcal{H}^2(E^\nu)$. Then:*

(1) *The derivative of the phase function is bounded. In fact we have*

$$\varphi'(x) \leq D\sqrt{\nu} \tag{3.12}$$

for all real x.

(2) *The zeros of $B(z)$ are separated. In fact, if $t > s$ are two consecutive real zeros of $B(z)$ then*

$$t - s \geq \frac{\pi}{\sqrt{\nu}D}.$$

(3) *For all real x and real t with $B(t) = 0$ we have*

$$\left| \frac{B(x)}{E(x)(x-t)} \right| \leq D\sqrt{\nu}.$$

Proof. Item (1). Recall that, by the remarks after Theorem 3.1.1 the function $E(z)$ has no real zeros. Using the reproducing kernel definition (2.6) we deduce that

$$K_\nu(x,x) = \nu|E(x)|^{2(\nu-1)}K(x,x)$$

for all real x, where $K_\nu(w,z)$ and $K(w,z)$ are respectively the reproducing kernels associated with $\mathcal{H}^2(E^\nu)$ and $\mathcal{H}^2(E)$. This implies that

$$K_\nu(t,t) = \frac{\nu}{\pi}A(t)^{2\nu-1}B'(t)$$

whenever $B(t) = 0$. We prove first (3.12) for $x = t$ where t is a zero of $B(z)$. Consider the entire function $F(z)$ defined by

$$F(z) = \pi E(z)^{\nu-2}B(z)K(t,z).$$

Evidently, $F \in \mathcal{H}^2(E^\nu)$ and by (2.8) we obtain

$$|F'(t)|^2 \leq \|F'\|^2_{E^\nu}K_\nu(t,t) \leq D^2\|F\|^2_{E^\nu}K_\nu(t,t).$$

Now observe that

$$\|F\|_{E^\nu}^2 \leq \pi^2 \|K(t,z)\|_E^2 = \pi B'(t)A(t)$$

and that $F'(t) = A(t)^{\nu-1}B'(t)^2$. By identity (2.9), $\varphi'(t) = B'(t)/A(t)$ and we obtain (3.12) for $x = t$.

Now, let $\theta \in \mathbb{R}$ be arbitrary and denote by $\varphi_\theta(x)$ the phase of $E_{(\theta)}(z) = e^{i\theta}E(z) = A_{(\theta)}(z) - B_{(\theta)}(z)$. Observe that $\varphi_\theta(x)$ and $\varphi(x) - \theta$ differ only by an integer multiple of 2π, hence $\varphi'(x) = \varphi_\theta'(x)$. Moreover, the functions $E(z)^\nu$ and $(E_{(\theta)}(z))^\nu$ generate the same space and the real zeros of $B_{(\theta)}(z)$ coincide with the points $\varphi(x) \equiv \theta \mod \pi$. Hence, the above argument for the space $\mathcal{H}^2(E_{(\theta)}^\nu)$ gives the claim for arbitrary $x \in \mathbb{R}$.

Item (2). Since the zeros of $B(z)$ coincide with the points t such that $\varphi(t) \equiv 0 \mod \pi$ we deduce that

$$\pi = \varphi(t) - \varphi(s) \leq D\sqrt{\nu}(t-s)$$

if $t > s$ are two consecutive zeros of $B(z)$.

Item (3). By inequality (2.8) we obtain $|K(w,z)|^2 \leq K(w,w)K(z,z)$ for all $w, z \in \mathbb{C}$. We obtain

$$\left|\frac{B(x)}{E(x)(x-t)}\right|^2 = \pi^2 \frac{K(t,x)^2}{|E(t)|^2|E(x)|^2} \leq \pi^2 \frac{K(x,x)K(t,t)}{|E(t)|^2|E(x)|^2} = \varphi'(t)\varphi'(x) \leq D^2\nu.$$

This concludes the lemma.

□

Lemma 3.2.3. *Let $E(z)$ be a Hermite–Biehler function with no real zeros, and let $\nu \geq 1$ be an integer. Then for any two distinct zeros s, t of $B(z)$ we have*

$$\langle B_{\nu,1}(\cdot, s), B_{\nu,1}(\cdot, t)\rangle_{E^\nu} = 0.$$

Proof. Define an entire function $I(z)$ by

$$I(z) = \frac{B(z)^{2\nu}}{(E(z)E^*(z))^\nu (z-s)(z-t)},$$

where s and t are two zeros of $B(z)$. We need to show that

$$\int_{\mathbb{R}} I(x)\mathrm{d}x = 0.$$

Let $K > 0$. Define a contour Γ_K in $\mathbb{C}$ by replacing in $[-K, K]$ the segments $[t-\delta, t+\delta]$ and $[s-\delta, s+\delta]$ with semicircles in the lower half–plane of radius δ and centers s and t, respectively, traced counterclockwise (here δ is chosen so small that the disks of radius δ about s and t contain no zero of $E(z)$ or $E^*(z)$). Since $I(z)$ is analytic in a neighborhood of $\mathbb{R}$, the integrals of $I(z)$ over $[-K, K]$ and over Γ_K are equal by the residue theorem.

We note that

$$\begin{aligned}\frac{B(z)^{2\nu}}{(E(z)E^*(z))^\nu} &= \frac{i^\nu}{2^\nu}\sum_{j=0}^{2\nu}\binom{2\nu}{j}\frac{E(z)^j E^*(z)^{2\nu-j}}{(E(z)E^*(z))^\nu} \\ &= \frac{i^\nu}{2^\nu}\sum_{j=0}^{2\nu}\binom{2\nu}{j}\left(\frac{E(z)}{E^*(z)}\right)^{j-\nu}.\end{aligned}$$

Define $I_j(z)$ for $j \in \{0, \ldots, 2\nu\}$ by

$$I_j(z) = \frac{1}{(z-s)(z-t)}\left(\frac{E(z)}{E^*(z)}\right)^{j-\nu}.$$

Each $I_j(z)$ is a meromorphic function in $\mathbb{C}$ with poles at $z = s$ and $z = t$, and with additional poles in the lower or upper half–plane depending on whether $j < \nu$ or $j > \nu$. For $j > \nu$ the function $(z-t)(z-s)I_j(z)$ is bounded and analytic in the lower half–plane. We close the contour Γ_K by a semicircle with center at the origin and radius K in the lower half–plane, traced clockwise.

Since none of the poles of $I_j(z)$ are contained in the region enclosed by this contour, the integral of $I_j(z)$ over this contour is equal to zero.

For $j < \nu$ (where $(z-s)(z-t)I_j(z)$ is bounded in the upper half–plane) we close the contour in the upper half–plane with a semicircle of radius K and center at the origin traced counterclockwise. We call this contour C_K. The poles at s and t of I_j are in the region enclosed by C_K, while all other poles are in the unbounded component of $\mathbb{C}\backslash C_K$. A partial fraction decomposition, the residue theorem, and the identities $E(s) = E^*(s) = A(s)$ and $E(t) = E^*(t) = A(t)$ give

$$\frac{1}{2\pi i}\int_{C_K} I_j(z)dz = \frac{1}{s-t}\left[\left(\frac{E(s)}{E^*(s)}\right)^{j-\nu} - \left(\frac{E(t)}{E^*(t)}\right)^{j-\nu}\right] = 0.$$

An analogous calculation shows that the integral of $I_\nu(z)$ over C_K equals zero. Letting K go to infinity gives the claim. □

Lemma 3.2.4. *Assume all hypotheses of Theorem 3.1.1. Then the following statements hold:*

(1) *For all $s \neq t$ with $B(s) = B(t) = 0$ and any $j = 2, \ldots, \nu$ we have*

$$|\langle B_{\nu,j}(\cdot, s), B_{\nu,j}(\cdot, t)\rangle_{E^\nu}| \ll_{D,\nu} \frac{1}{(s-t)^2}.$$

(2) *For all t with $B(t) = 0$ and $j = 1, \ldots, \nu$ we have*

$$\|B_{\nu,j}(\cdot, t)\|_{E^\nu} \ll_{D,\nu} 1.$$

(3) *Denote by $a_{\nu,j}(t)$ the coefficient of $(z-t)^j$ in the Taylor series expansion of $B_{\nu,\nu}(z,t)^{-1}$ about the point $z = t$. Then*

$$|a_{\nu,j}(t)|^2 \ll_{D,\nu,\delta} \frac{1}{K_\nu(t,t)}.$$

Proof. Item (1). Using the fact that $|B(x)| \leq |E(x)|$ for all real x we deduce that

$$\langle B_{2,2}(\cdot,s), B_{2,2}(\cdot,t)\rangle_{E^2} \leq \left\| \frac{B(\cdot)}{(\cdot - t)(\cdot - s)} \right\|_E^2 = \pi\left(\frac{\varphi'(t)+\varphi'(s)}{(t-s)^2}\right)$$
$$\ll_{D,\nu} (t-s)^{-2},$$

where the identity above is due to formula (2.10) and the last inequality due to (3.12). Now, for any $j = 2, \ldots, \nu$ we have the following sequence of estimates

$$\left|\frac{B_{\nu,j}(x,t)B_{\nu,j}(x,s)}{|E(x)|^{2\nu}}\right| = \frac{B(x)^{2\nu}}{|x-t|^j|x-s|^j|E(x)|^{2\nu}}$$
$$\leq \frac{B(x)^{2j}}{|x-t|^j|x-s|^j|E(x)|^{2j}}$$
$$\ll_{D,\nu} \frac{B(x)^4}{|x-t|^2|x-s|^2|E(x)|^4},$$

where the last inequality above is due to item (3) of Lemma 3.2.2. We conclude that

$$|\langle B_{\nu,j}(\cdot,s), B_{\nu,t}(\cdot,t)\rangle_{E^\nu}| \ll_{D,\nu} \langle B_{2,2}(\cdot,s), B_{2,2}(\cdot,t)\rangle_{E^2} \ll_{D,\nu} (t-s)^{-2}.$$

Item (2). We can apply item (2) of Lemma 3.2.2 to deduce that

$$\left|\frac{B(x)^\nu}{(x-t)^j E(x)^\nu}\right| \ll_{D,\nu} \left|\frac{B(x)}{(x-t)E(x)}\right|$$

for all real x and $j = 1, \ldots, \nu$. Since $B(z)/(z-t) = \pi K(t,z)/A(t)$, it follows that

$$\|B_{\nu,j}(z,t)\|_{E^\nu}^2 \ll_{D,\nu} \|\pi K(t,z)/A(t)\|_E^2 = \pi B'(t)/A(t) = \pi\varphi'(t) \leq \pi D\sqrt{\nu}.$$

Item (3). Denote by $b_{\nu,j}(t)$ the coefficients of the power series expansion of $B_{\nu,\nu}(z,t)$ as a function of z about $z = t$. The assumption that $\varphi'(t) \geq \delta$ whenever $B(t) = 0$ in conjunction with (3.12) implies that

$$|b_{\nu,0}(t)|^2 = |B'(t)^\nu|^2 \simeq_{D,\nu,\delta} K_\nu(t,t).$$

Also, for $j = 1, \ldots, \nu$ we have

$$|b_{\nu,j}(t)|^2 = \frac{1}{(j!)^2}|B^{(j)}(t,t)|^2 \leq \frac{1}{(j!)^2}\|B^{(j)}_{\nu,\nu}(\cdot,t)\|^2_{E^\nu} K_\nu(t,t) \ll_{D,\nu} K_\nu(t,t). \tag{3.13}$$

We obtain

$$\frac{|b_{\nu,j}(t)|^2}{|b_{\nu,0}(t)|^2} \ll_{D,\nu,\delta} 1. \tag{3.14}$$

Now note that for $\ell = 1, \ldots, \nu - 1$

$$0 = \partial_z^\ell \left[\frac{B_{\nu,\nu}(z,t)}{B_{\nu,\nu}(z,t)}\right]_{z=t} = \ell! \sum_{j=0}^{\ell} a_{\nu,j}(t) b_{\nu,\ell-j}(t).$$

Hence the relation between $a_{\nu,j}(t)$ and $b_{\nu,j}(t)$ is given by a triangular matrix with diagonal terms equal to $b_{\nu,0}(t)$. Using (3.13) and (3.14) we conclude that

$$|a_{\nu,j}(t)|^2 \ll_{D,\nu,\delta} 1/K_\nu(t,t).$$

This concludes the lemma. □

For the sake of completeness we state here a result about Hilbert–type inequalities proved in [18, Corollary 22].

Proposition 3.2.5 (Carneiro, Littmann and Vaaler)**.** *Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $|\lambda_n - \lambda_m| \geq \sigma$ whenever $m \neq n$, for some $\sigma > 0$. Let $a_1, a_2, \ldots, a_N$ be complex numbers. Then*

$$-\frac{\pi^2}{6\sigma^2}\sum_{n=1}^{N}|a_n|^2 \leq \sum_{\substack{m,n=1\\ m\neq n}}^{N} \frac{a_n\overline{a_m}}{(\lambda_n - \lambda_m)^2} \leq \frac{\pi^2}{3\sigma^2}\sum_{n=1}^{N}|a_n|^2.$$

The constants appearing in these inequalities are the best possible (as $N \to \infty$).

The next lemma estimates the norm of the linear combination of elements from $\mathcal{G}_\nu$. This is one of the two inequalities needed to show that this collection is a (weighted) frame.

Lemma 3.2.6. *Assume all hypotheses of Theorem 3.1.1. Let $c_j(t) \in \mathbb{C}$ for every zero t of $B(z)$ and $j \in \{0, \ldots, \nu - 1\}$ be such that*

$$\sum_{B(t)=0} \sum_{j=0}^{\nu-1} \frac{|c_j(t)|^2}{K_\nu(t,t)} < \infty.$$

Then

$$\Big\| \sum_{B(t)=0} \sum_{j=0}^{\nu-1} c_j(t) G_{\nu,j}(\cdot, t) \Big\|_{\mathcal{H}^2(E^\nu)} \ll_{D,\nu,\delta} \sum_{B(t)=0} \sum_{j=0}^{\nu-1} \frac{|c_j(t)|^2}{K_\nu(t,t)}.$$

Proof. Define

$$d_m(t) = \sum_{j=0}^{\nu-m} a_{\nu,\nu-m-j}(t) \frac{c_j(t)}{j!},$$

where $a_{\nu,j}(t)$ denotes the coefficient of $(z-t)^j$ in the Taylor series expansion of $B_{\nu,\nu}(z,t)^{-1}$ about the point $z = t$. By item (3) of Lemma 3.2.4 we trivially obtain

$$\sum_{m=1}^{\nu} |d_m(t)|^2 \ll_{D,\nu,\delta} \sum_{j=0}^{\nu-1} \frac{|c_j(t)|^2}{K_\nu(t,t)}. \tag{3.15}$$

For a given $T > 0$ we obtain

$$\begin{aligned}
\Big\| \sum_{\substack{B(t)=0\\|t|\le T}} \sum_{j=0}^{\nu-1} c_j(t) G_{\nu,j}(z,t) \Big\|_{E^\nu}^2 &= \Big\| \sum_{\substack{B(t)=0\\|t|\le T}} \sum_{j=0}^{\nu-1} \frac{c_j(t)}{j!} \sum_{m=1}^{\nu-j} a_{\nu,\nu-m-j}(t) B_{\nu,m}(z,t) \Big\|_{E^\nu}^2 \\
&= \Big\| \sum_{\substack{B(t)=0\\|t|\le T}} \sum_{m=1}^{\nu} d_m(t) B_{\nu,m}(z,t) \Big\|_{E^\nu}^2 \\
&\ll \sum_{m=1}^{\nu} \sum_{\substack{B(t)=0\\|t|\le T}} |d_m(t)|^2 + \sum_{m=2}^{\nu} \sum_{\substack{B(s)=0\\B(t)=0\\|s|,|t|\le T\\s\neq t}} \frac{|d_m(t)||d_m(s)|}{(t-s)^2} \\
&\ll \sum_{m=1}^{\nu} \sum_{\substack{B(t)=0\\|t|\le T}} |d_m(t)|^2 \\
&\ll \sum_{\substack{B(t)=0\\|t|\le T}} \sum_{j=0}^{\nu-1} \frac{|c_j(t)|^2}{K_\nu(t,t)},
\end{aligned}$$

where the first inequality is due to Lemmas 3.2.3 and 3.2.4, and the third inequality follows from (3.15). The second term on the right–hand side of the third line in the above calculation is in the form of a Hilbert–type inequality. By item (2) of Lemma 3.2.2 the zeros of $B(z)$ are uniformly separated, hence the second inequality above follows by Proposition 3.2.5. The implied constants above depend only on D, ν and δ. The statement follows by letting T to infinity. □

3.2.1 Proof of Theorem 3.1.1

Step 1. Let $F \in \mathcal{H}^2(E^\nu)$. Since $\mathcal{H}^2(E^\nu)$ is closed under differentiation, it follows that $F^{(j)} \in \mathcal{H}^2(E^\nu)$ for every $j \ge 0$ and using the fact that every zero of $B(z)$ is also a zero of $B_\nu(z)$, we may apply Theorem 2.1.1 to obtain

$$\sum_{j=0}^{\nu-1} \sum_{B(t)=0} \frac{|F^{(j)}(t)|^2}{K_\nu(t,t)} \ll_D \|F\|_{E^\nu}^2. \tag{3.16}$$

Thus, we may apply Lemma 3.2.6 to deduce that the interpolation formula (3.8) converges in $\mathcal{H}^2(E^\nu)$ to a function $F_0(z)$, hence also uniformly in compact sets of $\mathbb{C}$. We also conclude by Lemma 3.2.6 and property (3.7) that for every function $H \in \mathcal{H}^2(E^\nu)$ such that the interpolation formula (3.8) holds we must have

$$\|H\|_{E^\nu} \ll_{D,\nu,\delta} \sum_{j=0}^{\nu-1} \sum_{B(t)=0} \frac{|H^{(j)}(t)|^2}{K_\nu(t,t)}. \tag{3.17}$$

We claim that $F_0(z) = F(z)$. By Lemma 3.2.1 the span of $\mathcal{G}_\nu$ is dense in $\mathcal{H}^2(E^\nu)$, hence for any given $\varepsilon > 0$ there exists a function $H \in \mathcal{H}^2(E^\nu)$ such that the interpolation formula holds and $\|F - H\|_{E^\nu} < \varepsilon$. We obtain

$$\begin{aligned}
\|F - F_0\|_{E^\nu}^2 &\leq 2\varepsilon^2 + 2\|H - F_0\|_{E^\nu}^2 \\
&\ll \varepsilon^2 + \sum_{j=0}^{\nu-1} \sum_{B(t)=0} \frac{|H^{(j)}(t) - F^{(j)}(t)|^2}{K_\nu(t,t)} \\
&\ll \varepsilon^2 + \|H - F\|_{E^\nu}^2 \\
&< 2\varepsilon^2,
\end{aligned}$$

where the second inequality is due to (3.17) and the third one due to (3.16). Since $\varepsilon > 0$ is arbitrarily, the claim follows. This proves item (1) and inequality (3.9) of item (2).

Step 2. We prove next inequality (3.10) of Item (2). Define

$$\Delta_{\nu,j,t}(z) = K_\nu(t,t)^{-\frac{1}{2}} D_{\nu,j}(z,t).$$

Equation (3.9) implies that $\{\Delta_{\nu,j,t} : j = 0, \ldots, \nu-1; B(t) = 0\}$ is an (unweighted) frame for $\mathcal{H}^2(E^\nu)$. Consider the frame operator $U : \mathcal{H}^2(E^\nu) \to$

$\mathcal{H}^2(E^\nu)$ defined by

$$UF(z) = \sum_{B(t)=0} \sum_{j=0}^{\nu-1} \langle F, \Delta_{\nu,j,t} \rangle_{\mathcal{H}^2(E^\nu)} \Delta_{\nu,j,t}(z).$$

It is a basic result of frame theory (see [41, Corollary 5.1.3]) that U is invertible and positive, and that the collection $\{U^{-1}\Delta_{\nu,j,t} : j = 0, \ldots, \nu - 1; B(t) = 0\}$ is also a frame, sometimes called the canonical dual frame. It follows immediately from (3.7) that

$$UG_{\nu,j} = K_\nu(t,t)^{-\frac{1}{2}} \Delta_{\nu,j,t}(z),$$

which implies that $\{K_\nu(t,t)^{\frac{1}{2}} G_{\nu,j}(\cdot, t) : j = 0, \ldots, \nu - 1; B(t) = 0\}$ is the dual frame of $\mathcal{D}_\nu$. This implies (3.10). We remark that since for every fixed t the functions $G_{\nu,j}(z,t)$ and $B_{\nu,j}(z,t)$ are connected via an invertible matrix transformation, the inequalities can also be shown from the bounds for $\mathcal{B}_\nu$ established in Lemma 3.2.4. Finally, Item (3) is a direct consequence of property (3.7). The proof of Theorem 3.1.1 is complete.

3.3 The L^p–case

Before proving Theorem 3.1.2 we need some technical lemmas. We refer to Appendix 6.1 for further information about the L^p version of de Branges spaces and its connection with Hardy spaces. The next results indicate that when the derivative of the associated phase function is bounded, the L^p de Branges space in question behaves similarly to a Paley–Wiener space with respect to inclusion and summability issues.

Lemma 3.3.1. *Let $E(z)$ be a Hermite–Biehler function such that the associated phase function $\varphi(x)$ has bounded derivative on $\mathbb{R}$. Then for $1 \leq p < q < \infty$ we have $\mathcal{H}^p(E) \subset \mathcal{H}^q(E)$ continuously. Also, if $p > 1$ then $\mathcal{H}^p(E)$ is dense in $\mathcal{H}^q(E)$.*

Proof. Recall that $\varphi'(x) = \pi K(x,x)/|E(x)|^2$ and denote by τ its supremum. By the reproducing kernel property we obtain

$$\int_{\mathbb{R}} \left| \frac{K(\xi,x)}{E(x)} \right|^2 \mathrm{d}x = K(\xi,\xi) \le \tau |E(\xi)|^2/\pi$$

and

$$K(\xi,x)^2 \le K(\xi,\xi)K(x,x) \le |\tau E(\xi)E(x)/\pi|^2.$$

Thus, we obtain that for all $q \in [2,\infty)$

$$\int_{\mathbb{R}} \left| \frac{K(\xi,x)}{E(x)} \right|^q \mathrm{d}x \le (\tau/\pi)^{q-1} |E(\xi)|^q.$$

If $p \in [1,2]$ and $F \in \mathcal{H}^p(E)$, we can apply (6.7) to obtain that

$$|F(\xi)/E(\xi)| \le \|F\|_{E,p} \|K(\xi,\cdot)\|_{E,p'}/|E(\xi)| \le \|F\|_{E,p}(\tau/\pi)^{1/p}$$

for all $\xi \in \mathbb{R}$.

This implies the proposed inclusions for $1 \le p < q < \infty$ and $p \le 2$. By (6.8) the dual space of $\mathcal{H}^p(E)$ can be identified with $\mathcal{H}^{p'}(E)$ if $1 < p < \infty$. This implies the remaining inclusions. Since convergence in the space implies convergence on compacts sets of $\mathbb{C}$ we conclude that the identity map from $\mathcal{H}^p(E)$ to $\mathcal{H}^q(E)$ is closed, hence continuous by the Closed Graph Theorem.

The density part follows by an application of the Hahn-Banach Theorem in conjunction with the duality characterization (6.8) and the reproducing kernel property (6.6). □

Lemma 3.3.2. *Let $E(z)$ be a Hermite–Biehler function with no real zeros and such that $\tau = \sup_x \varphi'(x) < \infty$. Then for $p \in [1,2]$ and $F \in \mathcal{H}^p(E)$ we have*

$$\sum_{B(t)=0} \frac{|F(t)|}{(1+|t|)K(t,t)^{1/2}} \ll_{\tau,p} \|F\|_{E,p}.$$

The above estimate is also valid in the case $p \in (2,\infty)$ if we additionally assume that $E(z)$ satisfies condition (C1) (*that is $v(E^*/E) < 0$*).

Proof. **Step 1.** We start with the case $1 \leq p \leq 2$. By Lemma 3.3.1 we have $\mathcal{H}^p(E) \subset \mathcal{H}^2(E)$ continuously, thus the case $p < 2$ follows directly from the case $p = 2$.

The boundedness of $\varphi'(x)$ implies that the zeros of $B(z)$ are separated, because they coincide with the set of points t such that $\varphi(t) \equiv 0 \mod \pi$. Let $F \in \mathcal{H}^2(E)$, by the Cauchy-Schwarz inequality we have

$$\sum_{B(t)=0} \frac{|F(t)|}{(1+|t|)K(t,t)^{1/2}} \leq \left(\sum_{B(t)=0} \frac{|F(t)|^2}{K(t,t)} \right)^{1/2} \left(\sum_{B(t)=0} (1+|t|)^{-2} \right)^{1/2} \ll_\tau \|F\|_E,$$

where the last inequality is due to (2.10) and the separability of the zeros of $B(z)$.

Step 2. We now deal with the case $p > 2$. By hypothesis we have $v(E^*/E) = -2a < 0$. Fix a real number α such that $\alpha \in (-1/p, 0)$. Let $E_\alpha(z)$ be the function defined in Section 6.2 associated with homogeneous spaces and define the operator $\mathcal{L} : \mathcal{H}^p(E) \to \mathcal{H}^2(E^2)$ by $\mathcal{L}F(z) = e^{-iaz}E_\alpha(az)E^*(z)F(z)$. By the properties described in Section 6.2 we have:

(*i*) $v(E_\alpha^*) \leq v(E_\alpha) = \tau(E_\alpha) = 1$;

(*ii*) $|E_\alpha(t)| \simeq 1/|t|^{\alpha+1/2}$, for $|t| \geq 1$.

Hence, if $G(z) = \mathcal{L}F(z)$ we obtain

$$v(G/E^2) = v(F/E) + v(E^*/E) + v(E_\alpha(az)) + v(e^{-iaz}) \leq 0 - 2a + a + a = 0$$

and

$$v(G^*/E^2) = v(F^*/E) + v(E_\alpha^*(az)) + v(e^{iaz}) \leq 0 + a - a = 0.$$

By Holder's inequality we also have

$$\int_{\mathbb{R}} |G(x)/E(x)^2|^2 \, \mathrm{d}x \leq \left(\int_{\mathbb{R}} |E_\alpha(ax)|^{2q} \mathrm{d}x \right)^{1/q} \|F\|_{E,p}^2 \ll_{a,p} \|F\|_{E,p}^2, \quad (3.18)$$

where $q = (p/2)' = p/(p-2)$. Note that $q > p > 2$ and $2q(\alpha + 1/2) > 1$.

We conclude that the operator $\mathcal{L}$ is well-defined and continuous. Denoting by $K_2(w,z)$ the reproducing kernel of $\mathcal{H}^2(E^2)$ and $K(w,z)$ the reproducing kernel of $\mathcal{H}^2(E)$ we obtain $K_2(t,t) = 2|E(t)|^2 K(t,t)$. We have

$$\begin{aligned}
\sum_{B(t)=0} \frac{|F(t)|}{(1+|t|)K(t,t)^{1/2}} &= \sum_{B(t)=0} \frac{\sqrt{2}|G(t)|}{|E_\alpha(at)|(1+|t|)K_2(t,t)^{1/2}} \\
&\leq \left(\sum_{B(t)=0} \frac{2|G(t)|^2}{K_2(t,t)} \right)^{\frac{1}{2}} \left(\sum_{B(t)=0} \frac{1}{|E_\alpha(at)|^2(1+|t|)^2)} \right)^{\frac{1}{2}} \\
&\ll_{\tau,a,p} \|G\|_{E^2} \\
&\ll_{\tau,a,p} \|F\|_{E,p}\,,
\end{aligned}$$

where the first inequality is due to the Cauchy-Schwarz inequality, the second inequality due to (2.10) together with the choice of $\alpha < 0$ and the last one is due to (3.18). □

We now prove a generalized (weighted) version of the Pólya-Plancherel theorem (see [65]).

Proposition 3.3.3. *Let $E(z)$ be a Hermite–Biehler function satisfying condition* (C2) *for some $h > 0$ and such that $\tau = \sup_x \varphi'(x) < \infty$. Let $\{\lambda_n\}$ be a sequence of real numbers such that $|\lambda_n - \lambda_m| \geq \sigma$ whenever $n \neq m$, for some $\sigma > 0$. Then for any $p \in [1,\infty)$ and $F \in \mathcal{H}^p(E)$ we have*

$$\sum_n \left| \frac{F(\lambda_n)}{E(\lambda_n)} \right|^p \leq 2\, \frac{1 + e^{4\tau p \eta}}{\pi \eta} \int_{\mathbb{R}} \left| \frac{F(x)}{E(x)} \right|^p \mathrm{d}x,$$

where $\eta = \min\{\sigma, h\}$.

Proof. **Step 1.** Since E^*/E is bounded on the upper half–plane and has modulo one in the real line, by Nevanlinna's factorization (see [6, Theorem 8]) we obtain

$$\Theta(z) := \frac{E^*(z)}{E(z)} = e^{2aiz} \prod_n \frac{1 - z/w_n}{1 - z/\overline{w}_n}$$

where $\overline{w}_n = x_n - iy_n$ with $y_n \geq h$ are the zeros of $E(z)$ and $2a = -\nu(E^*/E) \geq 0$. If $z = x + iy$ with $y \geq 0$ we have the following identity

$$\frac{1}{2}\partial_y \log |\Theta^*(z)| = a + \sum_n \frac{y_n[(x - x_n)^2 + y_n^2 - y^2]}{|z - \overline{w}_n|^2 |z - w_n|^2}.$$

If $0 \leq y \leq h/2$ then $y_n^2 - y^2 \leq 4(y_n - y)^2$ and we deduce that

$$\begin{aligned}\frac{1}{2}\partial_y \log |\Theta^*(z)| \leq a + 4\sum_n \frac{y_n}{(x - x_n^2) + y_n^2} &= -3a + 4\frac{1}{2}\partial_y \log |\Theta^*(x)| \\ &= -3a + 4\varphi'(x) \leq 4\tau.\end{aligned}$$

Integrating in y we obtain $|\Theta^*(z)| \leq e^{8\tau y}$ if $0 \leq y \leq h/2$.

Step 2. Let $\eta = \min\{h, \sigma\}$. Since the function $|F(z)/E(z)|^p$ is sub-harmonic in the half–plane $\mathrm{Im}\, z > -h$, its value at the center of a disk is not greater than its mean value over the disk. We obtain

$$\begin{aligned}|F(\lambda_n)/E(\lambda_n)|^p \leq & \frac{4}{\pi\eta^2} \int_0^{\eta/2} \int_0^{2\pi} |F(\lambda_n + \rho e^{i\theta})/E(\lambda_n + \rho e^{i\theta})|^p \mathrm{d}\theta \rho \mathrm{d}\rho \\ \leq & \frac{4}{\pi\eta^2} \int_{-\eta/2}^{\eta/2} \int_{\lambda_n - \eta/2}^{\lambda_n + \eta/2} |F(x + iy)/E(x + iy)|^p \mathrm{d}x \mathrm{d}y.\end{aligned}$$

Using the separability of $\{\lambda_n\}$ we can sum the above inequality for all values of n to obtain

$$\begin{aligned}\sum_n |F(\lambda_n)/E(\lambda_n)|^p \leq & \frac{4}{\pi\eta^2} \int_{-\eta/2}^{\eta/2} \int_{\mathbb{R}} |F(x + iy)/E(x + iy)|^p \mathrm{d}x \mathrm{d}y \\ = & \frac{4}{\pi\eta^2} \int_0^{\eta/2} \int_{\mathbb{R}} |F(x + iy)/E(x + iy)|^p \mathrm{d}x \mathrm{d}y \\ & + \frac{4}{\pi\eta^2} \int_0^{\eta/2} \int_{\mathbb{R}} |F^*(x + iy)/E^*(x + iy)|^p \mathrm{d}x \mathrm{d}y.\end{aligned}$$

Since $|\Theta^*(z)| \leq e^{8\tau y}$ for $0 < y < h/2$ we conclude that

$$\sum_n |F(\lambda_n)/E(\lambda_n)|^p \leq \frac{4}{\pi\eta^2} \int_0^{\eta/2} \int_{\mathbb{R}} \frac{|F(x+iy)|^p + e^{4\tau p\eta}|F^*(x+iy)|^p}{|E(x+iy)|^p} \mathrm{d}x\mathrm{d}y.$$

By the discussion of Appendix 6.1, F/E and F^*/E belong to the Hardy space $H^p(\mathbb{C}^+)$. We can apply (6.5) to deduce that

$$\int_{\mathbb{R}} \frac{|F(x+iy)|^p + e^{4\tau p\eta}|F^*(x+iy)|^p}{|E(x+iy)|^p} \mathrm{d}x \leq (1 + e^{4\tau p\eta}) \int_{\mathbb{R}} \left|\frac{F(x)}{E(x)}\right|^p \mathrm{d}x$$

for every $y > 0$. This concludes the proof. □

For the sake of completeness we state a result about boundeness of the differentiation operator due to Baranov (see [2, Theorem A]). For similar results in model spaces of Hardy spaces we refer to Dyakonov papers [28, 29].

Theorem 3.3.4 (Baranov)**.** *Let $E(z)$ be a Hermite–Biehler function satisfying condition* (C1) *or* (C2)*. For $p \in (1, \infty)$ the following statements are equivalent:*

(1) *$\mathcal{H}^p(E)$ is closed under differentiation.*

(2) *$E(z)$ is of exponential type and $E'(z)/E(z) \in H^\infty(\mathbb{C}^+)$.*

Remark. The fact that condition (2) above is independent of $p \in (1, \infty)$ implies that if $E(z)$ satisfies condition (C1) or (C2) then $\mathcal{H}^p(E)$ is closed under differentiation if and only if $\mathcal{H}^2(E)$ is closed under differentiation.

3.3.1 Proof of Theorem 3.1.2

The following is a very technical proof and for this reason we split the proof into several steps. Steps 0, 1 and 2 are the crucial ones.

Step 0. A simple, but crucial consequence of formula (3.8) is that the singular part of the function $F(z)/B(z)^\nu$ at a given zero t of $B(z)$ is

$$\sum_{j=0}^{\nu-1} F^{(j)}(t)\frac{G_{\nu,j}(z,t)}{B^\nu(z)} = \sum_{j=0}^{\nu-1} F^{(j)}(t)\frac{P_{\nu,\nu-j-1}(z,t)}{j!(z-t)^{\nu-j}}.$$

To see this, define for any complex number w a linear operator $\mathcal{S}_w$ on the space of meromorphic functions $G(z)$ by

$$\mathcal{S}_w(G)(z) = \sum_{n\leq -1} \frac{g_n}{(z-w)^n}$$

if $G(z)$ has the series representation

$$G(z) = \sum_{n\in\mathbb{Z}} \frac{g_n}{(z-w)^n}$$

about $z = w$. That is, $\mathcal{S}_w(G)$ is defined as the singular part of the function $G(z)$ at the point $z = w$. Since $G(z)$ is meromorphic, $\mathcal{S}_w(G)(z)$ is always a rational function. It is a simple, but useful characterization that $S_w(G)(z)$ is the unique rational function $R(z)$ having exactly one pole which is located at $z = w$, such that $G(z) - R(z)$ has a removable singularity at the point $z = w$ and

$$\lim_{|z|\to\infty} \frac{R(z)}{z^j} = 0 \tag{3.19}$$

for every integer $j \geq 0$. Recalling that

$$\frac{(z-t)^\nu}{B(z)^\nu} = \sum_{n=0}^{\infty} a_{\nu,n}(t)(z-t)^n \quad \text{and} \quad P_{\nu,j}(z,t) = \sum_{n=0}^{j} a_{\nu,n}(t)(z-t)^n$$

we obtain

$$\mathcal{S}_t\left(\frac{F}{B^\nu}\right)(z) = \sum_{j=0}^{\nu-1} F^{(j)}(t)\frac{P_{\nu,\nu-j-1}(z,t)}{j!(z-t)^{\nu-j}} = \sum_{j=0}^{\nu-1} F^{(j)}(t)\frac{G_{\nu,j}(z,t)}{B(z)^\nu}. \tag{3.20}$$

Now, given a complex number $w \in \mathbb{C}$ we define another linear operator $\mathcal{M}_w$ on the space of entire functions $F(z)$ by

$$\mathcal{M}_w(F)(z) = \frac{F(z)B(w)^\nu - B(z)^\nu F(w)}{z-w}.$$

We observe that for every zero t of $B(z)$ and every $w \in \mathbb{C}$ not a zero of $B(z)$ we have

$$\mathcal{S}_t\left(\frac{\mathcal{M}_w(F)(\cdot)}{B(w)^\nu B(\cdot)^\nu}\right)(z) = \frac{\mathcal{S}_t(F/B^\nu)(z) - \mathcal{S}_t(F/B^\nu)(w)}{z-w}. \tag{3.21}$$

One can deduce this last identity by observing that

$$\frac{\mathcal{M}_w(F)(z)}{B(w)^\nu B(z)^\nu} - \frac{\mathcal{S}_t(F/B^\nu)(z) - \mathcal{S}_t(F/B^\nu)(w)}{z-w}$$

has a removable singularity at the point $z = w$ and also that the right–hand side of (3.21) is a rational function in the variable z with exactly one pole, which is located at $z = t$ and satisfies condition (3.19).

Step 1. We begin with the case $p \in [1, 2]$. By Lemma 3.2.2, the assumption that $\mathcal{H}^2(E^\nu)$ is closed under differentiation implies that $\tau = \sup_x \varphi'(x) < \infty$. We can apply Lemma 3.3.1 to conclude that $\mathcal{H}^p(E^\nu) \subset \mathcal{H}^2(E^\nu)$, hence formula (3.11) is a direct consequence of Theorem 3.1.1.

Step 2. Now, we deal with the case $p \in (2, \infty)$. A crucial observation is that if $F \in \mathcal{H}^p(E^\nu)$ then $\mathcal{M}_w(F) \in \mathcal{H}^2(E^\nu)$. Thus, we can apply Theorem 3.1.1 together with (3.20) to obtain

$$\mathcal{M}_w(F)(z) = \sum_{B(t)=0} B(z)^\nu \mathcal{S}_t\left(\frac{\mathcal{M}_w(F)(\cdot)}{B(\cdot)^\nu}\right)(z),$$

where the last sum converges uniformly in the variable z in every compact subset of $\mathbb{C}$ for every $w \in \mathbb{C}$. By (3.20) and (3.21) we conclude that

$$\frac{F(z)}{B(z)^\nu} - \frac{F(w)}{B(w)^\nu} = \sum_{B(t)=0} \left\{ \sum_{j=0}^{\nu-1} F^{(j)}(t)\frac{G_{\nu,j}(z,t)}{B(z)^\nu} - \sum_{j=0}^{\nu-1} F^{(j)}(t)\frac{G_{\nu,j}(w,t)}{B(w)^\nu} \right\} \tag{3.22}$$

for every $w, z \in \mathbb{C}$ that are not zeros of $B(z)$.

We claim that

$$F(z) = \Lambda(F)B(z)^\nu + \sum_{B(t)=0} \sum_{j=0}^{\nu-1} F^{(j)}(t)G_{\nu,j}(z,t) \tag{3.23}$$

for some constant $\Lambda(F)$, where the sum converges uniformly in compact sets of $\mathbb{C}$. In Steps 4 and 5 we show that under condition (C1) or (C2) the formula (3.23) holds and that $F \mapsto \Lambda(F)$ defines a continuous functional that vanishes in a dense set of functions in $\mathcal{H}^p(E^\nu)$ hence it vanishes identically. In either case, Λ is identically zero which concludes the proof.

Step 3. Recall that $a_{\nu,j}(t)$ are defined as the coefficients of the Taylor expansion of $B_{\nu,\nu}(z,t)^{-1}$ at the point $z = t$. By Lemma 3 these coefficients satisfy the estimate

$$|a_{\nu,j}(t)|^2 \ll_{D,\nu} \frac{1}{K_\nu(t,t)}$$

for $j = 0, \ldots, \nu - 1$, where D is the norm of the differentiation operator in $\mathcal{H}^2(E^\nu)$ and $K_\nu(w, z)$ is the reproducing kernel associated with $E(z)^\nu$. Since

$$G_{\nu,j}(z,t) = \frac{B(z)^\nu}{j!} \sum_{\ell=0}^{\nu-j-1} \frac{a_{\nu,\ell}(t)}{(z-t)^{\nu-\ell-j}},$$

we obtain

$$K_\nu(t,t)|G_{\nu,j}(i,t)|^2 \ll_{D,\nu} \frac{1}{1+t^2} \tag{3.24}$$

for every zero t of $B(z)$ and $j = 0, \ldots, \nu - 1$.

Step 4. Assume that $E(z)$ satisfies condition (C2) for some $h > 0$. By the remark after Theorem 3.3.4 we have that $\mathcal{H}^p(E^\nu)$ is closed under differentiation. Also, the assumption that $\varphi'(t) \geq \delta$ whenever $B(t) = 0$ in conjunction with formula (2.9) implies that

$$|E(t)|^{2\nu} \ll_\delta K_\nu(t,t) \tag{3.25}$$

whenever $B(t)=0$. By Lemma 3.2.2 items (1) and (2) we have that $\varphi'(x) \ll 1$ and that the zeros of $B(z)$ are separated. Thus, we can apply Proposition 3.3.3 together with (3.25) to obtain

$$\sum_{B(t)=0} \left| \frac{F(t)}{K_\nu(t,t)^{1/2}} \right|^p \ll_{D,\delta,h,p} \int_{\mathbb{R}} \left| \frac{F(x)}{E(x)^\nu} \right|^p \mathrm{d}x \tag{3.26}$$

for every $F \in \mathcal{H}^p(E^\nu)$. Finally, we obtain the following estimate

$$\begin{aligned} \sum_{j=0}^{\nu-1} \sum_{B(t)=0} |F^{(j)}(t) G_{\nu,j}(i,t)| &\le \sum_{j=0}^{\nu-1} \left(\sum_{B(t)=0} \frac{|F^{(j)}(t)|^p}{K_\nu(t,t)^{p/2}} \right)^{\frac{1}{p}} \left(\sum_{B(t)=0} \frac{|G_{\nu,j}(i,t)|^{p'}}{K_\nu(t,t)^{-p'/2}} \right)^{\frac{1}{p'}} \\ &\ll \sum_{j=0}^{\nu-1} \left(\sum_{B(t)=0} \frac{|F^{(j)|^p}(t)}{K_\nu(t,t)^{p/2}} \right)^{\frac{1}{p}} \\ &\ll \|F/E^\nu\|_{L^p(\mathbb{R})}, \end{aligned} \tag{3.27}$$

where the first inequality is Holder's inequality, the second one is due to (3.24) and the separation of the zeros of $B(z)$, the third one is due to (3.26) and the closure under differentiation of $\mathcal{H}^p(E^\nu)$.

Estimate (3.27) together with formula (3.22) clearly implies that (3.23) is valid and, by using estimate (6.7) we can easily deduce that $F \mapsto \Lambda(F)$ is a continuous functional over $\mathcal{H}^p(E^\nu)$. By Lemmas 3.2.1 and 3.3.1 the span of the functions $\{G_{\nu,j}(z,t)\}$ for $j=0,\ldots,\nu-1$ and $B(t)=0$ is dense in $\mathcal{H}^p(E^\nu)$ and trivially $\Lambda(G_{\nu,j}(\cdot,t))=0$. Hence Λ vanishes identically.

Step 5. Assume that $E(z)$ satisfies (C1) with $-a = v(E^*/E)$. By Theorem 3.3.4 we again conclude that $\mathcal{H}^p(E^\nu)$ is closed under differentiation. We can apply Lemma 3.3.2 together with (3.24) to deduce that

$$\sum_{j=0}^{\nu-1} \sum_{B(t)=0} |F^{(j)}(t) G_{\nu,j}(i,t)| \ll \|F(x)/E(x)^\nu\|_{L^p(\mathbb{R})} \tag{3.28}$$

for every $F \in \mathcal{H}^p(E^\nu)$. In the same way as in the previous step, we deduce that formula (3.23) is valid and $F \mapsto \Lambda(F)$ is a continuous functional over $\mathcal{H}^p(E^\nu)$ that is identically zero in a dense set of functions. Hence Λ vanishes identically. This concludes the proof.

Remark. We note that due to estimate (3.24), whenever the space $\mathcal{H}^2(E^\nu)$ is closed by differentiation, the estimate (3.26) implies estimate (3.28) by an application of Holder's inequality. Therefore, we conclude that the result of Theorem 3.1.2 (for some $p > 2$) will remain valid if we drop conditions (C1) and (C2) and replace them by the following by condition

$$\sum_{j=0}^{\nu-1} \sum_{B(t)=0} \frac{|F^{(j)}(t)|}{K_\nu(t,t)^{1/2}(1+t^2)} \ll \|F(x)/E(x)^\nu\|_{L^p(\mathbb{R})} \tag{3.29}$$

for all $F \in \mathcal{H}^p(E^\nu)$.

3.4 Applications

3.4.1 Weights Given by Powers of $|x|$

There is a variety of examples of de Branges spaces [6, Chapter 3] for which Theorems 3.1.1 and 3.1.2 may be applied. A basic example would be the classical Paley–Wiener space $\mathcal{H}^2(e^{-i\tau z})$ which gives us the previous results obtained by Vaaler in [73, Theorem 9]. Another interesting family arises in the discussion of [47, Section 5]. These are called homogeneous spaces. The definition of such spaces and some crucial results are presented in Appendix 6.2. In what follows we only define the objects needed to state our results.

Let $\alpha > -1$ be a parameter and consider the real entire function $A_\alpha(z)$ given by

$$A_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n}}{n!(\alpha+1)(\alpha+2)\ldots(\alpha+n)} = \Gamma(\alpha+1)\left(\tfrac{1}{2}z\right)^{-\alpha} J_\alpha(z),$$

where $J_\alpha(z)$ denotes the classical Bessel function of the first kind. For a given integer $\nu > 0$ define the functions $G_{\nu,j}(\alpha; z, t)$ as in (3.6) by replacing the function $B(z)$ by $A_\alpha(z)$ and the zeros of $B(z)$ by the zeros of $A_\alpha(z)$. Notice that these zeros are nothing but the positive real zeros of $J_\alpha(z)$ symmetrized about the origin. Also, define the following weight

$$\omega_{\alpha,\nu}(x) = \begin{cases} |x|^{\nu(\alpha+1/2)}, & \text{if } |x| > 1; \\ 1, & \text{if } |x| \leq 1. \end{cases}$$

The following results are applications of Theorems 3.1.1 and 3.1.2 to homogeneous spaces as described in Appendix 6.2. For the case $p > 2$ we use the alternative condition (3.29) pointed out by the remark in the end of Section 3.3.1 and which is proved in Lemma 6.2.1.

Theorem 3.4.1. *Let $\alpha > -1$ be a real number and $\nu > 0$ be an integer. Then there exists a constant $C > 0$ such that*

$$\frac{1}{C}\int_{\mathbb{R}} |F(x)\omega_{\alpha,\nu}(x)|^2 \mathrm{d}x \leq \sum_{A_\alpha(t)=0} \sum_{j=0}^{\nu-1} |F^{(j)}(t)\omega_{\alpha,\nu}(t)|^2 \leq C \int_{\mathbb{R}} |F(x)\omega_{\alpha,\nu}(x)|^2 \mathrm{d}x$$

for every entire function $F(z)$ of exponential type at most ν. Furthermore, if the above quantity is finite then

$$F(z) = \sum_{A_\alpha(t)=0} \sum_{j=0}^{\nu-1} F^{(j)}(t) G_{\nu,j}(\alpha; z, t),$$

where this series converges to $F(z)$ uniformly in compact sets of $\mathbb{C}$ and in the $L^2(\mathbb{R}, \omega_{\alpha,\nu}(x)^2\mathrm{d}x)$–norm.

Theorem 3.4.2. *Let $\alpha > -1$ be a real number and $\nu > 0$ be an integer. Let $F(z)$ be an entire function of exponential type at most ν such that*

$$\int_{\mathbb{R}} |F(x)\omega_{\alpha,\nu}(x)|^p \mathrm{d}x < \infty$$

for some $p \in [1, \infty)$. *Then*

$$F(z) = \sum_{A_\alpha(t)=0} \sum_{j=0}^{\nu-1} F^{(j)}(t) G_{\nu,j}(\alpha; z, t),$$

where this series converges uniformly in compact sets of $\mathbb{C}$.

3.4.2 Extremal Band–limited Functions

The purpose of this section is to prove a uniqueness result for some extremal problems described below. A set $K \subset \mathbb{R}^N$ is called a convex body if it is compact, convex, symmetric around the origin and with non-empty interior. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^N$ and $\mathcal{B}^N$ the compact Euclidean unit ball. Given a non-negative Borel measure $\mathrm{d}\mu$ on $\mathbb{R}^N$ and a real–valued function $g(\boldsymbol{x})$ we denote by $\mathcal{M}(g, K, \mathrm{d}\mu)$ the set of measurable real–valued functions $M(\boldsymbol{x})$ defined on $\mathbb{R}^N$ satisfying the following conditions:

(i) $M(\boldsymbol{x})$ defines a tempered distribution such that its distributional Fourier transform $\widehat{M}$ is supported on K.

(ii) $g(\boldsymbol{x}) \le M(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^N$.

(iii) $M - g \in L^1(\mathbb{R}^N, \mathrm{d}\mu)$.

In this case, we say that $M(\boldsymbol{x})$ is a band–limited majorant of $g(\boldsymbol{x})$. In an analogous way we define $\mathcal{L}(g, K, \mathrm{d}\mu)$ as the set of minorants. We are asked to minimize the quantities

$$\int_{\mathbb{R}^N} \big\{M(\boldsymbol{x}) - g(\boldsymbol{x})\big\} \,\mathrm{d}\mu(\boldsymbol{x}) \quad \text{and} \quad \int_{\mathbb{R}^N} \big\{g(\boldsymbol{x}) - L(\boldsymbol{x})\big\} \,\mathrm{d}\mu(\boldsymbol{x}) \qquad (3.30)$$

among all functions $M \in \mathcal{M}(g, K, \mathrm{d}\mu)$ and $L \in \mathcal{L}(g, K, \mathrm{d}\mu)$. And, if the minimum is attained, characterize the set of extremal functions. We call $M(\boldsymbol{x})$ (or $L(\boldsymbol{x})$) an *extremal function* if it minimizes the quantity (3.30).

The problem becomes treatable if we consider radial functions. For instance, we consider the situation where $K = \mathcal{B}^N$, the function $g(\boldsymbol{x})$ is radial, and

$$\mathrm{d}\mu_E(\boldsymbol{x}) = 2\Big(|E(|\boldsymbol{x}|)|^2|\boldsymbol{x}|^{N-1}\big|S^{N-1}\big|\Big)^{-1}\mathrm{d}\boldsymbol{x}\,, \tag{3.31}$$

where $\big|S^{N-1}\big|$ denotes the area of the $(N-1)$-dimensional sphere and $E(z)$ is of Hermite–Biehler class.

In this section $E(z) = A(z) - iB(z)$ will denote a Hermite–Biehler function of bounded type and mean type equal to π. Moreover, we also assume that $\mathcal{H}^2(E^2)$ is closed under differentiation and $1 \ll \varphi'(t)$ over the zero set of $A(z)$ and $B(z)$. We also assume that $E^*(-z) = E(z)$ and $A, B \notin \mathcal{H}^2(E)$. These assumptions imply that the companion functions $A(z)$ and $B(z)$ are respectively even and odd and, by Krein's Theorem 6.1.2, $E(z)$ is of exponential type with $\tau(E) = v(E) = \pi$. Moreover, $F \in \mathcal{H}^2(E)$ if and only if $F(z)$ is of exponential type at most π and $F/E \in L^2(\mathbb{R}, \mathrm{d}\boldsymbol{x})$.

These restrictions allow us to reduce the multidimensional problem to an one–dimensional problem and to use de Branges space techniques. Constructions of extremal band–limited approximations of radial functions in several variables were studied in [16, 17, 47]. In particular, Carneiro and Littmann [16, 17] were able to explicitly construct a pair of radial functions $M \in \mathcal{M}(g, \mathcal{B}^N, \mathrm{d}\mu_E)$ and $L \in \mathcal{L}(g, \mathcal{B}^N, \mathrm{d}\mu_E)$ that minimize the quantities in (3.30), where $\mathrm{d}\mu_E$ is given by (3.31) with $E(z) = E_\alpha(z)$ (see Appendix 6.2) and $g(\boldsymbol{x})$ belongs to a vast class of radial functions with exponential or Gaussian subordination.

For the sake of completeness we state here a classical theorem about tempered distributions with Fourier transform supported on a ball (see [48, Theorem 7.3.1]).

Theorem 3.4.3 (Paley–Wiener–Schwartz)**.** *Let F be a tempered distribution such that the support of $\widehat{F}$ is contained in $\mathcal{B}^N$. Then $F : \mathbb{C}^d \to \mathbb{C}$ is an entire function and there exists $C > 0$ such that*

$$|F(\boldsymbol{x} + i\boldsymbol{y})| \leq C(1 + |\boldsymbol{x} + i\boldsymbol{y}|)^C e^{2\pi|\boldsymbol{y}|}$$

for every $\boldsymbol{x}+i\boldsymbol{y} \in \mathbb{C}^d$. Conversely, every entire function $F : \mathbb{C}^d \to \mathbb{C}$ satisfying an estimate of this form defines a tempered distribution with Fourier transform supported on $\mathcal{B}^N$.

The next propositions give an interpolation condition for a majorant or minorant to be extremal and unique in the radial case. We highlight the fact that the uniqueness part below is a novelty in this multidimensional theory, and makes a crucial use of our interpolation formulas. This enhances the extremal results obtained in [16, 17].

Proposition 3.4.4. *Let $g(\boldsymbol{x})$ be a radial function that is differentiable for $\boldsymbol{x} \neq 0$. Suppose that $\mathcal{M}(g, \mathcal{B}^N, \mathrm{d}\mu_E) \neq \emptyset$ and there exists a radial function $L \in \mathcal{L}(g, \mathcal{B}^N, \mathrm{d}\mu_E)$ such that $L(\boldsymbol{x}) = g(\boldsymbol{x})$ whenever $A(|\boldsymbol{x}|) = 0$. Then $L(\boldsymbol{x})$ is extremal and unique among the set of entire functions on $\mathbb{C}^d$ whose restriction to $\mathbb{R}^N$ is radial.*

Proposition 3.4.5. *Let $g(\boldsymbol{x})$ be a radial function that is differentiable for $\boldsymbol{x} \neq 0$. Suppose that $\mathcal{L}(g, \mathcal{B}^N, \mathrm{d}\mu_E) \neq \emptyset$ and there exists a radial function $M \in \mathcal{M}(g, \mathcal{B}^N, \mathrm{d}\mu_E)$ such that $M(\boldsymbol{x}) = g(\boldsymbol{x})$ whenever $B(|\boldsymbol{x}|) = 0$. Then $M(\boldsymbol{x})$ is extremal and unique among the set of entire functions on $\mathbb{C}^d$ whose restriction to $\mathbb{R}^N$ is radial.*

We only prove Proposition 3.4.5 since the other is analogous.

Proof. **Optimality.** Fix $L \in \mathcal{L}(g, \mathcal{B}^N, \mathrm{d}\mu_E)$. Let $SO(N)$ denote the compact topological group of real orthogonal $N \times N$ matrices with determinant 1, with associated Haar probability measure $\mathrm{d}\sigma$.

For a given function $F(\boldsymbol{x})$ define its radial symmetrization by

$$F_r(\boldsymbol{x}) = \int_{SO(N)} F(\rho\boldsymbol{x})\,\mathrm{d}\sigma(\rho).$$

It is not difficult to see that if $F \in \mathcal{M}(g, \mathcal{B}^N, \mathrm{d}\mu_E)$ then $F_r \in \mathcal{M}(g, \mathcal{B}^N, \mathrm{d}\mu_E)$. Furthermore, since $M(\boldsymbol{x})$ is radial we also have

$$\int_{\mathbb{R}^N} \big\{F(\boldsymbol{x}) - M(\boldsymbol{x})\big\}\,\mathrm{d}\mu_E(\boldsymbol{x}) = \int_{\mathbb{R}^N} \big\{F_r(\boldsymbol{x}) - M(\boldsymbol{x})\big\}\,\mathrm{d}\mu_E(\boldsymbol{x}). \tag{3.32}$$

Let $F \in \mathcal{M}(g, \mathcal{B}^N, \mathrm{d}\mu_E)$ be given. Define $m(s) = M(se_1)$, $\ell(s) = L_r(se_1)$ and $f(s) = F_r(se_1)$ for all real s, where $e_1 = (1, 0, \ldots, 0)$. We can apply the Paley–Wiener-Schwartz Theorem to conclude that these functions extend to $\mathbb{C}$ as entire functions of exponential type at most 2π. By (3.31) and (3.32) we obtain that

$$\int_{\mathbb{R}^N} \big\{F(\boldsymbol{x}) - M(\boldsymbol{x})\big\}\mathrm{d}\mu_E(\boldsymbol{x}) = \int_{\mathbb{R}} \{f(s) - m(s)\}/|E(s)|^2\,\mathrm{d}s. \tag{3.33}$$

We claim that $f(s) - m(s) = |p(s)|^2 - |q(s)|^2$ for all real s for some $p, q \in \mathcal{H}^2(E)$. Since $m(s) - \ell(s) \geq 0$ and $f(s) - \ell(s) \geq 0$ for all real s and these functions belong to $\mathcal{H}^1(E^2)$, we conclude that there exist two entire functions $p, q \in \mathcal{H}^2(E)$ such that $m(s) - \ell(s) = |p(s)|^2$ and $r(s) - l(s) = |q(s)|^2$ for all real s (see [6, Theorem 13]). We can apply formula (2.10) to obtain that

$$\begin{aligned}\int_{\mathbb{R}} \{f(s) - m(s)\}|E(s)|^{-2}\mathrm{d}t &= \int_{\mathbb{R}} \frac{|p(s)|^2 - |q(s)|^2}{|E(s)|^2}\,\mathrm{d}t = \sum_{B(t)=0} \frac{|p(t)|^2 - |q(t)|^2}{K(t,t)} \\ &= \sum_{B(t)=0} \frac{f(t) - m(t)}{K(t,t)} = \sum_{B(t)=0} \frac{f(t) - g(te_1)}{K(t,t)} \geq 0\,,\end{aligned} \tag{3.34}$$

where the last equality is due to the interpolation condition of $M(\boldsymbol{x})$, that is, $M(\boldsymbol{x}) = g(\boldsymbol{x})$ whenever $B(|\boldsymbol{x}|) = 0$. By (3.33) and (3.34) we conclude that $M(\boldsymbol{x})$ is extremal.

Uniqueness. Let $F \in \mathcal{M}(g, \mathcal{B}^N, \mathrm{d}\mu_E)$ be radial and extremal. Inequality (3.34) implies that $f(t) := F(te_1) = g(te_1)$ whenever $B(t) = 0$. Since $\boldsymbol{x} \in \mathbb{R}^N \mapsto g(\boldsymbol{x})$ is radial and differentiable for $\boldsymbol{x} \neq 0$ we conclude that $f'(t) = \partial_1 g(te_1)$ if $B(t) = 0$ and $t \neq 0$ (recall that $B(z)$ is odd). Since $f(t)$ is even we also have $f'(0) = 0$. Since $f(s) - m(s)$ belongs to $\mathcal{H}^1(E^2)$ and $f(t) = m(t)$ and $f'(t) = m'(t)$ whenever $B(t) = 0$, by Theorem 3.1.2, we conclude that $f \equiv m$ and this concludes the proof. □

Remark. In some cases $g(\boldsymbol{x})$ may have a singularity at $\boldsymbol{x} = 0$, for instance if $\lim_{\boldsymbol{x} \to 0} g(\boldsymbol{x}) = \infty$. Thus, only the minorant problem is well-posed, that is $\mathcal{M}(g, \mathcal{B}^N, \mathrm{d}\mu_E) = \emptyset$. However, in the case of homogeneous spaces the previous proposition will still hold. In [16, Corollary 23], E. Carneiro and F. Littmann proved that every $f \in \mathcal{H}^1(E_\alpha^2)$, not necessarily non-negative on the real axis, can be represented as $f = pp^* - qq^*$ for $p, q \in \mathcal{H}^2(E_\alpha)$. We can easily see that this representation is sufficient to prove the previous propositions.

3.4.3 Sampling and Interpolation in Paley–Wiener Spaces

We give here one application of the results obtained for the L^2 case related to sampling/interpolation theory in Paley–Wiener spaces. We say that a sequence of real numbers $\{\lambda_m\}$ is *sampling of order* ν for $PW^2(\tau)$ if the norm

$$\eta_\nu(F) = \left[\sum_{j=0}^{\nu-1} \sum_m |F^{(j)}(\lambda_m)|^2 \right]^{1/2}$$

defines an equivalent norm in $PW^2(\tau)$ (that is, equivalent to the $L^2(\mathbb{R})$–norm). Recall that $PW^2(\tau)$ is defined as the space of entire functions $F(z)$ of exponential type at most $\tau > 0$ that belong to $L^2(\mathbb{R})$ when restricted to the real line. We say that $\{\lambda_m\}$ is a non-redundant sampling sequence if by extracting one element of $\{\lambda_m\}$ the norm η_ν is no longer equivalent to the $L^2(\mathbb{R})$–norm.

We say that $\{\lambda_m\}$ is *interpolating of order* ν in $PW^2(\tau)$ if the system

$$F^{(j)}(\lambda_m) = f_{j,m}$$

has a solution $F \in PW^2(\tau)$ for any

$$(\{f_{0,m}\}, \{f_{1,m}\}, \ldots, \{f_{\nu-1,m}\}) \in \overbrace{\ell^2(\mathbb{Z}) \times \ldots \times \ell^2(\mathbb{Z})}^{\nu \text{ times}}.$$

We say that $\{\lambda_m\}$ is a complete interpolating sequence if the solution is unique.

In [64, Theorem 1], Ortega–Cerdà and Seip gave necessary and sufficient conditions for a sequence be sampling with no derivatives in $PW^2(\tau)$. Their characterization relies heavily upon the representation $PW^2(\tau) = \mathcal{H}^2(E)$ for some Hermite–Biehler function $E(z)$. They proved the following result.

Theorem 3.4.6 (Ortega–Cerdà and Seip)**.** *Let* $\{\lambda_m\}$ *be a separated sequence of points. The following are equivalent:*

(i) $\{\lambda_m\}$ *is a sampling sequence without derivatives for* $PW^2(\tau)$*.*

(ii) There exists a Hermite–Biehler function $E(z)$ *and an entire function* $W(z)$ *with* $|W(\overline{z})| \leq |W(z)|$ *for all* $z \in \mathbb{C}^+$*, and such that* $PW^2(\tau) = \mathcal{H}^2(E)$ *as sets and* $\{\lambda_m\}$ *coincides with the real zeros of* $E^*W^* - EW$*.*

Remark. Ortega–Cerdà and Seip also proved that if $\{\lambda_m\}$ is a complete interpolating sequence without derivatives then there exists a Hermite–Biehler function $E(z) = A(z) - iB(z)$ such that $PW^2(\tau) = \mathcal{H}^2(E)$ and $\{\lambda_m\}$ are the real zeros of the function $B(z)$.

In this direction we derive the following result.

Theorem 3.4.7. *Let $E(z) = A(z) - iB(z)$ be a Hermite–Biehler function, $\nu \geq 2$ be an integer and denote by $\{t_n\}_{n\in\mathbb{Z}}$ the real zeros of $B(z)$. Assume that $PW^2(\tau) = \mathcal{H}^2(E^\nu)$ as sets and there exists $C > 0$ such that $|A(t_n)| \leq C$ for all n. Then the map*

$$F(z) \mapsto (F(t_n), F'(t_n), \cdots, F^{(\nu-1)}(t_n))$$

defines a continuous linear isomorphism from $PW^2(\tau)$ to $\overbrace{\ell^2(\mathbb{Z}) \times \ldots \times \ell^2(\mathbb{Z})}^{\nu \text{ times}}$. In particular, $\{t_n\}$ is a complete interpolating and a non-redundant sampling sequence of order ν.

Proof. Since convergence in the space implies uniform convergence in compacts of $\mathbb{C}$, we can apply the Closed Graph Theorem to obtain that

$$\int_{\mathbb{R}} |F(x)/E(x)^\nu|^2 \mathrm{d}x \simeq \int_{\mathbb{R}} |F(x)|^2 \mathrm{d}x$$

for all $F \in PW^2(\tau) = \mathcal{H}^2(E^\nu)$. Also, it is a known fact that the Paley–Wiener spaces are closed under differentiation (one can use Fourier inversion to see that). Thus, we can apply Lemma 3.2.2 to deduce that $\varphi'(x) \leq D\sqrt{\nu}$, where D is the norm of the differentiation operator in $PW^2(\tau)$.

Write $E(z)^\nu = A_\nu(z) - iB_\nu(z)$. Note that the zeros of $B_\nu(z)$ coincide with the points $\varphi(s) \equiv 0 \mod \pi/\nu$. Thus, if $s_1 < s_2$ are two consecutive zeros of $B_\nu(z)$ we obtain that

$$\pi/\nu = \varphi(s_2) - \varphi(s_1) \leq D\sqrt{\nu}(s_1 - s_2).$$

Hence the zeros of $B_\nu(z)$ are separated. By Theorem 3.4.6, the identification $\mathcal{H}^2(E^\nu) = \mathrm{PW}^2(\tau)$ implies that the zeros of the function $B_\nu(z)$ form a sampling sequence without derivates in $\mathrm{PW}^2(\tau)$, that is,

$$\sum_{B_\nu(t)=0} |F(t)|^2 \simeq \int_{\mathbb{R}} |F(x)|^2 \mathrm{d}x \tag{3.35}$$

for every $F \in PW^2(\tau)$. We conclude that $B \notin \mathcal{H}^2(E)$, otherwise, since

$$B_\nu(z) = \sum_{k=1}^{\nu} b_k A(z)^{\nu-k} B(z)^k$$

for some coefficients b_k, this would imply that $B_\nu(z) \in \mathcal{H}^2(E^\nu)$ and, by (3.35), $B_\nu(z)$ would have zero norm, a contradiction.

Finally, this theorem will follow from Theorem 3.1.1 items (1) and (2) once we verify the estimates

$$1 \ll \varphi'(t_n)$$

and

$$K_\nu(t_n, t_n) \simeq 1$$

for all n. Using the reproducing kernel structure of $PW^2(\tau)$ one can show that

$$K_\nu(x,x) = \sup_{\|F/E^\nu\|_{L^2} \leq 1} |F(x)|^2 \simeq \sup_{\|F\|_{L^2} \leq 1} |F(x)|^2 = \tau/\pi$$

for all real x. This last fact in conjunction with formula

$$\nu\varphi'(x)|E(x)|^{2\nu} = \pi K_\nu(x,x)$$

and the hypothesis that $|A(t_n)| \leq C$ for all n, proves the desired estimates and concludes the proof. □

Remark. In [64, Theorem 4], Lyubarskii and Seip give necessary and sufficient conditions for the representation $PW^2(\tau) = \mathcal{H}^2(E)$.

Chapter 4

Band–limited Approximations in Many Variables

4.1 Approximations of Functions Subordinated to Gaussians

In this section we study the Beurling–Selberg problem for multivariate Gaussian functions. We then generalize the *Gaussian subordination* and *distribution method*, originally developed in [18], to higher dimensions and apply the method to study the Beurling–Selberg problem for a class of radial functions. We conclude our investigations by adapting the construction to periodic functions (see Section 4.5.1). For further applications we refer to [37].

To state the first of our main results we will use the following notation (see Section 4.3.1 for additional information). For $\boldsymbol{a} = (a_1, \dots, a_N)$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ with only positive entries let

$$\boldsymbol{x} \in \mathbb{R}^N \mapsto G_{\boldsymbol{\lambda}}(\boldsymbol{x}) = \exp\left\{-\sum_{j=1}^{N} \lambda_j \pi x_j^2\right\}.$$

and let

$$Q(\boldsymbol{a}) = \prod_{j=1}^{N} [-a_j, a_j].$$

We also use the notation $Q(R) = Q((R, \dots, R))$ for a given $R > 0$ and $\mathbb{R}_+^N = (0, \infty)^N$.

Our first result is a solution to the Beurling–Selberg extremal problem of determining optimal *majorants* of the Gaussian function $G_{\boldsymbol{\lambda}}(\boldsymbol{x})$ that have Fourier transform supported in $Q(\boldsymbol{a})$.

Theorem 4.1.1. *Let $\boldsymbol{a}, \boldsymbol{\lambda} \in \mathbb{R}^N_+$. If $F : \mathbb{R}^N \to \mathbb{R}$ is an integrable function that satisfies:*

(*i*) $F(\boldsymbol{x}) \geq G_{\boldsymbol{\lambda}}(\boldsymbol{x})$ *for each* $x \in \mathbb{R}^N$, *and*

(*ii*) $\widehat{F}(\boldsymbol{\xi}) = 0$ *for each* $\boldsymbol{\xi} \notin Q(\boldsymbol{a})$,

then

$$\int_{\mathbb{R}^N} F(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \geq \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} \Theta(0; ia_j^2/\lambda_j).$$

where $\Theta(v;\tau)$ is Jacobi's theta function (see §4.3.1). Moreover, equality holds if $F(\boldsymbol{x}) = M_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})$ where $M_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})$ is defined by (4.23).

Remark. This theorem is essentially a corollary of Theorem 3 of [18]. The proof simply uses the product structure and positivity of $G_{\boldsymbol{\lambda}}(\boldsymbol{x})$ in conjunction with Theorem 3 of [18] and for this reason we omit the proof. However, it can be found in [37].

It would be interesting to determine the analogue of the above theorem for *minorants* of $G_{\boldsymbol{\lambda}}(\boldsymbol{x})$ (that is, the high dimensional analogue of Theorem 2 of [18]), where the extremal functions cannot be obtained by a tensor product of lower dimensional extremal functions. In our second result we address this problem by constructing minorants of the Gaussian function $G_{\boldsymbol{\lambda}}(\boldsymbol{x})$ that have Fourier transform supported in $Q(\boldsymbol{a})$ and that are *asymptotically extremal* as $\boldsymbol{a}$ becomes uniformly large in each coordinate.

Theorem 4.1.2. *Let $\boldsymbol{a}, \boldsymbol{\lambda} \in \mathbb{R}^N_+$. If $F : \mathbb{R}^N \to \mathbb{R}$ is an integrable function that satisfies* *(i)* $F(\boldsymbol{x}) \leq G_{\boldsymbol{\lambda}}(\boldsymbol{x})$ *for each* $\boldsymbol{x} \in \mathbb{R}^N$, *and*

(ii) $\widehat{F}(\boldsymbol{\xi}) = 0$ *for each* $\boldsymbol{\xi} \notin Q(\boldsymbol{a})$,

then

$$\int_{\mathbb{R}^N} F(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \leq \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} \Theta(\tfrac{1}{2}; ia_j^2/\lambda_j). \tag{4.1}$$

Furthermore, there exists a positive constant $\gamma_0 = \gamma_0(N)$ *such that if* $\gamma := \min_j\{a_j^2/\lambda_j\} \geq \gamma_0$, *then*

$$\prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} \Theta(\tfrac{1}{2}; ia_j^2/\lambda_j) \leq (1 + 5Ne^{-\pi\gamma}) \int_{\mathbb{R}^N} L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{4.2}$$

where $L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})$ *is the minorant defined by* (4.22).

Remark.

(1) For a fixed $\boldsymbol{a}$, if $\boldsymbol{\lambda}$ is large enough then the right–hand side of (4.2) would be negative and the inequality would not hold, but this is not true for large values of γ. In fact, this happens because the zero function would be a better minorant.

(2) Inequality (4.2) implies that if $\boldsymbol{\lambda}$ is fixed and $\boldsymbol{a}$ is large, then $\int_{\mathbb{R}^N} L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ approaches exponentially fast the optimal answer. In this sense, we say that $L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})$ is *asymptotically optimal* with respect to the type.

4.1.1 Gaussian Subordination Method

Our next set of results are Theorems 4.1.3, 4.1.4, and 4.1.5. These theorems generalize the so called *distribution* and *Gaussian subordination* methods of [18]. The main idea behind these methods goes back to the paper of Graham and Vaaler [40]. We will describe a "watered down" version of the approach here. Let us begin with the inequality

$$G_{\boldsymbol{\lambda}}(\boldsymbol{x}) \leq M_{\boldsymbol{\lambda}}(\boldsymbol{x})$$

where $F_{\boldsymbol{\lambda}}(\boldsymbol{x})$ is defined by (4.23). The idea is to *integrate the free parameter* $\boldsymbol{\lambda}$ in the function $G_{\boldsymbol{\lambda}}(\boldsymbol{x})$ with respect to a (positive) measure $\mathrm{d}\mu$ on $\mathbb{R}_+^N = (0,\infty)^N$ to obtain a pair of new functions of $\boldsymbol{x}$:

$$\mathcal{G}(\boldsymbol{x}) = \int_{\mathbb{R}_+^N} G_{\boldsymbol{\lambda}}(\boldsymbol{x}) d\mu(\boldsymbol{\lambda}) \leq \int_{\mathbb{R}_+^N} M_{\boldsymbol{\lambda}}(\boldsymbol{x}) d\mu(\boldsymbol{\lambda}) = \mathcal{M}(\boldsymbol{x}).$$

The process simultaneously produces a function $\mathcal{G}(\boldsymbol{x})$ and a majorant $\mathcal{M}(\boldsymbol{x})$ having $\widehat{\mathcal{M}}(\boldsymbol{\xi})$ supported in $Q(\boldsymbol{a})$. The difference of the functions in L^1−norm is similarly obtained by integrating against $\mathrm{d}\mu$.

Our next result is a generalization of the distribution method developed in [18] for existence of majorants and minorants. In what follows we say that an entire function $F : \mathbb{C}^N \to \mathbb{C}$ is of *exponential type* with respect to a compact convex set $K \subset \mathbb{R}^N$ if it has Fourier transform (in the sense of tempered distribution) supported in K.

Theorem 4.1.3 (Distribution Method – Existence)**.** *Let $K \subset \mathbb{R}^N$ be a compact convex set, Λ be a measurable space of parameters, and for each $\boldsymbol{\lambda} \in \Lambda$ let $G(\boldsymbol{x};\boldsymbol{\lambda}) \in L^1(\mathbb{R}^N)$ be a real–valued function. For each $\boldsymbol{\lambda}$ let $F(\boldsymbol{z};\boldsymbol{\lambda})$ be an entire function defined for $\boldsymbol{z} \in \mathbb{C}^N$ of exponential type with respect to K. Let $\mathrm{d}\mu$ be a non–negative measure on Λ that satisfies*

$$\int_\Lambda \int_{\mathbb{R}^N} |F(\boldsymbol{x};\boldsymbol{\lambda}) - G(\boldsymbol{x};\boldsymbol{\lambda})| \mathrm{d}\boldsymbol{x} \mathrm{d}\mu(\boldsymbol{\lambda}) < \infty. \tag{4.3}$$

and

$$\int_\Lambda \int_{\mathbb{R}^N} |\widehat{G}(\boldsymbol{x};\boldsymbol{\lambda})\varphi(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} \mathrm{d}\mu(\boldsymbol{\lambda}) < \infty \tag{4.4}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$ *supported in* K^c.

Let $\mathcal{G} \in \mathcal{S}'(\mathbb{R}^N)$ *be a real–valued continuous function such that*

$$\widehat{\mathcal{G}}(\varphi) = \int_{\mathbb{R}^N} \int_{\Lambda} \widehat{G}(\boldsymbol{x};\boldsymbol{\lambda}) d\mu(\boldsymbol{\lambda})\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \tag{4.5}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^N)$ *supported in* K^c.

(*i*) *If* $G(\boldsymbol{x};\boldsymbol{\lambda}) \leq F(\boldsymbol{x};\boldsymbol{\lambda})$ *for each* $\boldsymbol{x} \in \mathbb{R}^N$ *and* $\boldsymbol{\lambda} \in \Lambda$, *then there exists a real entire majorant* $\mathcal{M}(\boldsymbol{x})$ *for* $\mathcal{G}(\boldsymbol{x})$ *of exponential type with respect to* K *and*

$$\int_{\mathbb{R}^N} \{\mathcal{M}(\boldsymbol{x}) - \mathcal{G}(\boldsymbol{x})\}\mathrm{d}\boldsymbol{x}$$

is equal to the quantity in (4.3).

(*ii*) *If* $F(\boldsymbol{x};\boldsymbol{\lambda}) \leq G(\boldsymbol{x};\boldsymbol{\lambda})$ *for each* $\boldsymbol{x} \in \mathbb{R}^N$ *and* $\boldsymbol{\lambda} \in \Lambda$, *then there exists a real entire minorant* $\mathcal{L}(\boldsymbol{x})$ *for* $\mathcal{G}(\boldsymbol{x})$ *of exponential type with respect to* K *and*

$$\int_{\mathbb{R}^N} \{\mathcal{G}(\boldsymbol{x}) - \mathcal{L}(\boldsymbol{x})\}d\boldsymbol{x}$$

is equal to the quantity in (4.3).

Remark. With the exception of Theorem 4.1.3, Λ will always stand for $(0,\infty)^N$.

For a given $\boldsymbol{a} \in \mathbb{R}^N_+$ define $\mathfrak{G}^N_+(\boldsymbol{a})$ as the set of ordered pairs $(\mathcal{G}, \mathrm{d}\mu)$ where $\mathcal{G} : \mathbb{R}^N \to \mathbb{R}$ is a function and $\mathrm{d}\mu$ is a non-negative Borel measure in $\mathbb{R}^N_+$ such that:

(C1) $\mathcal{G}(\boldsymbol{x})$ is a continuous function that also defines a *tempered distribution* (that is, $\mathcal{G} \in \mathcal{S}'(\mathbb{R}^N)$).

(C2) For all $\varphi \in \mathcal{S}(\mathbb{R}^N)$ supported in $Q(\boldsymbol{a})^c$ we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N_+} |\widehat{G}_{\boldsymbol{\lambda}}(\boldsymbol{x})\varphi(\boldsymbol{x})| \mathrm{d}\mu(\boldsymbol{\lambda}) \mathrm{d}\boldsymbol{x} < \infty.$$

(C3) For all $\varphi \in \mathcal{S}(\mathbb{R}^N)$ supported in $Q(\boldsymbol{a})^c$ we have

$$\int_{\mathbb{R}^N} \mathcal{G}(\boldsymbol{x})\widehat{\varphi}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N_+} \widehat{G}_{\boldsymbol{\lambda}}(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{\lambda})\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

(C4+) The following integrability condition holds

$$\int_{\mathbb{R}^N_+} \prod_{k=1}^{N} \lambda_k^{-\frac{1}{2}} \left\{ \prod_{j=1}^{N} \Theta(0; ia_j^2/\lambda_j) - 1 \right\} \mathrm{d}\mu(\boldsymbol{\lambda}) < \infty.$$

In an analogous way, we define the class $\mathfrak{G}^N_-(\boldsymbol{a})$ by replacing condition **(C4+)** by

(C4-) The following integrability condition holds

$$\int_{\mathbb{R}^N_+} \left\{ 1 - \left(\sum_{j=1}^{N} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (N-1) \right) \prod_{j=1}^{N} \Theta(0; ia_j^2/\lambda_j) \right\} \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} \mathrm{d}\mu(\boldsymbol{\lambda}) < \infty. \tag{4.6}$$

Theorem 4.1.4 offers an optimal resolution of the Majorization Problem for the class of functions $\mathfrak{G}^N_+(\boldsymbol{a})$ and Theorem 4.1.5 offers an asymptotically optimal resolution of the Minorization Problem for the class of functions $\mathfrak{G}^N_-(\boldsymbol{a})$. In what follows we use the notation of Section 4.3.1.

Theorem 4.1.4 (Gaussian Subordination – Majorant)**.** *For a given $\boldsymbol{a} \in \mathbb{R}^N_+$, let $(\mathcal{G}, \mathrm{d}\mu) \in \mathfrak{G}^N_+(\boldsymbol{a})$. Then there exists an extremal majorant $\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{x})$ of exponential type with respect to $Q(\boldsymbol{a})$ for $\mathcal{G}(\boldsymbol{x})$. Furthermore, $\mathcal{M}_{\boldsymbol{a}}(\boldsymbol{x})$ interpolates $\mathcal{G}(\boldsymbol{x})$ on $\mathbb{Z}^N/\boldsymbol{a}$ and satisfies*

$$\int_{\mathbb{R}^N} \mathcal{M}(\boldsymbol{x}) - \mathcal{G}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^N_+} \prod_{k=1}^{N} \lambda_k^{-\frac{1}{2}} \left\{ \prod_{j=1}^{N} \Theta(0; ia_j^2/\lambda_j) - 1 \right\} \mathrm{d}\mu(\boldsymbol{\lambda}).$$

Theorem 4.1.5 (Gaussian Subordination – Minorant)**.** *For a given* $\boldsymbol{a} \in \mathbb{R}_+^N$*, let* $(\mathcal{G}, \mathrm{d}\mu) \in \mathfrak{G}_-^N(\boldsymbol{a})$*. Then, if* $\mathcal{F}(\boldsymbol{z})$ *is a real entire minorant of* $\mathcal{G}(\boldsymbol{x})$ *of exponential type with respect to* $Q(\boldsymbol{a})$*, we have*

$$\int_{\mathbb{R}^N} \{\mathcal{G}(x) - \mathcal{F}(\boldsymbol{x})\}\mathrm{d}\boldsymbol{x} \geq \int_{\mathbb{R}_+^N} \prod_{j=1}^{N} \lambda_k^{-\frac{1}{2}} \left\{1 - \prod_{j=1}^{N} \Theta(\tfrac{1}{2}; ia_j^2/\lambda_j)\right\}\mathrm{d}\mu(\boldsymbol{\lambda}).$$

Furthermore, there exists a family of minorants $\{\mathcal{L}_{\boldsymbol{a}}(\boldsymbol{z}) : \boldsymbol{a} \in \mathbb{R}_+^N\}$ *where* $\mathcal{L}_a(\boldsymbol{z})$ *is of exponential type with respect to* $Q(\boldsymbol{a})$ *such that*

$$\int_{\mathbb{R}^N} \{\mathcal{G}(\boldsymbol{x}) - \mathcal{L}_a(\boldsymbol{x})\}\mathrm{d}\boldsymbol{x}$$

is equal to the left-hand side of (4.6)*. Also*

$$\lim_{\boldsymbol{a}\uparrow\infty} \int_{\mathbb{R}^N} \{\mathcal{G}(\boldsymbol{x}) - \mathcal{L}_a(\boldsymbol{x})\}\mathrm{d}\boldsymbol{x} = 0,$$

where $\boldsymbol{a} \uparrow \infty$ *means* $a_j \uparrow \infty$ *for each* j*.*

Corollary 4.1.6. *Assume all the hypotheses of Theorem 4.1.5. Suppose also that exists a positive number* $R > 0$ *such that* $\mathrm{supp}(\mathrm{d}\mu) \subset \mathbb{R}_+^N \cap Q(R)$*,* $\mathcal{G} \in L^1(\mathbb{R}^N)$ *and*

$$\int_{\mathbb{R}^N} \mathcal{G}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}_+^N} \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} \mathrm{d}\mu(\boldsymbol{\lambda}) < \infty.$$

Then, there exists a constant $\alpha_0 > 0$ *such that, if* $\alpha := \min_j\{a_j\} \geq \alpha_0$ *and if* $\mathcal{F}(\boldsymbol{x})$ *is a real entire minorant of* $\mathcal{G}(\boldsymbol{x})$ *of exponential type with respect to* $Q(\boldsymbol{a})$*, then*

$$\int_{\mathbb{R}^N} \mathcal{F}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \leq (1 + 5Ne^{-\pi\alpha^2/R}) \int_{\mathbb{R}^N} \mathcal{L}_a(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Remark. We will prove only Theorem 4.1.3, since the other ones are a direct application of Theorems 4.1.1, 4.1.2 and 4.1.3. For the interested reader we refer to [37] for the complete proofs.

4.1.2 The Class of Admissible Functions

We define the class

$$\mathfrak{G}^N = \bigcap_{\boldsymbol{a}\in\mathbb{R}^N_+} \mathfrak{G}^N_-(\boldsymbol{a}) \cap \mathfrak{G}^N_+(\boldsymbol{a}).$$

This is the class of pairs such that Theorems 4.1.4 and 4.1.5 are applicable for every $\boldsymbol{a} \in \mathbb{R}^N_+$. In this subsection we present conditions for a pair $(\mathcal{G}, \mathrm{d}\mu)$ to belong to this class. Some interesting properties arise when the measure $\mathrm{d}\mu$ is concentrated in the diagonal of $\mathbb{R}^N_+$. For every $\eta \in [0,1]$ we define $(\mathbb{R}^N_+)_\eta = \{\boldsymbol{\lambda} \in \mathbb{R}^N_+ : \eta\lambda_j \leq \lambda_k \ \forall j,k\}$ and we note that $(\mathbb{R}^N_+)_0 = \mathbb{R}^N_+$ and $(\mathbb{R}^N_+)_1 = \{(t,\dots,t) : t > 0\}$ is the diagonal.

Proposition 4.1.7. *Let* $(\mathcal{G}, \mathrm{d}\mu)$ *be a pair that satisfies conditions* (C1), (C2) *and* (C3) *for every* $\boldsymbol{a} \in \mathbb{R}^N_+$. *Suppose that* $\mathrm{supp}(\mathrm{d}\mu) \subset (\mathbb{R}^N_+)_\eta$ *for some* $\eta \in (0,1]$ *and* $\mathrm{d}\mu((\mathbb{R}^N_+)_\eta \setminus Q(R)) < \infty$ *for every* $R > 0$. *Then* $(\mathcal{G}, \mathrm{d}\mu) \in \mathfrak{G}^N$.

Proof. We only prove that condition **(C4-)** holds, the condition **(C4+)** is analogous. For a given $\boldsymbol{a} \in \mathbb{R}^N_+$, define the function

$$\phi_{\boldsymbol{a}} : \boldsymbol{\lambda} \in \mathbb{R}^N_+ \mapsto \left(\sum_{j=1}^N \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (N-1) \right) \prod_{j=1}^N \Theta(0; ia_j^2/\lambda_j).$$

By (4.15) and the Poisson summation formula, we have the following estimates

$$\begin{aligned} 1 - \Theta(\tfrac{1}{2}; i/t) &\sim 2e^{-\pi/t} \quad \text{as } t \to 0, \\ \Theta(\tfrac{1}{2}; i/t) &\sim 2t^{\frac{1}{2}} e^{-\pi t/4} \quad \text{as } t \to \infty \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \Theta(0; i/t) - 1 &\sim 2e^{-\pi/t} \quad \text{as } t \to 0, \\ \Theta(0; i/t) &\sim t^{\frac{1}{2}} \quad \text{as } t \to \infty, \end{aligned} \tag{4.8}$$

where the symbol $\sim$ means that the quotient converges to 1. Using the left-hand side inequality of Lemma 4.3.4 we conclude that exists an $R > 0$ and a $C > 0$ such that

$$\phi_{\boldsymbol{a}}(\boldsymbol{\lambda}) \geq 1 - C \sum_{j=1}^{N} e^{-\pi a_j^2/\lambda_j}$$

for every $\boldsymbol{\lambda} \in \mathbb{R}_+^N \cap Q(R)$. Choose $\ell \in \{1, \dots, N\}$ such that $a_\ell \leq a_j$ for every j. If $\boldsymbol{\lambda} \in (\mathbb{R}_+^N)_\eta \cap Q(R)$ we have

$$\begin{aligned}\big\{1 - \phi_{\boldsymbol{a}}(\boldsymbol{\lambda})\big\} \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} &\leq C\bigg(\sum_{j=1}^{N} e^{-\pi a_j^2/\lambda_j}\bigg) \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} \leq dCe^{-\pi a_\ell^2 \eta/\lambda_\ell} \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} \\ &\leq NC \prod_{j=1}^{N} e^{-\pi a_\ell^2 \eta^2/(N\lambda_j)} \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} = NC\widehat{G}_{\boldsymbol{\lambda}}(\beta \boldsymbol{u}),\end{aligned}$$

where $\beta = a_\ell \eta N^{-1/2}$ and $\boldsymbol{u} = (1, \dots, 1)$.

By estimates (4.7)-(4.8) we see that the functions $\Theta(\frac{1}{2}; i/t)t^{-\frac{1}{2}}$ and $\Theta(0; i/t)t^{-\frac{1}{2}}$ are bounded for $t \in [\eta R, \infty)$, and thus, we conclude that the function

$$\boldsymbol{\lambda} \in \mathbb{R}_+^N \mapsto \phi_{\boldsymbol{a}}(\boldsymbol{\lambda}) \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}}$$

is bounded on $(\mathbb{R}_+^N)_\eta \setminus Q(R)$, since it is a finite sum of products of these theta functions. Since $\eta > 0$, we obtain that the function

$$\boldsymbol{\lambda} \in \mathbb{R}_+^N \mapsto \big\{1 - \phi_{\boldsymbol{a}}(\boldsymbol{\lambda})\big\} \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}}$$

is bounded in $(\mathbb{R}_+^N)_\eta \setminus Q(R)$ by a constant C'. Therefore, we have

$$\begin{aligned}&\int_{(\mathbb{R}_+^N)_\eta} \big\{1 - \phi_{\boldsymbol{a}}(\boldsymbol{\lambda})\big\} \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} \mathrm{d}\mu(\boldsymbol{\lambda}) \\ &\qquad \leq NC \int_{(\mathbb{R}_+^N)_\eta \cap Q(R)} \widehat{G}_{\boldsymbol{\lambda}}(\beta u) \mathrm{d}\mu(\boldsymbol{\lambda}) + C' \mathrm{d}\mu((\mathbb{R}_+^N)_\eta \setminus Q(R)) < \infty,\end{aligned}$$

which is finite by condition **(C2)** and the hypotheses of this lemma. Thus $\mathrm{d}\mu$ satisfies condition **(C4-)** and this concludes the proof. $\square$

Corollary 4.1.8. *Let* $\mathrm{d}\mu$ *be a probability measure on* $\mathbb{R}_+^N$ *with* $\mathrm{supp}(\mathrm{d}\mu) \subset (\mathbb{R}_+^N)_\eta$ *for some* $\eta \in (0,1]$. *Define the function*

$$\mathcal{G}(\boldsymbol{x}) = \int_{\mathbb{R}_+^N} G_{\boldsymbol{\lambda}}(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{\lambda}) \tag{4.9}$$

for all $\boldsymbol{x} \in \mathbb{R}^N$. *Then* $(\mathcal{G}, \mathrm{d}\mu) \in \mathfrak{G}^N$.

Due to a classical result of Schoenberg (see [66]), a radial function $\mathcal{G}(\boldsymbol{x}) = \mathcal{G}(|\boldsymbol{x}|)$ admits the representation (4.9) for a probability $\mathrm{d}\mu$ supported on the diagonal $(\mathbb{R}_+^N)_1$ if and only if the radial extension to $\mathbb{R}^n$ of $\mathcal{G}(r)$ is positive definite, for all $n > 0$. And this occurs if and only if the function $\mathcal{G}(\sqrt{r})$ is completely monotone. As a consequence of this fact and Corollary 4.1.8 the following multidimensional versions of the functions in [18, Section 11] are admissible

Example 1.

$$\mathcal{G}(\boldsymbol{x}) = e^{-\alpha|\boldsymbol{x}|^r} \in \mathfrak{G}^N, \;\; \alpha > 0 \;\; \textit{and} \;\; 0 < r \leq 2.$$

Example 2.

$$\mathcal{G}(\boldsymbol{x}) = (|\boldsymbol{x}|^2 + \alpha^2)^{-\beta} \in \mathfrak{G}^N, \;\; \alpha > 0 \textit{ and } \beta > 0.$$

Example 3.

$$\mathcal{G}(\boldsymbol{x}) = -\log\left(\frac{|\boldsymbol{x}|^2 + \alpha^2}{|\boldsymbol{x}|^2 + \beta^2}\right) \in \mathfrak{G}^N, \;\; \textit{for } 0 < \alpha < \beta.$$

The next example is a high dimensional analogue of [18, Corollary 21] which is admissible by using Proposition 4.1.7 with the measure $\mathrm{d}\mu(t) = t^{-\sigma/2-1}$ restricted to the diagonal $(\mathbb{R}_+^N)_1$.

Example 4.

$$\mathcal{G}_\sigma(\boldsymbol{x}) = |x|^\sigma, \text{ for } \sigma \in (0,\infty) \setminus 2\mathbb{Z}_+$$

For further examples and details we refer to [37].

4.2 Selberg's Box–Minorant Problem

In the 1970's Selberg introduced some approximations to the indicator function of an interval that quickly inherited the name "Selberg's magic functions". If I is an interval in $\mathbb{R}$ of finite length, $\mathbf{1}_I(x)$ is the indicator of I, and $\delta > 0$, then Selberg's functions $x \in \mathbb{R} \mapsto M(x)$ and $x \in \mathbb{R} \mapsto L(x)$ satisfy:

(*i*) $\widehat{M}(\xi) = \widehat{L}(\xi) = 0$ if $|\xi| > \delta$,

(*ii*) $L(x) \leq \mathbf{1}_I(x) \leq M(x)$ for each $x \in \mathbb{R}$,

(*iii*) $\int_{\mathbb{R}} \{M(x) - \mathbf{1}_I(x)\} \mathrm{d}x = \int_{\mathbb{R}} \{\mathbf{1}_I(x) - L(x)\} \mathrm{d}x = \delta^{-1}$.

Furthermore, among all functions that satisfy (*i*) and (*ii*) above, Selberg's functions minimize the integrals appearing in (*iii*) if and only if $\delta|I| \in \mathbb{Z}$. When $I = [0,\infty)$ is a half-open integral, the analogous result was proven 40 years before by Beurling, and Beurling's functions are always extremal, regardless of the value of δ. See [69, 73] for a survey.

Selberg originally constructed his functions so that he could use them to prove a sharp form of the large sieve inequality. Today, however, there is a large number of applications of Selberg's functions (and their generalizations) in various areas of mathematics, including number theory, dynamical systems, optics, combinatorics, sampling theory, and beyond. While the single variable

Beurling–Selberg extremal theory is relatively well understood, the multivariable theory in its current state is much less so (see [39]).

In unpublished work, Selberg was able to use his functions $M(x)$ and $L(x)$ to construct majorants and minorants of the box $I \times \cdots \times I \subset \mathbb{R}^N$ whose Fourier transforms are supported in the box $[-\delta, \delta]^N$. The multivariable majorant $\mathcal{M}(\boldsymbol{x})$ is just tensors of $M(x)$, however the multivariable minorant Selberg obtained was

$$\mathcal{L}(\boldsymbol{x}) = -(N-1)\prod_{n=1}^{N} M(x_n) + \sum_{n=1}^{N} L(x_n) \prod_{m \neq n} M(x_m). \tag{4.10}$$

A simple calculation shows that

$$\int_{\mathbb{R}^N} \mathcal{L}(\boldsymbol{x}) d\boldsymbol{x} = (|I| + \delta^{-1})^{N-1} \left(|I| - \delta^{-1}(2N-1)\right).$$

We see that, when N is sufficiently large, or $|I|$ and δ are sufficiently small, then Selberg's minorants have *negative* integrals. Therefore, the zero function is a better minorant. Nevertheless, it can be shown that Selberg's minorants are asymptotically extremal as $\delta \to \infty$.

These questions beg us to study the simplest version of Selberg's minorant problem in several variables, that we call the *Box–Minorant problem*, and which is described as follows:

Let $Q_N = [-1,1]^N$ denote the N–dimensional box in $\mathbb{R}^N$ and $\mathbf{1}_{Q_N}(\boldsymbol{x})$ denote the indicator function of Q_N. For every integer $N \geq 1$ define the following quantity

$$\nu(N) = \sup \int_{\mathbb{R}^N} F(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{4.11}$$

where the supremum is taken over functions $F \in PW^1(Q_N)$ (that is, $F \in L^1(\mathbb{R}^N)$ and $\widehat{F}(\boldsymbol{\xi})$ is supported in Q_N) that satisfy

$$F(\boldsymbol{x}) \leq \mathbf{1}_{Q_N}(\boldsymbol{x}) \tag{4.12}$$

for (almost) every $\boldsymbol{x} \in \mathbb{R}^N$.

A function $F(\boldsymbol{x})$ satisfying the above conditions will be called *admissible for* $\nu(N)$ (or $\nu(N)$–admissible) and if it achieves equality in (4.11), then it is said to be *extremal*. It is shown in Lemma 4.4.4 that extremal functions for $\nu(N)$ always exist. Moreover, by condition (4.12) and an application of the Poisson summation formula (4.35) we obtain the trivial bound

$$\nu(N) \leq 1,$$

for all $N \geq 1$. Selberg (see [69, 73]) was able to show that $\nu(1) = 1$ and that

$$\frac{\sin^2 \pi x}{(\pi x)^2(1 - x^2)} \tag{4.13}$$

is an extremal function (this is not the unique extremal function).

Our first result states that in fact $\nu(N)$ is a decreasing function of N that converges to zero as $N \to \infty$.

Theorem 4.2.1. *The following statements hold:*

(i) $\nu(2) < 1$.

(ii) *If* $\nu(N) > 0$ *then* $\nu(N+1) < \nu(N)$.

(iii) $\lim_{N\to\infty} \nu(N) = 0$.

We deduce Theorem 4.2.1 from a much more unexpected fact regarding the behavior of the values $F(\mathbf{0})$ for $\nu(N)$–admissible functions $F(\boldsymbol{x})$. To motivate the result we observe that if an extremal function $F(\boldsymbol{x})$ has positive integral ($\widehat{F}(\mathbf{0}) > 0$), then its maximum must be equal to 1. To see this, suppose $\rho \in (0, 1]$ is the maximum of $F(\boldsymbol{x})$, then the function $\rho^{-1}F(\boldsymbol{x})$ would then be $\nu(N)$–admissible and, seeing that $F(\boldsymbol{x})$ is extremal, it must have integral equal to $F(\boldsymbol{x})$ and so the $\rho = 1$. Intuitively one might guess that this maximum will occur at the origin, which is the center of mass of Q_N. However, as often happens in mathematics, our next theorem demonstrates a counter-intuitive result. It shows that, for large N, no admissible function for $\nu(N)$ can achieve its maximum at the origin.

Theorem 4.2.2. *For every $\varepsilon > 0$ there exists a dimension $N(\varepsilon) > 0$ such that for every $N \geq N(\varepsilon)$, any $\nu(N)$–admissible function $F(\boldsymbol{x})$ with non-negative integral satisfies $F(\mathbf{0}) < \varepsilon$.*

Remark. It was not previously known, however, whether $\nu(N) > 0$ for any $N > 1$. One of our main contributions of is the development of a method (Section 4.5.2) to produce non-trivial admissible functions for $\nu(N)$, and thereby establishing that $\nu(N) \geq 0.12$ for $N \leq 5$.

Let $F(\boldsymbol{x})$ be a $\nu(N)$–admissible function. It follows from the Poisson summation formula (4.35) that

$$\widehat{F}(\mathbf{0}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \widehat{F}(\boldsymbol{n}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} F(\boldsymbol{n}) \leq F(\mathbf{0}).$$

Thus we have the fundamental inequality

$$\widehat{F}(\mathbf{0}) \leq F(\mathbf{0}). \tag{4.14}$$

Evidently, there is equality in (4.14) if, and only if, $F(\boldsymbol{n}) = 0$ for each nonzero $\boldsymbol{n} \in \mathbb{Z}^N$. If $N = 1$ then, by using the interpolation formula (4.36), this is precisely how Selberg was able to construct the minorant (4.13) and show that $\nu(1) = 1$. Another one of our main results demonstrates that this line of attack fails in higher dimensions.

Theorem 4.2.3. *Let $N > 1$. Let $F(\boldsymbol{x})$ be an admissible function for $\nu(N)$. Assume that $F(\boldsymbol{n}) = 0$ for every nonzero $\boldsymbol{n} \in \mathbb{Z}^N$ and $F(\mathbf{0}) \geq 0$ (or equivalently, $\widehat{F}(\mathbf{0}) = F(\mathbf{0}) \geq 0$). Then $F(\boldsymbol{x}) = 0$ for any $\boldsymbol{x} \in \mathbb{R}^N$ that has $N-2$ integers entries. We conclude that $\widehat{F}(\mathbf{0}) = F(\mathbf{0}) = 0$.*

Remark. The proof of the above theorem follows easily by Proposition 4.4.6, where we deal with the case $N = 2$, in conjunction with a slicing argument presented in Lemma 4.4.2.

It would be interesting to know the rate at which $\nu(N)$ tends to 0. Indeed, as a first step it would be interesting to know whether $\nu(N) > 0$ for all $N > 1$. In this direction we have the following result.

Theorem 4.2.4. *We have the following lower bounds for $\nu(N)$:*

(*i*) $\nu(2) \geq \frac{63}{64} = 0.984375$,

(*ii*) $\nu(3) \geq \frac{119}{128} = 0.9296875$,

(*iii*) $\nu(4) \geq \frac{95}{128} = 0.7421875$,

(*iv*) $\nu(5) \geq \frac{31}{256} = 0.12109375$.

Remark. The above result is a direct consequence of Theorem 4.5.6.

The above results lead to the important question on whether or not $\nu(N)$ vanishes in finite time. In Section 4.5.2 we estimate the critical dimension for which a quantity closely related to $\nu(N)$ vanishes. However, at the present moment we do not have sufficient information to state any conjecture.

4.3 Proofs of Theorems 4.1.2, 4.1.1 and 4.1.3

4.3.1 Background

One of the main objects of study in this chapter is the *Fourier transform*. Given an integrable function $F(\boldsymbol{x})$ on $\mathbb{R}^N$, we define the Fourier transform of $F(\boldsymbol{x})$ by

$$\widehat{F}(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} e(\boldsymbol{x}\cdot\boldsymbol{\xi})F(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{\xi} \in \mathbb{R}^N$ and $e(\theta) = e^{-2\pi i\theta}$. We extend the definition in the usual way to tempered distributions (see for instance [71]). We will mainly be considering functions whose Fourier transforms are supported in a bounded subset of $\mathbb{R}^N$. Such functions are called *band–limited*. It is well-known that band–limited functions can be extended to an entire functions on $\mathbb{C}^N$ satisfying an exponential growth condition. We will now state a generalization of the Paley–Wiener theorem which can be found in [48, Theorem 7.3.1].

Theorem 4.3.1 (Paley–Wiener–Schwartz)**.** *Let K be a convex compact subset of $\mathbb{R}^N$ with supporting function*

$$H(\boldsymbol{x}) = \sup_{\boldsymbol{y}\in K} |\boldsymbol{x}\cdot\boldsymbol{y}|.$$

If F is a tempered distribution such that the support of $\widehat{F}$ is contained in K, then $F : \mathbb{C}^N \to \mathbb{C}$ is an entire function and exists $C > 0$ such that

$$|F(\boldsymbol{x}+i\boldsymbol{y})| \leq C(1+|\boldsymbol{x}+i\boldsymbol{y}|)^C e^{2\pi H(\boldsymbol{y})}.$$

for every $\boldsymbol{x}+i\boldsymbol{y} \in \mathbb{C}^N$. Conversely, every entire function $F : \mathbb{C}^N \to \mathbb{C}$ satisfying an estimate of this form defines a tempered distribution with Fourier transform supported on K.

We will now define and compile some results about Gaussians and theta functions that we will need in the sequel. Given a positive real number $\delta > 0$

the Gaussian $g_\delta : \mathbb{R} \to \mathbb{R}$ is defined by

$$g_\delta(t) = e^{-\delta\pi t^2},$$

and its Fourier transform is given by $\widehat{g}_\delta(\xi) = \delta^{-1/2} g_{1/\delta}(\xi)$. For $\tau = \sigma + it$ with $t > 0$, denote $q = e^{\pi i \tau}$. The *Jacobi's theta function* (see [23]) is defined by

$$\Theta(v;\tau) = \sum_{n\in\mathbb{Z}} e(nv) q^{n^2}. \tag{4.15}$$

These functions are related through the Poisson summation formula by

$$\sum_{m\in\mathbb{Z}} g_\delta(v+m) = \sum_{n\in\mathbb{Z}} e(nv)\widehat{g}_\delta(n) = \delta^{-1/2}\Theta(v; i\delta^{-1}). \tag{4.16}$$

The one dimensional case of Theorems 4.1.1 and 4.1.2 were proven in [18]. They showed that the functions

$$\ell_\delta(z) = \Big(\frac{\cos \pi z}{\pi}\Big)^2 \Bigg\{ \sum_{k\in\mathbb{Z}} \frac{g_\delta(k+\frac{1}{2})}{(z-k-\frac{1}{2})^2} + \sum_{k\in\mathbb{Z}} \frac{g_\delta'(k+\frac{1}{2})}{(z-k-\frac{1}{2})} \Bigg\} \tag{4.17}$$

and

$$m_\delta(z) = \Big(\frac{\sin \pi z}{\pi}\Big)^2 \Bigg\{ \sum_{k\in\mathbb{Z}} \frac{g_\delta(k)}{(z-k)^2} + \sum_{k\in\mathbb{Z}} \frac{g_\delta'(k)}{(z-k)} \Bigg\} \tag{4.18}$$

are entire functions of exponential type at most 2π (that is, their Fourier transforms are supported in $[-1,1]$) and they satisfy

$$\ell_\delta(\boldsymbol{x}) \le g_\delta(\boldsymbol{x}) \le m_\delta(\boldsymbol{x}) \tag{4.19}$$

for all real x. Moreover,

$$\int_{\mathbb{R}} m_\delta(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \delta^{-\frac{1}{2}}\Theta(0; i/\delta) \tag{4.20}$$

and

$$\int_{\mathbb{R}} \ell_\delta(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \delta^{-\frac{1}{2}}\Theta(\tfrac{1}{2}; i/\delta). \tag{4.21}$$

Also, in view of (4.16)–(4.21), the functions $\ell_\delta(z)$ and $m_\delta(z)$ are the best one–sided L^1-approximations of $g_\delta(x)$ having exponential type at most 2π.

To construct a minorant of the Gaussian, we begin with the functions $m_\delta(z)$ and $\ell_\delta(z)$ defined by (4.17) and (4.18) and use Selberg's bootstrapping technique to obtain multidimensional minorants. The majorant is constructed by taking $m_\delta(z)$ tensored with itself N times.

The following proposition is due to Selberg, but it was never published. We call it *Selberg's bootstrapping method* because it enables us to construct a minorant for a tensor product of functions provided that we have majorants and minorants of each component at our disposal. This method has been used in [3, 27, 43, 44, 45].

Proposition 4.3.2. *Let $N > 0$ be natural number and $f_j : \mathbb{R} \to (0, \infty)$ be functions for every $j = 1, \ldots, N$. Let $l_j, m_j : \mathbb{R} \to \mathbb{R}$ be real–valued functions such that*

$$l_j(x) \le f_j(x) \le m_j(x)$$

for every real x and j. Then

$$-(N-1)\prod_{k=1}^{N} m_k(x_k) + \sum_{k=1}^{N} l_k(x_k)\prod_{\substack{j=1\\ j\neq k}}^{N} m_j(x_j) \le \prod_{k=1}^{N} f_k(x_k).$$

This proposition is easily deduced from the following inequality.

Lemma 4.3.3. *If $\beta_1, \ldots, \beta_N \ge 1$, then*

$$\sum_{k=1}^{N}\prod_{\substack{j=1\\ j\neq k}}^{N}\beta_j \le 1 + (N-1)\prod_{k=1}^{N}\beta_k.$$

Proof. We give a proof by induction, starting with the inductive step since the base case is simple. Suppose that the claim is true for $N = 1, \dots, L$. Let $\beta_1, \dots, \beta_L, \beta_{L+1}$ be a sequence of real numbers not less than one and write $\beta_j = 1 + \varepsilon_j$. We obtain

$$\begin{aligned}
\sum_{k=1}^{L+1} \prod_{\substack{j=1 \\ j \neq k}}^{L+1} \beta_j &= \prod_{j=1}^{L} \beta_j + (1 + \varepsilon_{L+1}) \sum_{k=1}^{L} \prod_{\substack{j=1 \\ j \neq k}}^{L} \beta_j \\
&\leq \prod_{j=1}^{L} \beta_j + (1 + \varepsilon_{L+1}) \left\{ 1 + (L-1) \prod_{j=1}^{L} \beta_j \right\} \\
&= 1 + \varepsilon_{L+1} + \prod_{j=1}^{L} \beta_j + (L-1) \prod_{j=1}^{L+1} \beta_j \\
&\leq 1 + \varepsilon_{L+1} \prod_{j=1}^{L} \beta_j + \prod_{j=1}^{L} \beta_j + (L-1) \prod_{j=1}^{L+1} \beta_j \\
&= 1 + L \prod_{j=1}^{L+1} \beta_j.
\end{aligned}$$

□

Now we can define our candidates for majorant and minorant of $G_{\boldsymbol{\lambda}}(\boldsymbol{x})$. For a given $\boldsymbol{\lambda} \in \mathbb{R}_+^N$ define the functions

$$\boldsymbol{z} \in \mathbb{C}^N \mapsto L_{\boldsymbol{\lambda}}(\boldsymbol{z}) = -(N-1) \prod_{j=1}^{N} m_{\lambda_j}(z_j) + \sum_{k=1}^{N} \ell_{\lambda_k}(z_k) \prod_{\substack{j=1 \\ j \neq k}}^{N} m_{\lambda_j}(z_j) \tag{4.22}$$

and

$$\boldsymbol{z} \in \mathbb{C}^N \mapsto M_{\boldsymbol{\lambda}}(\boldsymbol{z}) = \prod_{j=1}^{N} m_{\lambda_j}(z_j). \tag{4.23}$$

It follows from Proposition 4.3.2 and (4.19) that

$$L_{\boldsymbol{\lambda}}(\boldsymbol{x}) \leq G_{\boldsymbol{\lambda}}(\boldsymbol{x}) \leq M_{\lambda}(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathbb{R}^N. \tag{4.24}$$

Moreover, since $\ell_\delta(x)$ and $m_\delta(x)$ have exponential type at most 2π, we conclude that the Fourier transforms of $L_{\boldsymbol{\lambda}}(\boldsymbol{x})$ and $M_{\boldsymbol{\lambda}}(\boldsymbol{x})$ are supported on $Q(1)$. We modify $L_{\boldsymbol{\lambda}}(\boldsymbol{z})$ and $M_{\boldsymbol{\lambda}}(\boldsymbol{z})$ to have exponential type with respect to $Q(\boldsymbol{a})$ in the following way. Given $\boldsymbol{a}, \boldsymbol{\lambda} \in \mathbb{R}^N_+$ we define the functions

$$L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{z}) = L_{\boldsymbol{\lambda}/\boldsymbol{a}^2}(\boldsymbol{a}\boldsymbol{z}) \tag{4.25}$$

and

$$M_{\boldsymbol{\lambda},\boldsymbol{a}}(z) = M_{\boldsymbol{\lambda}/\boldsymbol{a}^2}(\boldsymbol{a}\boldsymbol{z}). \tag{4.26}$$

Here we use the (non-standard) notation $\boldsymbol{x}\boldsymbol{y} = (x_1y_1, \ldots, x_Ny_N)$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. We also write $\boldsymbol{x}/\boldsymbol{y} = (x_1/y_1, \ldots, x_N/y_N)$ if all the entries of $\boldsymbol{y}$ are nonzero.

By (4.24) we obtain

$$L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x}) \leq G_{\boldsymbol{\lambda}}(\boldsymbol{x}) \leq M_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x}) \text{ for all } \boldsymbol{x} \in \mathbb{R}^N$$

and using the scaling properties of the Fourier transform, we conclude that $L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})$ and $M_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})$ have exponential type with respect to $Q(\boldsymbol{a})$. By formula (4.18), we have $m_\delta(k) = g_\delta(k)$ for all integers k, hence we obtain

$$M_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{k}/\boldsymbol{a}) = G_{\boldsymbol{\lambda}}(\boldsymbol{k}/\boldsymbol{a})$$

for all $\boldsymbol{k} \in \mathbb{Z}^N$ (recall that $\boldsymbol{k}/\boldsymbol{a} = (k_1/a_1, \ldots, k_d/a_d)$).

We are now in a position to prove Theorems 4.1.1 and 4.1.2.

Proof of Theorem 4.1.1. It follows from (4.24) and (4.26) that the function $M_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})$ is majorant of $G_{\boldsymbol{\lambda}}(\boldsymbol{x})$ of exponential type with respect to $Q(\boldsymbol{a})$. Define $\alpha = a_1 \ldots a_N$. Using definition (4.23) and (4.26) we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} M_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \;=\; \alpha^{-1}\int_{\mathbb{R}^N} M_{\boldsymbol{\lambda}/a^2}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \;&=\; \alpha^{-1}\prod_{j=1}^N \int_{\mathbb{R}^N} m_{\lambda_j/a_j^2}(x_j)\mathrm{d}x_j \\ &=\; \prod_{j=1}^N \lambda_j^{-\frac{1}{2}}\Theta(0; ia_j^2/\lambda_j),\end{aligned}$$

where the first equality is due to a change of variables, the second one due to the product structure and the third one due to (4.20).

Now we will prove that (4.26) is extremal. Suppose that $F(z)$ is an entire majorant of $G_{\boldsymbol{\lambda}}(\boldsymbol{x})$ of exponential type with respect to $Q(\boldsymbol{a})$ and integrable on $\mathbb{R}^N$ (and therefore absolutely integrable on $\mathbb{R}^N$). We then have

$$\begin{aligned}\int_{\mathbb{R}^N} F(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \alpha^{-1}\sum_{k\in\mathbb{Z}^N} F(\boldsymbol{k}/\boldsymbol{a}) \geq \alpha^{-1}\sum_{k\in\mathbb{Z}^N} G_{\boldsymbol{\lambda}}(\boldsymbol{k}/\boldsymbol{a}) \\ = \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}}\Theta(0; ia_j^2/\lambda_j)\end{aligned}$$

because $M_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})$ majorizes $G_{\boldsymbol{\lambda}}(\boldsymbol{x})$, and the rightmost equality is given by (4.16). □

Proof of Theorem 4.1.2. Before we turn to the proof of Theorem 4.1.2 we need a technical lemma whose proof we postpone.

Lemma 4.3.4. *For all $t > 0$ we have*

$$1 - 4q/(1-q)^2 < \frac{\Theta(\frac{1}{2}; it)}{\Theta(0; it)} < e^{-2q}, \tag{4.27}$$

where $q = e^{-\pi t}$.

Suppose that $F(\boldsymbol{z})$ is an entire minorant of exponential type with respect to $Q(\boldsymbol{a})$ and absolutely integrable on $\mathbb{R}^N$. Denote $\boldsymbol{u} = (1, \ldots, 1)$ and $\alpha = a_1 \ldots a_N$. We can apply the Poisson summation formula to deduce that

$$\begin{aligned}\int_{\mathbb{R}^N} F(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \alpha^{-1}\sum_{\boldsymbol{k}\in\mathbb{Z}^N} F(\boldsymbol{k}/\boldsymbol{a} + \boldsymbol{u}/2\boldsymbol{a}) \;&\leq\; \alpha^{-1}\sum_{\boldsymbol{k}\in\mathbb{Z}^N} G_{\lambda}(\boldsymbol{k}/\boldsymbol{a} + \boldsymbol{u}/2\boldsymbol{a}) \\ &=\; \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}}\Theta(\tfrac{1}{2}; ia_j^2/\lambda_j),\end{aligned}$$

where the last equality is given by (4.16). This proves (4.1). By construction, $L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{z})$ is an entire minorant of exponential type with respect to $Q(\boldsymbol{a})$. Using definitions (4.22) and (4.25) we conclude that

$$\int_{\mathbb{R}^N} L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \left\{ \sum_{j=1}^{N} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (N-1) \right\} \prod_{j=1}^{N} \Theta(0; ia_j^2/\lambda_j)\lambda_j^{-\frac{1}{2}}.$$

Thus, to deduce (4.2), we only need to prove that

$$(1+5Ne^{-\pi\gamma})\left\{ \sum_{j=1}^{N} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (N-1) \right\} \geq \prod_{j=1}^{N} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} \tag{4.28}$$

for large γ (recall that $\gamma = \min\{\lambda_j/a_j^2\}$).

If we let $q_j = e^{-\pi a_j^2/\lambda_j}$ and γ sufficiently large such that $(1-e^{-\pi\gamma})^2 > 4/5$, we can use Lemma 4.3.4 to obtain

$$\sum_{j=1}^{N} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (N-1) \geq 1 - 4\sum_{j=1}^{N} q_j/(1-q_j)^2 \geq 1 - 5\sum_{j=1}^{N} q_j. \tag{4.29}$$

Applying Lemma 4.3.4 for a sufficiently large γ we obtain

$$\prod_{j=1}^{N} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} \leq \exp\left\{ -2\sum_{j=1}^{N} q_j \right\} \leq 1 - \sum_{j=1}^{N} q_j, \tag{4.30}$$

where the last inequality holds if $\sum_{j=1}^{N} q_j < \log 2$. Write $\beta = \sum_{j=1}^{N} q_j$ and note that

$$1-\beta \leq (1-5\beta)(1+5\beta)$$

if β is sufficiently small, for instance if $\beta \in [0, 1/25)$. Therefore, if γ is sufficiently large such that $\sum_{j=1}^{N} q_j < 1/25$ we obtain

$$1 - \sum_{j=1}^{N} q_j \leq \big(1 - 5\sum_{j=1}^{N} q_j\big)\big(1 + 5\sum_{j=1}^{N} q_j\big) < \big(1 - 5\sum_{j=1}^{N} q_j\big)(1+5Ne^{-\pi\gamma}). \tag{4.31}$$

By (4.29), (4.30) and (4.31) we conclude that there exists $\gamma_0 = \gamma_0(N) > 0$ such that if $\gamma \geq \gamma_0$ then (4.28) holds. This completes the proof. □

We now prove Lemma 4.3.4.

Proof of Lemma 4.3.4. Recall that $e(v) = e^{2\pi i v}$ and $q = e^{\pi i \tau}$. By [70, Chapter 10, Theorem 1.3] the theta function has the following product representation

$$\Theta(v;\tau) = \prod_{n=1}^{\infty} \big(1 - q^{2n}\big)\big(1 + q^{2n-1}e(v)\big)\big(1 + q^{2n-1}e(-v)\big). \tag{4.32}$$

It follows from (4.32) that

$$\frac{\Theta(\frac{1}{2}; it)}{\Theta(0; it)} = \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n-1}}{1 + q^{2n-1}}\right)^2 = \exp\left\{2\sum_{n=0}^{\infty} \log\left(1 - \frac{2q^{2n+1}}{1 + q^{2n+1}}\right)\right\}.$$

Using the inequality $\log(1 - x) \geq -x/(1 - x)$ for all $x \in [0, 1)$ we obtain

$$\begin{aligned}
\frac{\Theta(\frac{1}{2}; it)}{\Theta(0; it)} \geq \exp\left\{-4\sum_{n=0}^{\infty} \frac{q^{2n+1}}{1 - q^{2n+1}}\right\} \;&\geq\; \exp\left\{-\frac{4}{1-q}\sum_{n=0}^{\infty} q^{2n+1}\right\} \\
&=\; \exp\left\{-\frac{4q}{(1-q)(1-q^2)}\right\} \\
&>\; e^{-4q/(1-q)^2} \\
&>\; 1 - 4q/(1-q)^2
\end{aligned}$$

and this proves the left-hand side inequality in (4.27). The right-hand side inequality in (4.27) is deduced by a similar argument using the inequality $\log(1 - x) \leq -x$ for all $x \in [0, 1)$. $\square$

Proof of theorem 4.1.3. We follow the proof of [18, Theorem 14] proving only the majorant case, since the minorant case is nearly identical. Let

$$D(\boldsymbol{x}; \boldsymbol{\lambda}) = F(\boldsymbol{x}; \boldsymbol{\lambda}) - G(\boldsymbol{x}; \boldsymbol{\lambda}) \geq 0.$$

By condition (4.3) and Fubini's theorem the function

$$\mathcal{D}(\boldsymbol{x}) = \int_{\Lambda} D(\boldsymbol{x}; \boldsymbol{\lambda}) \mathrm{d}\mu(\boldsymbol{\lambda}) \geq 0$$

is defined for almost all $\boldsymbol{x} \in \mathbb{R}^N$ and $\mathcal{D} \in L^1(\mathbb{R}^N)$. The Fourier transform of $\mathcal{D}(\boldsymbol{x})$ is a continuous function given by

$$\widehat{\mathcal{D}}(\boldsymbol{\xi}) = \int_{\Lambda} \widehat{D}(\boldsymbol{\xi}; \boldsymbol{\lambda}) \mathrm{d}\mu(\boldsymbol{\lambda}),$$

and, due to (4.4), for almost every $\boldsymbol{\xi} \notin K$ we have the alternative representation

$$\widehat{\mathcal{D}}(\boldsymbol{\xi}) = -\int_{\Lambda} \widehat{G}(\boldsymbol{\xi}; \boldsymbol{\lambda}) \mathrm{d}\mu(\boldsymbol{\lambda}). \tag{4.33}$$

Let $\mathcal{M}$ be the tempered distribution given by

$$\mathcal{M}(\varphi) = \int_{\mathbb{R}^N} \{\mathcal{D}(\boldsymbol{x}) + \mathcal{G}(\boldsymbol{x})\} \varphi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

Now for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$ supported in K^c, we have by combining (4.5) and (4.33)

$$\widehat{\mathcal{M}}(\varphi) = \widehat{\mathcal{D}}(\varphi) + \widehat{\mathcal{G}}(\varphi) = 0.$$

Hence $\widehat{\mathcal{M}}$ is supported on K, in the distributional sense. By the Theorem 4.3.1, it follows that the distribution $\mathcal{M}$ is identified with an entire function $\mathcal{M} : \mathbb{C}^d \to \mathbb{C}$ of exponential type with respect to K and that

$$\mathcal{M}(\varphi) = \int_{\mathbb{R}^N} \mathcal{M}(\boldsymbol{x}) \varphi(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \tag{4.34}$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^N)$. It then follows from the definition of $\mathcal{M}$ and (4.34) that for almost every $\boldsymbol{x} \in \mathbb{R}^N$

$$\mathcal{M}(\boldsymbol{x}) = \mathcal{D}(\boldsymbol{x}) + \mathcal{G}(\boldsymbol{x}),$$

which implies $\mathcal{M}(\boldsymbol{x}) \geq \mathcal{G}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^N$ since $\mathcal{G}(\boldsymbol{x})$ is continuous, and

$$\int_{\mathbb{R}^N} \{\mathcal{M}(\boldsymbol{x}) - \mathcal{G}(\boldsymbol{x})\} \mathrm{d}\boldsymbol{x} = \int_{\Lambda} \int_{\mathbb{R}^N} \{F(\boldsymbol{x}; \boldsymbol{\lambda}) - G(\boldsymbol{x}; \boldsymbol{\lambda})\} \mathrm{d}\boldsymbol{x} \mathrm{d}\mu(\boldsymbol{\lambda}) < \infty.$$

□

4.4 Proofs of Theorems 4.2.1, 4.2.2 and 4.2.3

4.4.1 Preliminary Results

In this section we recall the crucial results needed to demonstrate our main results as well as some basic facts about the theory of Paley–Wiener spaces and extremal functions.

For a given function $F : \mathbb{R}^N \to \mathbb{R}$ we define its Fourier transform as

$$\widehat{F}(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} F(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{x}.$$

In this part we will almost always deal with functions $F(\boldsymbol{x})$ that are integrable and whose Fourier transforms are supported in the box

$$Q_N = [-1, 1]^N.$$

For this reason, given a $p \in [1, 2]$ we define $PW^p(Q_N)$ as the set of functions $F \in L^p(\mathbb{R}^N)$ such that their Fourier transform is supported in Q_N. By Fourier inversion these functions can be identified with analytic functions that extend to $\mathbb{C}^N$ as entire functions.

The following is a special case of Stein's generalization of the Paley–Wiener theorem (see [71]).

Theorem 4.4.1 (Stein). *Let $p \in [1, 2]$ and $F \in L^p(\mathbb{R}^N)$. The following statements are equivalent:*

(i) $F \in PW^p(Q_N)$.

(ii) $F(\boldsymbol{x})$ *is the restriction to* $\mathbb{R}^N$ *of an entire function defined in* $\mathbb{C}^N$ *with the property that there exists a constant* $C > 0$ *such that*

$$|F(\boldsymbol{x} + i\boldsymbol{y})| \le C \exp\left[2\pi \sum_{n=1}^{N} |y_n|\right]$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$.

Remark. In particular the theorem implies that $PW^1(Q_N) \subset PW^2(Q_N)$.

The Pólya-Plancharel Theorem [65], states that if $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$ is a sequence in $\mathbb{R}^N$ satisfying that $|\boldsymbol{\xi}_n - \boldsymbol{\xi}_m|_{\ell^\infty} \geq \varepsilon$ for all $m \neq n$ for some $\varepsilon > 0$ then

$$\sum_n |F(\boldsymbol{\xi}_n)|^p \leq C(p,\varepsilon) \int_{\mathbb{R}^N} |F(\boldsymbol{\xi})|^p \mathrm{d}\boldsymbol{\xi}$$

for every $F \in PW^p(Q_N)$. The Poisson summation formula states that for all $F \in PW^1(Q_N)$ we have

$$\int_{\mathbb{R}^N} F(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} F(\boldsymbol{n}). \tag{4.35}$$

Let $F \in PW^2(Q_N)$. If $\boldsymbol{t} \in \mathbb{C}^{N-k}$, then the function $\boldsymbol{y} \in \mathbb{R}^k \mapsto G_{\boldsymbol{t}}(\boldsymbol{y}) = F(\boldsymbol{y}, \mathbf{t})$ is the inverse Fourier transform of the following function

$$\boldsymbol{\xi} \in \mathbb{R}^k \mapsto \int_{Q_{N-k}} \widehat{F}(\boldsymbol{\xi}, \boldsymbol{u}) e^{2\pi i \boldsymbol{t} \cdot \boldsymbol{u}} \mathrm{d}\boldsymbol{u}.$$

Since $\widehat{F} \in L^2(\mathbb{R}^N)$, we conclude that the above function has finite $L^2(\mathbb{R}^k)$–norm and as a consequence $G_{\boldsymbol{t}} \in PW^2(Q_k)$. A similar result is valid for $p = 1$ but only for $\nu(N)$–admissible functions.

Lemma 4.4.2. *Let $N > k > 0$ be integers. If $F(\boldsymbol{x})$ is $\nu(N)$–admissible then the function $\boldsymbol{y} \in \mathbb{R}^k \mapsto F(\boldsymbol{y}, \mathbf{0})$ with $\mathbf{0} \in \mathbb{R}^{N-k}$ is $\nu(k)$-admissible and*

$$\int_{\mathbb{R}^N} F(\boldsymbol{x}) d\boldsymbol{x} \leq \int_{\mathbb{R}^k} F(\boldsymbol{y}, \mathbf{0}) \mathrm{d}\boldsymbol{y}.$$

Proof. We give a proof only for the case $N = 2$ since it will be clear that the general case follows by an adaption of the following argument.

Let $F(x, y)$ be a function admissible for $\nu(2)$ and define $G(x) = F(x, 0)$. Clearly $G(x)$ is a minorant of $\mathbf{1}_{Q_1}(x)$. By Fourier inversion we obtain that

$$G(x) = \int_{-1}^{1} \left(\int_{-1}^{1} \widehat{F}(s,t) \mathrm{d}t \right) e^{2\pi i s x} \mathrm{d}s.$$

This shows that $G \in PW^2(Q_1)$. Now, for every $a \in (0,1)$ define the functions

$$G_a(x) = G((1-a)x)\left(\frac{\sin(a\pi x)}{a\pi x}\right)^2$$

and

$$F_a(x,y) = F((1-a)x, y)\left(\frac{\sin(a\pi x)}{a\pi x}\right)^2.$$

By an application of Holder's inequality and Theorem 4.4.1, we deduce that $G_a \in PW^1(Q_1)$ and $F_a \in PW^1(Q_2)$ for all $a \in (0,1)$. Hence, we can apply Poisson summation to conclude that

$$\begin{aligned}\int_{\mathbb{R}} G_a(x)\mathrm{d}x &= \sum_{n\in\mathbb{Z}} G((1-a)n)\left(\frac{\sin(a\pi n)}{a\pi n}\right)^2 \\ &\geq \sum_{(n,m)\in\mathbb{Z}^2} F((1-a)n, m)\left(\frac{\sin(a\pi n)}{a\pi n}\right)^2 \\ &= \int_{\mathbb{R}^2} F((1-a)x, y)\left(\frac{\sin(a\pi x)}{a\pi x}\right)^2 \mathrm{d}x\mathrm{d}y,\end{aligned}$$

where the above inequality is valid because the function $F(x,y)$ is a minorant of the box Q_2. Observing that $G_a(x) \leq \mathbf{1}_{Q_{1/(1-a)}}(x)$ for every $x \in \mathbb{R}$, we can apply Fatou's lemma to conclude that

$$\begin{aligned}&\int_{\mathbb{R}} [\mathbf{1}_{Q_1}(x) - G(x)]\mathrm{d}x \\ &\qquad \leq \liminf_{a\to 0} \int_{\mathbb{R}} [\mathbf{1}_{Q_{1/(1-a)}}(x) - G_a(x)]\mathrm{d}x \\ &\qquad \leq \int_{\mathbb{R}} \mathbf{1}_{Q_1}(x)\mathrm{d}x - \limsup_{a\to 0} \int_{\mathbb{R}^2} F((1-a)x, y)\left(\frac{\sin(a\pi x)}{a\pi x}\right)^2 \mathrm{d}x\mathrm{d}y \\ &\qquad = \int_{\mathbb{R}} \mathbf{1}_{Q_1}(x)\mathrm{d}x - \int_{\mathbb{R}^2} F(x,y)\mathrm{d}x\mathrm{d}y < \infty.\end{aligned}$$

This concludes the proof. $\square$

We now introduce an interpolation theorem which has proven indispensable throughout our investigations.

Theorem 4.4.3. *For all* $F \in PW^2(Q_N)$ *we have*

$$F(\boldsymbol{x}) = \prod_{n=1}^{N} \left\{ \frac{\sin \pi x_n}{\pi} \right\}^2 \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \sum_{\boldsymbol{j} \in \{0,1\}^N} \frac{\partial_{\boldsymbol{j}} F(\boldsymbol{n})}{(\boldsymbol{x} - \boldsymbol{n})^{\boldsymbol{2}-\boldsymbol{j}}} \tag{4.36}$$

where $\partial_{\boldsymbol{j}} = \partial_{j_1 \dots j_N}$ *and* $(\boldsymbol{x} - \boldsymbol{n})^{\boldsymbol{2}-\boldsymbol{j}} = (x_1 - n_1)^{2-j_1} \dots (x_N - n_N)^{2-j_N}$ *and the right–hand side of* (4.36) *converges uniformly in compact subsets of* $\mathbb{R}^N$.

Proof (*by induction*). The base case $N = 1$ was established by Vaaler [73]. Suppose the theorem is valid for $N = 1, 2, \dots, L-1$, we need to show that it is valid for $N = L$. Suppose $F \in L^2(\mathbb{R}^L)$ and that $\widehat{F}(\boldsymbol{\xi})$ is supported in Q_L. Write $F(\boldsymbol{x}, x_L)$ where $\boldsymbol{x} \in \mathbb{R}^{L-1}$ and $x_L \in \mathbb{R}$. By Lemma 4.4.2 and the inductive hypothesis

$$F(\boldsymbol{x}, x_L) = \prod_{n=1}^{L-1} \left\{ \frac{\sin \pi x_n}{\pi} \right\}^2 \sum_{\boldsymbol{n} \in \mathbb{Z}^{L-1}} \sum_{\boldsymbol{j} \in \{0,1\}^{L-1}} \frac{\partial_{\boldsymbol{j}} F(\boldsymbol{n}, x_L)}{(\boldsymbol{x} - \boldsymbol{n})^{\boldsymbol{2}-\boldsymbol{j}}}. \tag{4.37}$$

By Fourier inversion $(\boldsymbol{x}, t) \mapsto \partial_{\boldsymbol{j}} F(\boldsymbol{x}, t)$ is in $L^2(\mathbb{R}^L)$ and its Fourier transform is supported in Q_L. Thus, by another application of Lemma 4.4.2 and the inductive hypothesis (but this time with respect to $\boldsymbol{t} = x_L$ and $N = 1$, respectively) we have for each $\boldsymbol{n} \in \mathbb{Z}^{L-1}$

$$\partial_{\boldsymbol{j}} F(\boldsymbol{n}, x_L) = \left\{ \frac{\sin \pi x_L}{\pi} \right\}^2 \sum_{n_L \in \mathbb{Z}} \sum_{j \in \{0,1\}} \frac{\partial_{\boldsymbol{j}} \partial_{x_L}^{j} F(\boldsymbol{n}, n_L)}{(x_L - n_L)^{2-j}}. \tag{4.38}$$

Combining (4.37) with (4.38) yields (4.36), completing the induction.

The theorem of Pólya-Plancherel [65] guarantees that the sequence $\{\partial_{\boldsymbol{j}} F(\boldsymbol{n}) : \boldsymbol{n} \in \mathbb{Z}^N\}$ is square summable for every $\boldsymbol{j} \in \mathbb{Z}_+^N$. Therefore the sum on the right–hand side of (4.36) converges uniformly in compact subsets of $\mathbb{R}^N$. □

Finally, the next lemma demonstrates that extremal functions always exist for $\nu(N)$.

Lemma 4.4.4. *Suppose $G \in L^1(\mathbb{R}^N)$ is a real valued function. Let $\{F_\ell(\boldsymbol{x})\}_\ell$ be a sequence in $PW^1(Q_N)$ such that $F_\ell(\boldsymbol{x}) \leq G(\boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{R}^N$, and there exists $A > 0$ such that $\widehat{F}_\ell(\mathbf{0}) \geq -A$ for each ℓ. Then there exists a subsequence $F_{\ell_k}(\boldsymbol{x})$ and a function $F \in PW^1(Q_N)$ such that $F_{\ell_k}(\boldsymbol{x})$ converges to $F(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^N$ as k tends to infinity. In particular, we deduce that $F(\boldsymbol{x}) \leq G(\boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{R}^N$ and $\limsup_{k\to\infty} \widehat{F}_{\ell_k}(\mathbf{0}) \leq \widehat{F}(\mathbf{0})$.*

Proof. By the remark after Theorem 4.4.1 each $F_k \in PW^2(Q_N)$ and we can bound their $L^2(\mathbb{R}^N)$- norms in the following way. Observe that

$$\|F_k\|_2 = \|\widehat{F}_k\|_2 \leq \mathrm{vol}_N(Q_N)^{1/2}\|\widehat{F}_k\|_\infty \leq 2^{N/2}\|F_k\|_1$$

and

$$\|F_k\|_1 \leq \|G - F_k\|_1 + \|G\|_1 = \int_{\mathbb{R}^N} (G(\boldsymbol{x}) - F_k(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} + \|G\|_1 \leq 2\|G\|_1 + A.$$

Hence the sequence $F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \ldots$ is uniformly bounded in $L^2(\mathbb{R}^N)$ and, by the Banach-Alaoglu Theorem, we may extract a subsequence (that we still denote by F_k) that converges weakly to a function $F \in PW^2(Q_N)$. By Theorem 4.4.1 we can assume that $F(\boldsymbol{x})$ is continuous. Also, by using Fourier inversion in conjunction with weak convergence we deduce that $F_k(\boldsymbol{x})$ converges to $F(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^N$. We conclude that $G(\boldsymbol{x}) \geq F(\boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{R}^N$. By applying Fatou's lemma to the sequence of functions $G(\boldsymbol{x}) - F_1(\boldsymbol{x}), G(\boldsymbol{x}) - F_2(\boldsymbol{x}), \ldots$ we find that $F \in L^1(\mathbb{R}^N)$ and

$$\limsup_{k\to\infty} \int_{\mathbb{R}^N} F_k(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{R}^N} F(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

This concludes the lemma. □

Corollary 4.4.5. *There are $\nu(N)$–admissible functions which verify the equality in (4.11). That is, extremal functions for $\nu(N)$ exist.*

The next theorem is the cornerstone in the proof of our main results. This theorem is in stark contrast with the one dimensional case. In the one dimensional case, Selberg's function interpolates at all lattice points, and is therefore extremal. In two dimensions, on the other hand, if a minorant interpolates everywhere except for possibly the origin, then it is identically zero. This theorem is therefore troublesome because it seems to disallow the possibility of using interpolation (in conjunction with Poisson summation) to prove an extremality result.

Theorem 4.4.6. *If $F(x, y)$ is admissible for $\nu(2)$ and $F(0,0) = \widehat{F}(0,0) \geq 0$ then $F(x, y)$ vanishes identically.*

Proof. **Step 1.** First we assume that the function $F(x, y)$ is invariant under the symmetries of the square, that is,

$$F(x, y) = F(y, x) = F(|x|, |y|) \tag{4.39}$$

for all $x, y \in \mathbb{R}$. We claim that for any $(m, n) \in \mathbb{Z}^2$ we have:

(a) $\partial_x F(m, n) = 0$ if $(m, n) \neq (\pm 1, 0)$ and $\partial_y F(m, n) = 0$ if $(m, n) \neq (0, \pm 1)$,

(b) $\partial_{xx} F(m, n) = 0$ if $n \neq 0$ and $\partial_{yy} F(m, n) = 0$ if $m \neq 0$,

(c) $\partial_{xy} F(m, n) = 0$ if $n \neq \pm 1$ or $m \neq \pm 1$.

First notice that the identity $F(0,0) = \widehat{F}(0,0)$ is equivalent, by Poisson summation, to $F(m, n) = 0$ for all nonzero pairs $(m, n) \in \mathbb{Z}^2$. Thus, we can apply Theorem 4.4.2 to deduce that, for each fixed nonzero integer m, the function $x \in \mathbb{R} \mapsto F(m, x)$ is a non–positive function belonging to $PW^1(Q_1)$ that vanishes in the integers, hence identically zero by formula (4.36). Also note that the points $(m, 0)$ for $m \in \mathbb{Z}$ with $|m| > 1$ are local maxima of the function $x \in \mathbb{R} \mapsto F(x,0)$. These facts in conjunction with the invariance property (4.39) imply items (a) and (b).

Finally, note that a point (m,n) with $|n| > 1$ has to be a local maximum of the function $F(x,y)$. Thus, the Jacobian determinant of $F(x,y)$ at such a point has to be non-negative. That is,

$$J_F(m,n) := \partial_{xx}F(m,n)\partial_{yy}F(m,n) - [\partial_{xy}F(m,n)]^2 \geq 0.$$

However, by item (b), $\partial_{xx}F(m,n) = 0$ and we conclude that $\partial_{xy}F(m,n) = 0$. This proves item (c) after using again the property (4.39).

Step 2. We can now apply formula (4.36) and deduce that $F(x,y)$ has to have the following form

$$F(x,y) = \left(\frac{\sin(\pi x)\sin(\pi y)}{\pi^2 xy}\right)^2 \left\{F(0,0) - \frac{ax^2}{x^2-1} - \frac{ay^2}{y^2-1} - \frac{bx^2y^2}{(x^2-1)(y^2-1)}\right\},$$

where $a = -2\partial_x F(1,0)$ and $b = -4\partial_{xy}F(1,1)$. Denote by $B(x,y)$ the expression in the brackets above and note that it should be non–positive if $|x| \geq 1$ or $|y| \geq 1$. We deduce that

$$F(0,0) - a - (a+b)\frac{x^2}{x^2-1} = B(x,\infty) \leq 0$$

for all real x. We conclude that $a + b = 0$, $F(0,0) \leq a$ and

$$B(x,y) = F(0,0) - a\left[1 - \frac{1}{(x^2-1)(y^2-1)}\right].$$

For each $t > 0$, the set of points $(x,y) \in \mathbb{R}^2 \backslash Q_2$ such that $(x^2-1)(y^2-1) = 1/t$ is non-empty and $B(x,y) = F(0,0) - a + at$ at such a point. Therefore $a \leq 0$ and we deduce that $F(0,0) \leq 0$. We conclude that $F(0,0) = 0$, which in turn implies that $a = 0$. Thus $F(x,y)$ vanishes identically.

Step 3. We now finish the proof. Let $F(x,y)$ be a $\nu(2)$–admissible function such that $F(0,0) = \widehat{F}(0,0) \geq 0$. Define the function

$$G_1(x,y) = \frac{F(x,y) + F(-x,y) + F(x,-y) + F(-x,-y)}{4}.$$

Clearly, the following function

$$G_0(x,y) = \frac{G_1(x,y) + G_1(y,x)}{2}$$

is also $\nu(2)$–admissible and $G_0(0,0) = \widehat{G}_0(0,0) \geq 0$. Moreover, $G_0(x,y)$ satisfies the symmetry property (4.39). By steps 1 and 2 the function $G_0(x,y)$ must vanish identically. Thus, we obtain that

$$G_1(x,y) = -G_1(y,x).$$

However, since $G_1(x,y)$ is also $\nu(2)$–admissible we conclude that $G_1(x,y)$ is identically zero outside the box Q_2, hence it vanishes identically. An analogous argument can be applied to the function $G_2(x,y) = [F(x,y) + F(-x,y)]/2$ to conclude that this function is identically zero outside the box Q_2, hence it vanishes identically. Using the same procedure again we finally conclude that $F(x,y)$ vanishes identically and the proof of the theorem is complete. □

Proof of Theorem 4.2.2. The theorem is proven by contradiction. Assume, by contradiction, that for some $\varepsilon > 0$ there exists a sequence $F_{N_k}(\boldsymbol{x})$ of $\nu(N_k)$–admissible functions such that $\widehat{F}_{N_k}(\mathbf{0}) \geq 0$ and $F_{N_k}(\mathbf{0}) \geq \varepsilon$ for all integers N_k. We can assume (by making a mean of $F_{N_k}(\boldsymbol{x})$ over the set of symmetries of the box Q_{N_k} if necessary) that $F_{N_k}(\boldsymbol{x})$ is invariant under the symmetries of Q_{N_k} for each N_k, that is, $F_{N_k}(\boldsymbol{x})$ is a symmetric function which is also even in each variable.

Define the slicing functions

$$G_{N_k}(x,y) = F_{N_k}(x,y,\mathbf{0}_{N_k-2}),$$

for $(x, y) \in \mathbb{R}^2$. By Lemma 4.4.2, the functions $G_{N_k}(x, y)$ are admissible for $\nu(2)$ and $\widehat{G}_{N_k}(0,0) \geq \widehat{F}_{N_k}(\mathbf{0}) \geq 0$. Also $G_{N_k}(0,0) = F_{N_k}(\mathbf{0}) \geq \varepsilon$ for all N_k. We can now apply Lemma 4.4.4 to conclude that (by taking a further subsequence of the N_k's if necessary) there exists a function $G(x, y)$ which is admissible for $\nu(2)$ and such that

$$\lim_{k\to\infty} G_{N_k}(x, y) = G(x, y)$$

for all $(x, y) \in \mathbb{R}^2$.

Now, since $F_{N_k}(\boldsymbol{x})$ is invariant under the symmetries of Q_{N_k}, we can apply Poisson summation to obtain that for every nonzero $(m, n) \in \mathbb{Z}^2$ we have

$$\widehat{F}_{N_k}(\mathbf{0}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^{N_k}} F_{N_k}(\boldsymbol{n}) \leq 1 + \#\left\{\sigma(m, n, \mathbf{0}_{N_k-2}) : \sigma \in \mathrm{Sym}(Q_{N_k})\right\} G_{N_k}(m, n),$$

where $\mathrm{Sym}(Q_N)$ is the symmetry group of Q_N. Since $G_{N_k}(m, n) \leq 0$ for a nonzero $(m, n) \in \mathbb{Z}^2$ and

$$\#\left\{\sigma(m, n, \mathbf{0}_{N-2}) : \sigma \in \mathrm{Sym}(Q_N)\right\} \to \infty$$

as $N \to \infty$, we conclude that $G_{N_k}(m, n) \to 0$ for each nonzero $(m, n) \in \mathbb{Z}^2$. We deduce that $G(m, n) = 0$ for each nonzero $(m, n) \in \mathbb{Z}^2$ and, by Theorem 4.4.6, we conclude that the function $G(x, y)$ vanishes identically. We have a contradiction, because $G(0, 0) \geq \varepsilon$. This completes the proof of the theorem. □

Proof of Theorem 4.2.1. Suppose that $\nu(N) = \nu(N+1)$. Let $(\boldsymbol{x}, t) \in \mathbb{R}^N \times \mathbb{R} \mapsto F(\boldsymbol{x}, t)$ be an extremal function for $\nu(N+1)$. Let $G_m(\boldsymbol{x}) = F(\boldsymbol{x}, m)$ for each $m \in \mathbb{Z}$. Lemma 4.4.2 implies that $G_m(\boldsymbol{x})$ is also admissible for $\nu(N)$. By the Poisson summation formula we have for each nonzero $m \in \mathbb{Z}$

$$\widehat{F}(\mathbf{0}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \sum_{k\in\mathbb{Z}} F(\boldsymbol{n}, k) \leq \sum_{\boldsymbol{n}\in\mathbb{Z}^N} (F(\boldsymbol{n}, m) + F(\boldsymbol{n}, 0)) = \widehat{G}_m(\mathbf{0}) + \widehat{G}_0(\mathbf{0}). \quad (4.40)$$

By assumption

$$\widehat{G}_0(\mathbf{0}) \le \nu(N) = \nu(N+1) = \widehat{F}(\mathbf{0}) \tag{4.41}$$

Combining (4.40) and (4.41) yields $0 \le \widehat{G}_m(\mathbf{0})$ for each $m \neq 0$. However, $G_m(\boldsymbol{x}) \le 0$ for each $\boldsymbol{x} \in \mathbb{R}^N$ whenever m is a nonzero integer. Consequently, $G_m(\boldsymbol{x})$ vanishes identically. It follows that $F(\boldsymbol{n}) = 0$ for each nonzero $\boldsymbol{n} \in \mathbb{Z}^{N+1}$. By Theorem 4.4.6, $F(\mathbf{0}) = \widehat{F}(\mathbf{0}) = 0$. Therefore $\nu(N+1) = \nu(N) = 0$. This proves item (ii). Item (i) is a direct consequence of item (ii) since $\nu(1) = 1$. Item (iii) is a consequence of Theorem 4.2.2 and the fundamental inequality (4.14). □

4.5 Further Results

4.5.1 Periodic Functions Subordinated to Theta Functions

In this subsection we find the best approximations by trigonometric polynomials for functions that are, in some sense, subordinated to theta functions. The proofs of the theorems in this section are almost identical to the proofs of the Section 4.3, and thus we state the theorems without proof. for details see [37].

Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}_+^N$ (that is, $a_j \ge 1\ \forall j$). We will say that the degree of a trigonometric polynomial $P(\boldsymbol{x})$ is less than $\boldsymbol{a}$ (degree $P < \boldsymbol{a}$) if

$$P(\boldsymbol{x}) = \sum_{-\boldsymbol{a}<\boldsymbol{n}<\boldsymbol{a}} \widehat{P}(\boldsymbol{n}) e(\boldsymbol{n}\cdot\boldsymbol{x}).$$

Here we use the the notation $\boldsymbol{x} < \boldsymbol{y}$ to say that $x_j < y_j$ for every $j = 1, \ldots, N$. The problems we are interested to solve have the following general form.

Periodic Majorization Problem. Fix an $\boldsymbol{a} \in \mathbb{Z}_+^N$ (called the *degree*) and a Lebesgue measurable real periodic function $g : \mathbb{T}^N \to \mathbb{R}$. Determine the value of

$$\inf \int_{\mathbb{T}^N} |F(\boldsymbol{x}) - g(\boldsymbol{x})| \mathrm{d}\boldsymbol{x},$$

where the infimum is taken over functions $F : \mathbb{T}^N \to \mathbb{R}$ satisfing:

(*i*) $F(\boldsymbol{x})$ is a real trigonometric polynomial.

(*ii*) Degree of $F(\boldsymbol{x})$ is less than $\boldsymbol{a}$.

(*iii*) $F(\boldsymbol{x}) \geq g(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbb{T}^N$.

If the infimum is achieved, then identify the extremal functions $F(\boldsymbol{x})$. Similarly, we consider the minorant problem.

Periodic Minorization Problem. Solve the previous problem with condition (*iii*) replaced by the condition

$$(iii')\ F(\boldsymbol{x}) \leq g(\boldsymbol{x}) \text{ for every } \boldsymbol{x} \in \mathbb{T}^N.$$

Now we define for every $\boldsymbol{\lambda} \in \mathbb{R}^N_+$, the periodization of the Gaussian function $G_{\boldsymbol{\lambda}}(\boldsymbol{x})$ by

$$f_{\boldsymbol{\lambda}}(\boldsymbol{x}) := \sum_{n \in \mathbb{Z}^N} G_{\boldsymbol{\lambda}}(x+n) = \sum_{n \in \mathbb{Z}^N} \prod_{j=1}^{N} e^{-\boldsymbol{\lambda}_j \pi (x_j + n_j)^2} = \prod_{j=1}^{N} \Theta(x_j; i/\lambda_j) \lambda_j^{-\frac{1}{2}}.$$

Theorem 4.5.1 (Existence)**.** *For a given* $\boldsymbol{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^N_+$*, let* Λ *be a measurable space of parameters, and for each* $\boldsymbol{\lambda} \in \Lambda$*, let* $R_{\boldsymbol{\lambda}}(\boldsymbol{x})$ *be a real trigonometric polynomial with degree less than* $\boldsymbol{a}$*. Let* $\mathrm{d}\mu$ *be a non–negative measure in* Λ *that satisfies*

$$\int_{\Lambda} \int_{\mathbb{T}^N} |R_{\boldsymbol{\lambda}}(\boldsymbol{x}) - f_{\boldsymbol{\lambda}}(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} \mathrm{d}\mu(\boldsymbol{\lambda}) < \infty. \tag{4.42}$$

Suppose that $g : \mathbb{T}^N \to \mathbb{R}$ *is a continuous periodic function such that*

$$\widehat{g}(\boldsymbol{k}) = \int_{\Lambda} \widehat{G}_{\boldsymbol{\lambda}}(\boldsymbol{k}) \mathrm{d}\mu(\boldsymbol{\lambda}),$$

for all $\boldsymbol{k} \in \mathbb{Z}^N$ *such that* $|k_j| \geq a_j$ *for some* $j \in \{1, 2, \ldots, N\}$*. Then:*

(i) *If* $f_{\boldsymbol{\lambda}}(\boldsymbol{x}) \leq R_{\boldsymbol{\lambda}}(\boldsymbol{x})$ *for each* $x \in \mathbb{T}^N$ *and* $\boldsymbol{\lambda} \in \Lambda$*, then there exists a trigonometric polynomial* $m_{\boldsymbol{a}}(\boldsymbol{x})$ *with degree* $m_{\boldsymbol{a}} < \boldsymbol{a}$*, such that of* $m_a(\boldsymbol{x}) \geq g(\boldsymbol{x})$ *for all* $x \in \mathbb{T}^N$ *and*

$$\int_{\mathbb{T}^N} \{m_{\boldsymbol{a}}(\boldsymbol{x}) - g(\boldsymbol{x})\} \mathrm{d}\boldsymbol{x}$$

is equal to the left-hand side of (4.42)*.*

(ii) *If* $R_{\boldsymbol{\lambda}}(\boldsymbol{x}) \leq f_{\boldsymbol{\lambda}}(\boldsymbol{x})$ *for each* $\boldsymbol{x} \in \mathbb{T}^N$ *and* $\boldsymbol{\lambda} \in \Lambda$*, then there exists a trigonometric polynomial* $\ell_{\boldsymbol{a}}(\boldsymbol{x})$ *with degree* $\ell_{\boldsymbol{a}} < \boldsymbol{a}$*, such that* $\ell_{\boldsymbol{a}}(\boldsymbol{x}) \leq g(\boldsymbol{x})$ *in* $\mathbb{T}^N$*, and*

$$\int_{\mathbb{T}^N} \{g(\boldsymbol{x}) - \ell_{\boldsymbol{a}}(\boldsymbol{x})\} \mathrm{d}\boldsymbol{x}$$

is equal to the left-hand side of (4.42)*.*

Before we state the main theorems of this section we need some definitions. The functions $M_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})$ and $L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x})$, defined in (4.23) and (4.22), belong to $L^1(\mathbb{R}^N)$, thus, by the Plancharel–Pólya theorem (see [65]) and the periodic Fourier inversion formula, their respective periodizations are trigonometric polynomials of degree less than $\boldsymbol{a}$, that is

$$m_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x}) := \sum_{\boldsymbol{n} \in \mathbb{Z}^N} M_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x} + \boldsymbol{n}) = \sum_{-\boldsymbol{a} < \boldsymbol{n} < \boldsymbol{a}} \widehat{M}_{\boldsymbol{\lambda},a}(\boldsymbol{n}) e^{2\pi i \boldsymbol{n} \cdot \boldsymbol{x}}$$

and

$$\ell_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x}) := \sum_{\boldsymbol{n} \in \mathbb{Z}^N} L_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{x} + \boldsymbol{n}) = \sum_{-\boldsymbol{a} < \boldsymbol{n} < \boldsymbol{a}} \widehat{L}_{\boldsymbol{\lambda},\boldsymbol{a}}(\boldsymbol{n}) e^{2\pi i \boldsymbol{n} \cdot \boldsymbol{x}}$$

hold for each $\boldsymbol{x} \in \mathbb{T}^N$. The following theorem offers a resolution to the Majorization Problem for a specific class of functions.

Theorem 4.5.2 (Gaussian Subordination – Periodic Majorant)**.** *Let* $\boldsymbol{a} \in \mathbb{Z}_+^N$ *and* $\mathrm{d}\mu$ *be non-negative Borel measure on* $\mathbb{R}_+^N$ *that satisfies*

$$\int_{\mathbb{R}_+^N} \prod_{j=1}^{N} \lambda_k^{-\frac{1}{2}} \left\{ \prod_{j=1}^{N} \Theta(0; i a_j^2/\lambda_j) - 1 \right\} \mathrm{d}\mu(\boldsymbol{\lambda}) < \infty.$$

Let $g : \mathbb{T}^N \to \mathbb{R}$ be a continuous periodic function such that

$$\widehat{g}(\boldsymbol{k}) = \int_{\mathbb{R}^N_+} \widehat{G}_{\boldsymbol{\lambda}}(\boldsymbol{k}) \mathrm{d}\mu(\boldsymbol{\lambda}),$$

for all $\boldsymbol{k} \in \mathbb{Z}^N$ such that $|k_j| \geq a_j$ for some $j \in \{1, 2, \ldots, N\}$. Then for every real trigonometric polynomial $P(\boldsymbol{x})$, with degree $P < \boldsymbol{a}$ and $P(\boldsymbol{x}) \geq g(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{T}^N$, we have

$$\int_{\mathbb{T}^N} \{P(\boldsymbol{x}) - g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}\} \geq \int_{\mathbb{R}^N_+} \prod_{j=1}^{N} \lambda_k^{-\frac{1}{2}} \left\{ \prod_{j=1}^{N} \Theta(0; ia_j^2/\lambda_j) - 1 \right\} \mathrm{d}\mu(\boldsymbol{\lambda}). \quad (4.43)$$

Moreover, there exists a real trigonometric polynomial $m_{\boldsymbol{a}}$, with degree $m_{\boldsymbol{a}} < \boldsymbol{a}$, such that $m_{\boldsymbol{a}}(\boldsymbol{x})$ is a majorant of $g(\boldsymbol{x})$ that interpolates $g(\boldsymbol{x})$ on the lattice $\mathbb{Z}^N/\boldsymbol{a}$ and equality at (4.43) holds.

Recall that $\mathbb{Z}^N/\boldsymbol{a} = \mathbb{Z}/a_1 \times \ldots \times \mathbb{Z}/a_N$.

Theorem 4.5.3 (Gaussian Subordination – Periodic Minorant)**.** *Let $\boldsymbol{a} \in \mathbb{Z}^N_+$ and $\mathrm{d}\mu$ be non-negative Borel measure on $\mathbb{R}^N_+$ such that*

$$\int_{\mathbb{R}^N_+} \left\{ 1 - \left\{ \sum_{j=1}^{N} \frac{\Theta(\frac{1}{2}; ia_j^2/\lambda_j)}{\Theta(0; ia_j^2/\lambda_j)} - (N-1) \right\} \prod_{j=1}^{N} \Theta(0; ia_j^2/\lambda_j) \right\} \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} \mathrm{d}\mu(\boldsymbol{\lambda}) < \infty. \quad (4.44)$$

Let $g : \mathbb{T}^N \to \mathbb{R}$ be continuous periodic function such that

$$\widehat{g}(\boldsymbol{k}) = \int_{\mathbb{R}^N_+} \widehat{G}_{\boldsymbol{\lambda}}(\boldsymbol{k}) \mathrm{d}\mu(\boldsymbol{\lambda})$$

for all $\boldsymbol{k} \in \mathbb{Z}^N$ such that $|k_j| \geq a_j$ for some $j \in \{1, 2, \ldots, N\}$. Then, if $P(\boldsymbol{x})$ is a real trigonometric polynomial with degree less than $\boldsymbol{a}$ that minorizes $g(\boldsymbol{x})$, we have

$$\int_{\mathbb{T}^N} \{g(\boldsymbol{x}) - P(\boldsymbol{x})\} \mathrm{d}\boldsymbol{x} \geq \int_{\mathbb{R}^N_+} \prod_{j=1}^{N} \lambda_k^{-\frac{1}{2}} \left\{ 1 - \prod_{j=1}^{N} \Theta(\tfrac{1}{2}; ia_j^2/\lambda_j) \right\} \mathrm{d}\mu(\boldsymbol{\lambda}).$$

Moreover, there exists a family of trigonometric polynomials minorants $\{\ell_{\boldsymbol{a}}(\boldsymbol{x}) : \boldsymbol{a} \in \mathbb{Z}_+^N\}$ *with degree* $\ell_{\boldsymbol{a}} < \boldsymbol{a}$*, such that the integral*

$$\int_{\mathbb{T}^N} \{g(\boldsymbol{x}) - \ell_{\boldsymbol{a}}(\boldsymbol{x})\}\mathrm{d}\boldsymbol{x}$$

is equal to the quantity in (4.44)*, and*

$$\lim_{a\uparrow\infty} \int_{\mathbb{T}^N} \{g(\boldsymbol{x}) - \ell_{\boldsymbol{a}}(\boldsymbol{x})\}\mathrm{d}\boldsymbol{x} = 0,$$

where $\boldsymbol{a} \uparrow \infty$ *means* $a_j \uparrow \infty$ *for every* j*.*

Corollary 4.5.4. *Assume all the hypotheses of Theorem 4.5.3. Suppose also that there exists* $R > 0$ *such that* $\mathrm{supp}(\mathrm{d}\mu) \subset \mathbb{R}_+^N \cap Q(R)$*, and*

$$\int_{\mathbb{T}^N} g(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}_+^N} \prod_{j=1}^{N} \lambda_j^{-\frac{1}{2}} \mathrm{d}\mu(\boldsymbol{\lambda}) < \infty.$$

Then, there exists a constant $\alpha_0 > 0$*, such that if* $\alpha := \min_j\{a_j\} \geq \alpha_0$ *and if* $P(\boldsymbol{x})$ *is a trigonometric polynomial with degree* $P < \boldsymbol{a}$ *that minorizes* $g(\boldsymbol{x})$*, we have*

$$\int_{\mathbb{T}^N} P(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \leq (1 + 5Ne^{-\alpha^2/R}) \int_{\mathbb{T}^N} \ell_{\boldsymbol{a}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Given a pair $(\mathcal{G}, \mathrm{d}\mu) \in \mathfrak{G}^N$, suppose that $\mathcal{G} \in L^1(\mathbb{R}^N)$ and the periodization

$$g(\boldsymbol{x}) := \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \mathcal{G}(\boldsymbol{n} + \boldsymbol{x})$$

is equal almost everywhere to a continuous function. We easily see that the pair $(g, \mathrm{d}\mu)$ is admissible by the Theorems 4.5.2 and 4.5.3 for every degree $\boldsymbol{a} \in \mathbb{Z}_+^N$. Thus, the periodic method contemplates the following functions

Example 5.

$$g(\boldsymbol{x}) = \prod_{j=1}^{N} \lambda_j^{-1/2} \Theta(x_j; i/\lambda_j), \text{ for all } \boldsymbol{\lambda} \in \mathbb{R}_+^N.$$

Example 6.

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} e^{-\alpha|\boldsymbol{n}+\boldsymbol{x}|^r}, \text{ for all } \alpha > 0 \text{ and } 0 < r \leq 2.$$

However, we cannot use this construction for the case of the functions $\mathcal{G}_\sigma(\boldsymbol{x}) = |x|^\sigma$ of Example 4. The next proposition tell us that if the Fourier coefficients of $\mathcal{G}(\boldsymbol{x})$ decay sufficiently fast, then the periodization of $\mathcal{G}(x)$ via Poisson summation formula is admissible by the periodic method.

Proposition 4.5.5. *Let $(\mathcal{G}, \mathrm{d}\mu) \in \mathfrak{G}^N$. Suppose that exist constants $C > 0$ and $\delta > N$ such that*

$$\int_{\mathbb{R}_+^N} \widehat{G}_{\boldsymbol{\lambda}}(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{\lambda}) \leq C|\boldsymbol{x}|^{-\delta}$$

if $|\boldsymbol{x}| \geq 1$. Define the function

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N\setminus\{\boldsymbol{0}\}} \int_{\mathbb{R}_+^N} \widehat{G}_{\boldsymbol{\lambda}}(\boldsymbol{n})\mathrm{d}\mu(\boldsymbol{\lambda})e(\boldsymbol{n}\cdot\boldsymbol{x}).$$

Then the pair $(g, \mathrm{d}\mu)$ satisfies all the conditions of Theorems 4.5.2 and 4.5.3 for every $\boldsymbol{a} \in \mathbb{Z}_+^N$.

With this last proposition we see that the following example is contemplated by the periodic method.

Example 7.

$$g_\sigma(\boldsymbol{x}) = \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^N \\ \boldsymbol{n}\neq 0}} |\boldsymbol{n}|^{-N-\sigma}e(\boldsymbol{n}\cdot\boldsymbol{x}), \text{ for all } \sigma > 0.$$

4.5.2 Explicit Lower Bounds for the Box–Minorant Problem

Lower Bounds for $\nu(N)$ is Low Dimensions

We define an auxiliary variational quantity $\lambda(N)$ defined over a more restrictive set of admissible functions than $\nu(N)$. Let

$$\lambda(N) = \sup \int_{\mathbb{R}^N} F(\boldsymbol{x}) d\boldsymbol{x},$$

where the supremum is taken over functions $F(\boldsymbol{x})$ that are admissible for $\nu(N)$ and, in addition, $F(\mathbf{0}) = 1$, and

$$F(\boldsymbol{n}) = 0$$

for each nonzero $\boldsymbol{n} \in \mathbb{Z}^N$ unless $\boldsymbol{n}$ is a corner of the box Q_N. Here, a corner of the box Q_N is a vector $\boldsymbol{n} \in \partial Q_N \cap \mathbb{Z}^N$ that does not belong to a smooth part of the boundary of the box, that is, there exists at most $N-2$ zero entries in $\boldsymbol{n}$ and all the nonzero entries are equal to ± 1. This definition makes any k–dimensional slice of an admissible function for $\lambda(N)$ ($k < N$) admissible for $\lambda(k)$, which in turn implies that

$$\lambda(N+1) \leq \lambda(N)$$

for all N. We note that Selberg's functions constructed via (4.10) are always admissible for $\lambda(N)$ but have negative integral. In this way, we want to construct minorants that resemble Selberg's construction but do a better job.

Making use of the interpolation formula (4.36) we conclude that every function $F(\boldsymbol{x})$ admissible for $\lambda(N)$ has the following useful representation

$$F(\boldsymbol{x}) = S(\boldsymbol{x}) P(\boldsymbol{x}), \tag{4.45}$$

where

$$S(\boldsymbol{x}) = \prod_{n=1}^{N} \left(\frac{\sin(\pi x_n)}{\pi x_n (x_n^2 - 1)} \right)^2$$

and $P(\boldsymbol{x})$ is a polynomial such that each variable x_n appearing in its expression has an exponent not greater than 4. Notice that, by Poisson summation, if $F(\boldsymbol{x})$ is admissible for $\lambda(N)$ and is also invariant under the symmetries of Q_N then

$$\int_{\mathbb{R}^N} F(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 1 + \sum_{k=2}^{N} \binom{N}{k} 2^{-k} P(\boldsymbol{u}_k), \tag{4.46}$$

where $\boldsymbol{u}_k = (\overbrace{1, 1, 1, .., 1}^{k \text{ times}}, 0, \ldots, 0)$.

In what follows will be useful to use a particular family of symmetric functions. For given integers $N \geq k \geq 1$ we define

$$\sigma_{N,k}(\boldsymbol{x}) = \sum_{1 \leq n_1 < n_2 < \ldots < n_k \leq N} x_{n_1}^2 x_{n_2}^2 \ldots x_{n_k}^2$$

and

$$\widetilde{\sigma}_{N,k}(\boldsymbol{x}) = \sum_{1 \leq n_1 < n_2 < \ldots < n_k \leq N} x_{n_1}^4 x_{n_2}^4 \ldots x_{n_k}^4.$$

Theorem 4.5.6. *Define the functions $\mathcal{F}_2(x_1, x_2)$, $\mathcal{F}_3(x_1, x_2, x_3)$, $\mathcal{F}_4(x_1, \ldots, x_4)$ and $\mathcal{F}_5(x_1, \ldots, x_5)$ by using representation* (4.45) *and the following polynomials respectively:*

- $P_2(x_1, x_2) = (1 - x_1^2)(1 - x_2^2) - \frac{1}{16}\widetilde{\sigma}_{2,2}(x_1, x_2)$,
- $P_3(x_1, x_2, x_3) = \prod_{n=1}^{3}(1 - x_n^2) - \frac{1}{16}\widetilde{\sigma}_{3,2}(x_1, x_2, x_3)$,
- $P_4(x_1, \ldots, x_4) = \prod_{n=1}^{4}(1 - x_n^2) - \frac{3}{4}\sigma_{4,4}(x_1, \ldots, x_4) - \frac{1}{16}\widetilde{\sigma}_{4,2}(x_1, \ldots, x_4)$,
- $P_5(x_1, \ldots, x_5) = \prod_{n=1}^{5}(1 - x_n^2) - \frac{3}{4}\sigma_{5,4}(x_1, \ldots, x_5) - \frac{1}{16}\widetilde{\sigma}_{5,2}(x_1, \ldots, x_5)$.

These functions are admissible for $\lambda(2)$, $\lambda(3)$, $\lambda(4)$ *and* $\lambda(5)$ *respectively and their respective integrals are equal to:* $63/64 = 0.984375$, $119/128 = 0.9296875$, $95/128 = 0.7421975$ *and* $31/256 = 0.12109375$.

Proof. The integrals of these functions can be easily calculated using formula (4.46), we prove only their admissibility. We start with $\mathcal{F}_2(\boldsymbol{x})$. Clearly, if $|x_1| > 1 > |x_2|$ then $P_2(x_1, x_2) < 0$. Also, writing $t = |x_1 x_2|$ we obtain

$$\begin{aligned} P_2(x_1, x_2) &= 1 + x_1^2 x_2^2 - x_1^2 - x_2^2 - x_1^4 x_2^4/16 \\ &\leq 1 + x_1^2 x_2^2 - 2|x_1 x_2| - x_1^4 x_2^4/16 \\ &= 1 + t^2 - 2t - t^4/16. \end{aligned}$$

On the other hand, we have

$$1 + t^2 - 2t - t^4/16 = (1-t)^2 - t^4/16 \tag{4.47}$$

and

$$1 + t^2 - 2t - t^4/16 = (t-2)^2(4 - 4t - t^2)/16. \tag{4.48}$$

If $|x_1|, |x_2| < 1$ then $0 \leq t < 1$, and by (4.47) we deduce that $P_2(x_1, x_2) < 1$. If $|x_1|, |x_2| > 1$ then $t > 1$, and by (4.48) we deduce that $P_2(x_1, x_2) \leq 0$. This proves that $\mathcal{F}_2(\boldsymbol{x})$ is $\lambda(2)$-admissible.

Observe that $P_3(x_1, x_2, x_3) < 1$ inside the box Q_3 and $P_3(x_1, x_2, x_2) < 0$ if exactly one or three variables have modulus greater than one. If exactly two variables have modulus greater than one, suppose for instance that $|x_1|, |x_2| > 1 > |x_3|$, then

$$P_3(x_1, x_2, x_3) \leq P_2(x_1, x_2) \leq 0.$$

This proves that $\mathcal{F}_3(\boldsymbol{x})$ is admissible for $\lambda(3)$.

In the same way, clearly $P_4(x_1,\ldots,x_4) < 1$ if all the variables have modulus less than one. If an odd number of variables have modulus greater than one then the function is trivially negative. If $|x_1|,|x_2| > 1 > |x_3|,|x_4|$ then

$$P_4(x_1,x_2,x_3,x_4) \le P_2(x_1,x_2) \le 0.$$

On the other hand, if $|x_1|,|x_2|,|x_3|,|x_4| > 1$ then, suppressing the variables, we have

$$P_4 = 1 - \sigma_{4,1} + \sigma_{4,2} - \sigma_{4,3} + \tfrac{1}{4}\sigma_{4,4} - \tfrac{1}{16}\widetilde{\sigma}_{4,2}.$$

Observing that

$$\sigma_{4,2} - \sigma_{4,3} \le x_1^2x_2^2 + x_3^2x_4^2,$$

we obtain

$$\begin{aligned} 1 - \sigma_{4,1} + \sigma_{4,2} - \sigma_{4,3} - \tfrac{1}{16}\widetilde{\sigma}_{4,2} \le & -1 + P_2(x_1,x_2) + P_2(x_3,x_4) \\ & - \tfrac{1}{16}[x_1^4x_3^4 + x_1^4x_4^4 + x_2^4x_3^4 + x_2^4x_4^4]. \end{aligned}$$

Since $P_2(x_1,x_2) \le 0$ and $P_2(x_3,x_4) \le 0$, we deduce that

$$\begin{aligned} P_3(x_1,\ldots,x_4) &\le -1 + \tfrac{1}{4}x_1^2x_2^2x_3^2x_4^2 - \tfrac{1}{16}[x_1^4x_3^4 + x_1^4x_4^4 + x_2^4x_3^4 + x_2^4x_4^4] \\ &\le -1 + \tfrac{1}{16}(x_1^4 + x_2^4)(x_3^4 + x_4^4) - \tfrac{1}{16}[x_1^4x_3^4 + x_1^4x_4^4 + x_2^4x_3^4 + x_2^4x_4^4] \\ &= -1. \end{aligned}$$

This proves that $\mathcal{F}_4(\boldsymbol{x})$ is admissible for $\lambda(4)$. By a similar argument one can prove that $\mathcal{F}_5(\boldsymbol{x})$ is admissible for $\lambda(5)$. □

Remark. We note that the following function

$$S(\boldsymbol{x}) \prod_{n=1}^{N} (1 - x_n^2) \tag{4.49}$$

is the N–fold product of Selberg's one–dimensional minorant, which is not a minorant for $N > 1$. In this sense, the minorants we have constructed above can be seen as corrections of (4.49) by subtracting higher order terms.

Vanishing in Finite Time

In this part we estimate the critical dimension N at which a quantity related to $\nu(N)$ vanishes. For a given integer $N > 0$ denote by $\mu(N)$ the supremum of

$$\int_{\mathbb{R}^N} F(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

over the set of functions admissible for $\nu(N)$ that also satisfy that

$$F(\mathbf{0}) = 1.$$

Clearly $\lambda(N) \leq \mu(N) \leq \nu(N)$ and $\mu(N)$ is a non-increasing function of N. Also, Theorem 4.2.2 demonstrates that no such function will exist if N is sufficiently large. The next proposition estimates the critical dimension.

Proposition 4.5.7. *We have* $\mu(N) = 0$ *if*

$$N \geq \frac{1}{1-\mu(2)}.$$

Proof. Let $F_N(\boldsymbol{x})$ be admissible for $\mu(N)$ and invariant under the symmetries of Q_N. Then $G_N(x,y) = F_N(x,y,0,\dots,0)$ is admissible for $\mu(2)$ and we have

$$\begin{aligned}
\int_{\mathbb{R}^N} F_N(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= 1 + \sum_{\boldsymbol{n}\in\mathbb{Z}^N\setminus\{\mathbf{0}\}} F(\boldsymbol{n}) \\
&\leq 1 + N\sum_{k\in\mathbb{Z}\setminus\{0\}} G_N(k,0) + \binom{N}{2}\sum_{\substack{(k,\ell)\in\mathbb{Z}^2\\ k,\ell\neq 0}} G_N(k,\ell) \\
&\leq 1 + N\sum_{(n,m)\in\mathbb{Z}^2\setminus\{(0,0)\}} G_N(n,m) \\
&= 1 + N\left(\int_{\mathbb{R}^2} G_N(x,y)\mathrm{d}x\mathrm{d}y - 1\right) \leq 1 + N(\mu(2)-1)
\end{aligned}$$

We conclude that

$$\mu(N) \leq 1 + N\left(\mu(2)-1\right),$$

and this finishes the proof. □

Chapter 5

A Central Limit Theorem for Operators

5.1 Preliminaries

In this chapter we study the intrinsic nature of the approximation method used by Beckner in [4] to prove the sharp form of the Hausdorff–Young inequality. Inspired by Beckner's approach, we demonstrate that Beckner's method is a special instance of a general approximation method (Theorem 5.1.1) that we see as an analogue of the Central Limit Theorem for operators and which leads to (Theorem 5.1.2) a *transference principle* for operators and hyper–contractive estimates. In particular, we characterize the Hermite semi–group as the limiting family of operators associated with any semi–group of operators.

For a given function $f : \mathbb{R} \to \mathbb{C}$, recall that the Fourier Transform of $f(x)$ is defined by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \mathrm{d}x.$$

The Hausdorff–Young inequalitystates that $\mathcal{F}$ maps $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$ boundedly if $1 \leq p \leq 2$. In 1975, Beckner was able to show that

$$\|\mathcal{F}f\|_{L^{p'}} \leq \left[p^{1/p}/p'^{1/p'}\right]^{1/2} \|f\|_{L^p} \tag{5.1}$$

for all $f \in L^p(\mathbb{R})$ and that equality occurs if $f(x)$ is a Gaussian . To prove this result, first he reduced the problem to a hyper–contractive estimate for the Hermite semi–group. He demonstrated that inequality (5.1) is equivalent to the fact that the following Hermite semi–group operator

$$T_\omega : H_\ell(x) \mapsto \omega^\ell H_\ell(x) \tag{5.2}$$

with $\omega = i\sqrt{p-1}$ defines a contraction from $L^p(\mathbb{R}, \mathrm{d}\gamma)$ to $L^{p'}(\mathbb{R}, \mathrm{d}\gamma)$, where $\mathrm{d}\gamma$ is the normal distribution on the real line and $\{H_\ell(x)\}$ is the set of Hermite polynomials associated with $\mathrm{d}\gamma$.

To prove this contraction estimate he proposed a new type of approximation method. Using the following two-point probability measure

$$\mathrm{d}\alpha = \frac{\delta_{-1} + \delta_1}{2} \tag{5.3}$$

and the operator

$$K(f)(x) = \int_{\mathbb{R}} f(t)\mathrm{d}\alpha(t) + xi\sqrt{p-1}\int_{\mathbb{R}} tf(t)\mathrm{d}\alpha(t), \tag{5.4}$$

defined for all $f \in \mathbb{C}[x]$, he constructed a sequence of operators $\{K_N\}$ ($K_1 = K$) and showed that K_N converges in some sense to the operator T_ω. Moreover, by using the approximating sequence $\{K_N\}$ in conjunction with the Central Limit Theorem he showed that if

$$\|Kf\|_{L^{p'}(\mathbb{R},\mathrm{d}\alpha)} \leq \|f\|_{L^p(\mathbb{R},\mathrm{d}\alpha)} \tag{5.5}$$

for all $f \in \mathbb{C}[x]$ then T_ω defines a contraction from $L^p(\mathbb{R}, \mathrm{d}\gamma)$ to $L^{p'}(\mathbb{R}, \mathrm{d}\gamma)$. Finally, he proved inequality (5.5), which became known as Beckner's two-point inequality [4, Lemma 2].

Our main result, Theorem 5.1.1, generalizes Beckner's approximation method in the following way. We show that for any given standardized probability measure $\mathrm{d}\alpha$ defined on $\mathbb{R}$ (that is, $\mathrm{d}\alpha$ has zero mean, unit variance and finite moments of all orders) and any linear operator K defined in $\mathbb{C}[x]$ satisfying a certain orthogonality condition, the sequence of operators $\{K_N\}$ defined in Section 5.1.2 converges in a weak sense to a unique operator $\mathcal{C}$, also defined in $\mathbb{C}[x]$, that belongs to a particular family of operators denoted by $\mathfrak{C}$ that we call *Centered Gaussian Operators*. As a particular case, we show that if K is a semi–group operator associated with the orthogonal polynomials generated by a given probability measure $\mathrm{d}\alpha$, then the mentioned orthogonality conditions are met and the Centered Gaussian Operator associated with K is a Hermite semi–group operator T_ω defined in (5.2).

5.1.1 Notation

Here we define the notation used throughout this chapter. We use the word *standardized* to say that a given probability measure has zero mean, unit variance and finite moments of all orders. We denote by

$$\mathrm{d}\gamma(x) = (2\pi)^{-1/2} \exp(-x^2/2)\mathrm{d}x$$

the normal distribution. For a given probability measure $\mathrm{d}\alpha$ defined on $\mathbb{R}$ and every positive integer N we denote by

$$\mathrm{d}\alpha_N(\boldsymbol{x}) = \mathrm{d}\alpha(x_1) \times \ldots \times \mathrm{d}\alpha(x_N)$$

(with a sub-index) the N-fold product of $\mathrm{d}\alpha$ with itself which is defined on $\mathbb{R}^N$. On the other hand, we denote by

$$\mathrm{d}\alpha^N(x) = \underbrace{\mathrm{d}\alpha * \ldots * \mathrm{d}\alpha(x)}_{N \text{ times}}$$

(with a super-index) the N-fold convolution of $\mathrm{d}\alpha$ with itself defined on $\mathbb{R}$. We always use bold letters to denote N–dimensional vectors when convenient, for instance $\boldsymbol{x} = (x_1, \ldots, x_N)$ or $\boldsymbol{y} = (y_1, \ldots, y_N)$.

Given a function $f(x)$ defined for real x we write

$$f_+(\boldsymbol{x}) = f(x_1 + \ldots + x_N)$$

(the dimension N will be clear by the context). We also denote by $\mathbb{C}[x]$ the ring of polynomials with complex coefficients and by $\mathbb{C}[x_1, \ldots, x_N]$ the several variables analogue.

5.1.2 The Aproximating Sequence

Let $\mathrm{d}\alpha$ be a standardized probability measure, $q > 1$ and $K : \mathbb{C}[x] \to L^q(\mathbb{R}, \mathrm{d}\alpha)$ be a linear operator. For a given integer $N > 0$ we define a linear operator $K_N : \mathbb{C}[x_1, \ldots, x_N] \to L^q(\mathbb{R}^N, \mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}))$ as follows

$$K_N = S_{N,\sqrt{N}} K_{N,N} K_{N,N-1} \ldots K_{N,1} S_{N,1/\sqrt{N}}, \tag{5.6}$$

where $K_{N,n}$ denotes the operator K applied only to the nth variable and

$$S_{N,\lambda} : f(x_1, \ldots, x_N) \mapsto f(\lambda x_1, \ldots, \lambda x_N) \tag{5.7}$$

is a scaling operator defined for all $\lambda \in \mathbb{C}$. In particular, if $p_j(x) = x^j$ for all real x and $f(\boldsymbol{x}) = p_{j_1}(x_1) \ldots p_{j_N}(x_N)$ we have

$$K_N(f)(\boldsymbol{x}) = \frac{K(p_{j_1})(\sqrt{N}x_1) \ldots K(p_{j_N})(\sqrt{N}x_N)}{N^{\frac{j_1 + \ldots + j_N}{2}}}.$$

The sequence $\{K_N\}_{N>0}$ defined by (5.6) is of a special type, it is generated by independently applying the given operator K in each variable, resembling the process of convolving a measure with itself or, in the point of view of probability theory, of making a normalized sum of random variables.

5.1.3 The Family of Centered Gaussian Operators

Here we define a family of operators that we call Centered Gaussian Operators. Let $\{H_\ell(x)\}_{\ell\geq 0}$ denote the sequence of Hermite polynomials associated with $\mathrm{d}\gamma(x)$ (see Appendix 6.3). The Hermite semi–group is a family of operators parametrized by $\omega \in \mathbb{C}$ and defined by

$$T_w : H_\ell(x) \mapsto \omega^\ell H_\ell(x).$$

Often this semi–group is denoted by e^{-zH} where $\omega = e^{-z}$.

We also need two other operators: the one–dimensional scaling operator $S_\lambda := S_{1,\lambda}$ already defined in (5.7) and a multiplication operator defined below

$$M_\tau : f(x) \mapsto \sqrt{1+\tau}e^{-\tau x^2/2} f(x).$$

Here we need the technical condition: $\mathrm{Re}\,\tau > -1$, which guarantees that $M_\tau(\mathbb{C}[x]) \subset L^{1+\varepsilon}(\mathbb{R}, \mathrm{d}\gamma)$ for some small $\varepsilon > 0$.

The family of Centered Gaussian operators will be denoted by $\mathfrak{C}$ and defined by

$$\mathfrak{C} = \{M_\tau T_\omega S_\lambda : \lambda, \omega, \tau \in \mathbb{C}, \quad \mathrm{Re}\,\tau > -1\}.$$

We now explain how this family coincides with the family operators given by centered Gaussian kernels. Let $\mathcal{C} \in \mathfrak{C}$ with $\mathcal{C} = M_\tau T_\omega S_\lambda$. Using the relation $T_\omega S_\lambda = S_b T_a$ if $ab = \omega\lambda$ and $\lambda^2(1-\omega^2) = 1 - a^2$, together with fact that the operator T_ω is given by the following Mehler kernel (see [4, p. 163])

$$T_\omega(x,y) = \frac{1}{\sqrt{1-\omega^2}} \exp\left[-\frac{\omega^2(x^2+y^2)}{2(1-\omega^2)} + \frac{\omega xy}{1-\omega^2}\right],$$

we conclude that

$$\mathcal{C}f(x) = \int_{\mathbb{R}} \mathcal{C}(x,y) f(y) \mathrm{d}\gamma(y)$$

for every $f \in \mathbb{C}[x]$, where

$$\begin{aligned}&\mathcal{C}(x,y)\\&=\sqrt{\frac{1+\tau}{\lambda^2(1-\omega^2)}}\exp\left[-\frac{\tau+(1-\tau)\omega^2}{2(1-\omega^2)}x^2-\frac{1-\lambda^2(1-\omega^2)}{2\lambda^2(1-\omega^2)}y^2+\frac{\omega xy}{\lambda(1-\omega^2)}\right].\end{aligned}$$

Therefore, by inverting the following system of equations

$$A=\frac{\tau+(1-\tau)\omega^2}{(1-\omega^2)},\qquad B=\frac{1-\lambda^2(1-\omega^2)}{\lambda^2(1-\omega^2)}\ \text{ and }\ C=\frac{\omega}{\lambda(1-\omega^2)}$$

we deduce that the class of Centered Gaussian Operators $\mathfrak{C}$ coincides with the class of operators given by centered Gaussian kernels of the following form

$$G(x,y)=\exp[-(A/2)x^2-(B/2)y^2+Cxy+D].$$

Remark. An interesting problem within this theory consists of studying for which parameters A, B and C an operator of this form is bounded from $L^p(\mathbb{R}, \mathrm{d}\gamma)$ to $L^q(\mathbb{R}, \mathrm{d}\gamma)$ and, if that is the case, classify the set of maximizers. This problem was studied by Lieb [52] where he showed that (in a much more general context) in most cases if $\mathcal{C}$ is bounded from $L^p(\mathbb{R}, \mathrm{d}\gamma)$ to $L^q(\mathbb{R}, \mathrm{d}\gamma)$ then it is a contraction.

5.1.4 Main Results

The following is the main result of this chapter.

Theorem 5.1.1. *Let* $\mathrm{d}\alpha$ *be a standardized probability measure defined on* $\mathbb{R}$*, let* $q > 1$ *be a real number and* $K : \mathbb{C}[x] \to L^q(\mathbb{R}, \mathrm{d}\alpha)$ *be a linear operator. Define the numbers*

$$K_{\ell,m}=\int_{\mathbb{R}}K(H_\ell)(x)H_m(x)\mathrm{d}\alpha(x).$$

Assume that:

(1) $K_{0,0} = 1$ *and* $K_{0,1} = K_{1,0} = 0$,

(2) $\operatorname{Re} K_{0,2} > -1$.

Then there exists a unique operator $\mathcal{C} \in \mathfrak{C}$ *such that*

$$\lim_{N\to\infty} \int_{\mathbb{R}^N} K_N(f_+)(\boldsymbol{x}) g_+(\boldsymbol{x}) \mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}) = \int_{\mathbb{R}} \mathcal{C}(f)(x) g(x) \mathrm{d}\gamma(x) \tag{5.8}$$

for every $f, g \in \mathbb{C}[x]$, *where* K_N *is the sequence of operators defined by* (5.6). *Furthermore, the representation* $\mathcal{C} = M_\tau T_\omega S_\lambda$ *is valid if and only if:*

(*i*) $\tau = \frac{-K_{0,2}}{1+K_{0,2}}$,

(*ii*) $\lambda^2 = 1 + K_{2,0} + \tau(1+\tau)K_{1,1}^2$,

(*iii*) $\lambda\omega = (1+\tau)K_{1,1}$.

Remarks.

(1) Observe that if $\lambda^2 \neq 0$ then the system (*i*)–(*iii*) always has two solutions of the form $(\tau, \pm\omega, \pm\lambda)$. However, these two triples define the same operator $\mathcal{C}$ since $T_\omega S_\lambda = T_{-\omega} S_{-\lambda}$. If $\lambda^2 = 0$ then the operator $\mathcal{C}$ is still uniquely defined since $T_\omega S_0 = S_0$ for every $\omega \in \mathbb{C}$. We also note that $T_\omega S_\lambda = S_b T_a$ if $ab = \omega\lambda$ and $\lambda^2(1-\omega^2) = 1 - a^2$.

(2) Condition (1) in the previous theorem is what we call *orthogonality condition*. This assumption is necessary for the existence and non-vanishing of the limit at (5.8) for $f(x)$ and $g(x)$ of the form $ax + b$. We also note that by equation (*i*), condition (2) is equivalent to $\operatorname{Re}\tau > -1$.

(3) In [4], Beckner proved the convergence result (5.8) only for the two-point measure $\mathrm{d}\alpha$ defined in (5.3) and the operator K_ω defined in (5.4). His proof, however, is very different than our proof (which works in the general setting). He exploited a special relation between Hermite polynomials and symmetric functions that does not exist in the general framework.

The next result shows that if the initial operator K is a contraction then its associated Centered Gaussian Operator $\mathcal{C}$ is also a contraction.

Theorem 5.1.2 (Transference Principle)**.** *Assume all the hypotheses of Theorem 5.1.1. Suppose in addition that there exists a standardized probability measure* $\mathrm{d}\beta$ *and a real number* $p \in [1, q]$ *such that*

$$\|Kf\|_{L^q(\mathrm{d}\alpha)} \leq \|f\|_{L^p(\mathrm{d}\beta)}$$

for every $f \in \mathbb{C}[x]$*. Then* $\mathcal{C}$ *extends to a bounded operator from* $L^p(\mathbb{R}, \mathrm{d}\gamma)$ *to* $L^q(\mathbb{R}, \mathrm{d}\gamma)$ *of unit norm. Moreover,* $\tau = 0$ *(or equivalently* $\mathcal{C}(1) = 1$*) and the limit* (5.8) *is also valid for all* $f \in \mathbb{C}[x]$ *and all continuous functions* $g(x)$ *satisfying an estimate of the form:* $|g(x)| \leq A(1 + |x|^A)$*, for some* $A > 0$*.*

The problem of hyper–contractive estimates for the Hermite semi–group was partially solved by Weissler in [76] and then completely solved by Epperson in [30]. They proved that for all $p, q > 0$ with $1 \leq p \leq q < \infty$, the Hermite semi–group operator T_ω defines a contraction from $L^p(\mathbb{R}, \mathrm{d}\gamma)$ to $L^q(\mathbb{R}, \mathrm{d}\gamma)$ if and only if

$$|p - 2 - \omega^2(q - 2)| \leq p - |\omega|^2 q. \tag{5.9}$$

We also refer to [25, 42] for more results on hypercontractivity.

The next corollary is a straightforward application of Theorem 5.1.2 to semi–groups.

Corollary 5.1.3 (Transference Principle for Semi–groups)**.** *Let* $\mathrm{d}\alpha$ *be a standardized probability measure and denote by* $\{P_\ell(x)\}_{\ell\geq 0}$ *the set of monic orthogonal polynomials associated with* $\mathrm{d}\alpha$ *(see [72, Chapter 2]). Define the following semi–group operator*

$$K_\omega : P_\ell(x) \mapsto \omega^\ell P_\ell(x)$$

for $\omega \in \mathbb{C}$*. Then* K_ω *satisfies all the hypotheses of Theorem 5.1.1 and* $\mathcal{C} = T_\omega$ *is the Centered Gaussian Operator associated with* K_ω*. Furthermore, if for some* $\omega \in \mathbb{C}$ *and* $p, q > 0$ *with* $1 \leq p \leq q < \infty$ *we have an estimate of the form*

$$\|K_\omega f\|_{L^q(\mathrm{d}\alpha)} \leq \|f\|_{L^p(\mathrm{d}\alpha)}$$

for every $f \in \mathbb{C}[x]$*, then the operator* T_ω *satisfies the analogous estimate*

$$\|T_\omega f\|_{L^q(\mathrm{d}\gamma)} \leq \|f\|_{L^p(\mathrm{d}\gamma)}$$

for every $f \in \mathbb{C}[x]$ *and condition* (5.9) *must be satisfied.*

5.2 Representation in Terms of Hermite Polynomials

The proof of Theorem 5.1.1 relies on the formal representation in terms of Hermite polynomials of an operator $\mathcal{C} \in \mathfrak{C}$. The next lemmas deal first with the convergence issues. We begin by compiling useful estimates.

Lemma 5.2.1. *We have the following estimates:*

(1) *For every* $q \geq 1$ *and* $B > 0$ *we have*

$$\lim_{N\to\infty} \frac{B^N}{N!} \left\| |x+iy|^N e^{B|x|} \right\|_{L^q(\mathbb{R}^2, \mathrm{d}\gamma(x)\times \mathrm{d}\gamma(y))} = 0. \tag{5.10}$$

(2) *For every* $t, x \in \mathbb{C}$ *we have*

$$\sum_{\ell=0}^{L} \left| \frac{t^\ell}{\ell!} H_\ell(x) \right| \leq e^{|tx|} \int_{\mathbb{R}} e^{|ty|} \mathrm{d}\gamma(y).$$

(3) *For every $t, x \in \mathbb{C}$ we have*

$$\sum_{\ell > L} \left| \frac{t^\ell}{\ell!} H_\ell(x) \right| \leq e^{|tx|} \int_{\mathbb{R}} \frac{|t(x+iy)|^{L+1}}{(L+1)!} \mathrm{d}\gamma(y).$$

Proof. Estimates (2) and (3) are consequences of the integral formula (6.20) and the following inequalities respectively

$$\sum_{\ell=0}^{L} \frac{s^\ell}{\ell!} \leq e^s \quad \text{and} \quad \sum_{\ell > L} \frac{s^\ell}{\ell!} \leq e^s \frac{s^{L+1}}{(L+1)!}, \qquad (s \geq 0). \tag{5.11}$$

Using the following inequalities

$$(a+b)^t \leq 2^{t-1}(a^t + b^t) \quad \text{and} \quad ab \leq \frac{2a^{3/2}}{3} + \frac{b^3}{3}, \qquad (a, b \geq 0,\ t \geq 1)$$

we deduce that

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} |x+iy|^{Nq} e^{Bq|x|} \mathrm{d}\gamma(x) \mathrm{d}\gamma(y) &\ll 2^{Nq} \left(1 + \int_{\mathbb{R}} |x|^{3Nq/2} \mathrm{d}\gamma(x) \right) \\ &= 2^{Nq} \left(1 + \pi^{-1/2} 2^{3Nq/4} \Gamma(3Nq/4 + 1/2) \right) \\ &\ll 4^{Nq} \left(1 + \Gamma(3Nq/4 + 1/2) \right),\end{aligned}$$

where the implied constants depend only on B and q. Using Stirling's formula

$$\Gamma(1+t) \sim \sqrt{2\pi}\, t^{t+1/2} e^{-t}, \qquad t \to \infty$$

the limit (5.10) follows. This completes the proof. □

Now we prove a useful inequality.

Lemma 5.2.2. *Let $\omega, \lambda \in \mathbb{C}$. Then for every $L' < L$ and every $t, x \in \mathbb{C}$ we have*

$$\begin{aligned}&\left| \exp\left[x\omega\lambda t - (1 - \lambda^2 + \omega^2\lambda^2) t^2/2 \right] - \sum_{\ell=0}^{L} \frac{t^\ell}{\ell!} T_\omega S_\lambda(H_\ell)(x) \right| \\ &\qquad \leq \left\{ \int_{\mathbb{R}} \frac{|t(x+iy)|^{L+1}}{(L+1)!} \mathrm{d}\gamma(y) + \frac{|(\lambda^2-1)t^2/2|^{\lfloor (L-L')/2 \rfloor + 1}}{(\lfloor (L-L')/2 \rfloor + 1)!} \int_{\mathbb{R}} e^{|\omega\lambda t y|} \mathrm{d}\gamma(y) \right. \\ &\qquad\quad \left. + \int_{\mathbb{R}} \frac{|\omega\lambda t(x+iy)|^{L'+1}}{(L'+1)!} \mathrm{d}\gamma(y) \right\} \exp\left[|(\lambda^2-1)t^2/2| + (1 + |\omega\lambda|)|tx| \right].\end{aligned}$$

Proof. Using the generating function (6.19) one can deduce that

$$H_\ell(\lambda x) = \sum_{k=0}^{\ell} \binom{\ell}{k} \lambda^k (1-\lambda^2)^{\frac{\ell-k}{2}} H_{\ell-k}(0) H_k(x).$$

Now, we can use (6.18) to obtain

$$\begin{aligned}
\sum_{\ell=0}^{L} \frac{t^\ell}{\ell!} T_\omega S_\lambda(H_\ell)(x) &= \sum_{k=0}^{L} \left(\sum_{\ell=0}^{\left\lfloor \frac{L-k}{2} \right\rfloor} \frac{((\lambda^2-1)t^2/2)^\ell}{\ell!} \right) \frac{(\omega\lambda t)^k}{k!} H_k(x) \\
&= e^{(\lambda^2-1)t^2/2} \sum_{k=0}^{L} \frac{(\omega\lambda t)^k}{k!} H_k(x) \\
&\quad - \sum_{k=0}^{L} \left(\sum_{\ell > \left\lfloor \frac{L-k}{2} \right\rfloor} \frac{((\lambda^2-1)t^2/2)^\ell}{\ell!} \right) \frac{(\omega\lambda t)^k}{k!} H_k(x).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\exp[x\omega\lambda t - (1-\lambda^2+\omega^2\lambda^2)t^2/2] - \sum_{\ell=0}^{L} \frac{t^\ell}{\ell!} T_\omega S_\lambda(H_\ell)(x) \\
&\quad = e^{(\lambda^2-1)t^2/2} \sum_{\ell > L} \frac{t^\ell}{\ell!} H_\ell(x) + \sum_{k=0}^{L} \left(\sum_{\ell > \left\lfloor \frac{L-k}{2} \right\rfloor} \frac{((\lambda^2-1)t^2/2)^\ell}{\ell!} \right) \frac{(\omega\lambda t)^k}{k!} H_k(x) \\
&\quad =: \mathcal{I}_1(t,x,L) + \mathcal{I}_2(t,x,L).
\end{aligned}$$

We now estimate quantities $\mathcal{I}_1$ and $\mathcal{I}_2$. Using the estimate (3) of Lemma 5.2.1 we obtain

$$|\mathcal{I}_1(t,x,L)| \leq e^{|(\lambda^2-1)t^2/2| + |tx|} \int_{\mathbb{R}} \frac{|t(x+iy)|^{L+1}}{(L+1)!} \mathrm{d}\gamma(y). \tag{5.12}$$

Now, we split quantity $\mathcal{I}_2$ into two parts

$$\mathcal{I}_2(t,x,L) = \mathcal{J}_1(t,x,L',L) + \mathcal{J}_2(t,x,L',L),$$

where $\mathcal{J}_1(t,x,L',L)$ denotes the sum from $k=0$ to $k=L'$ and $\mathcal{J}_2(t,x,L',L)$ the sum from $k=L'+1$ to L. Applying estimate (2) of Lemma 5.2.1 and inequality (5.11) we obtain

$$\begin{aligned}\mathcal{J}_1(t,x,L',L) &\leq \left(\sum_{\ell=0}^{L'} \frac{|\omega\lambda t|^k}{k!}|H_k(x)|\right) \sum_{\ell > \left\lfloor \frac{L-L'}{2} \right\rfloor} \frac{|(\lambda^2-1)t^2/2|^\ell}{\ell!} \\ &\leq e^{|(\lambda^2-1)t^2/2|+|\omega\lambda t x|} \frac{|(\lambda^2-1)t^2/2|^{\lfloor (L-L')/2\rfloor+1}}{(\lfloor (L-L')/2\rfloor+1)!} \int_{\mathbb{R}} e^{|\omega\lambda t y|}\mathrm{d}\gamma(y).\end{aligned} \tag{5.13}$$

By a similar method we obtain

$$\mathcal{J}_2(t,x,L',L) \leq e^{|(\lambda^2-1)t^2/2|+|\omega\lambda t x|} \int_{\mathbb{R}} \frac{|\omega\lambda t(x+iy)|^{L'+1}}{(L'+1)!}\mathrm{d}\gamma(y). \tag{5.14}$$

The lemma follows from (5.12), (5.13) and (5.14). □

Remark. Notice that, by taking $L' = \lfloor L/2 \rfloor$, the previous lemma implies that

$$\lim_{L\to\infty} \sum_{\ell=0}^{L} \frac{t^\ell}{\ell!} T_\omega S_\lambda(H_\ell)(x) = \exp\left[x\omega\lambda t - (1-\lambda^2+\omega^2\lambda^2)t^2/2\right],$$

where the convergence is uniform for t and x in any fixed compact set of $\mathbb{C}$.

Let $\mathcal{C} \in \mathfrak{C}$ with $\mathcal{C} = M_\tau T_\omega S_\lambda$. Since $\operatorname{Re}\tau > -1$, we can easily see that $\mathcal{C}(\mathbb{C}[x]) \subset L^{1+\varepsilon}(\mathbb{R},\mathrm{d}\gamma)$ for some small $\varepsilon > 0$. Therefore, the following coefficients

$$c_{\ell,m} = \int_{\mathbb{R}} \mathcal{C}(H_\ell)(x)H_m(x)\mathrm{d}\gamma(x) \tag{5.15}$$

are well defined and if $\operatorname{Re}\tau > -1/2$

$$\mathcal{C}(H_\ell)(x) = \sum_{m\geq 0} \frac{c_{\ell,m}}{m!} H_m(x)$$

in the $L^2(\mathbb{R},\mathrm{d}\gamma)-$sense. The next lemma gives an exact analytic expression for these coefficients. Below the operation $\wedge$ represents the minimum between two given numbers and $\vee$ represents the maximum.

Lemma 5.2.3. *Let $\tau, \omega, \lambda \in \mathbb{C}$ with $\operatorname{Re}\tau > -1$. Then*

$$\frac{c_{\ell,m}}{\ell! m!} = \sum_{\substack{n=0 \\ n \text{ even}}}^{\ell \wedge m} \frac{\left(\frac{-\tau}{\tau+1}\right)^{\frac{\ell \vee m - \ell + n}{2}} (-a)^{\frac{\ell \vee m - m + n}{2}} b^{\ell \wedge m - n}}{2^{\frac{|\ell - m|}{2} + n} (n/2)! \left(\frac{|m-\ell|+n}{2}\right)! (\ell \wedge m - n)!} \tag{5.16}$$

if $\ell + m$ is even and $c_{\ell,m} = 0$ if $\ell + m$ is odd. The quantities a and b are given by

$$a = 1 - \lambda^2 + \lambda^2 \omega^2 \frac{\tau}{\tau+1}, \qquad b = \frac{\lambda \omega}{\tau + 1}. \tag{5.17}$$

Proof. **Step 1.** Define for every $N > 0$ the following function

$$\begin{aligned} &F_N(s,t) \\ &= \sum_{\ell,m=0}^{N} c_{\ell,m} \frac{t^\ell s^m}{\ell! m!} = \int_{\mathbb{R}} M_\tau T_\omega S_\lambda \left(\sum_{\ell=0}^{N} \frac{t^\ell}{\ell!} H_\ell \right)(x) \left(\sum_{m=0}^{N} \frac{s^m}{m!} H_m(x) \right) \mathrm{d}\gamma(x) \end{aligned} \tag{5.18}$$

for every $s, t \in \mathbb{C}$. We claim that if $\operatorname{Re}\tau \geq 0$ and $q \in [1, \infty)$ then

$$\begin{aligned} \Bigg\| &\sqrt{1+\tau} \exp\left[x\omega\lambda t - (1 - \lambda^2 + \omega^2 \lambda^2) t^2/2 - \tau x^2/2 \right] \\ &\qquad - \sum_{\ell=0}^{L} \frac{t^\ell}{\ell!} M_\tau T_\omega S_\lambda (H_\ell)(x) \Bigg\|_{L^q(\mathrm{d}\gamma(x))} \end{aligned} \tag{5.19}$$

converges to zero uniformly in the variable t in any fixed compact set of $\mathbb{C}$.

Assuming the claim is true, we prove the lemma. First we deal with the case $\operatorname{Re}\tau \geq 0$. In this case, by an application of Hölder's inequality in (5.18) we deduce that

$$\begin{aligned} F(s,t) &:= \lim_{N \to \infty} F_N(s,t) \\ &= \sqrt{1+\tau} \int_{\mathbb{R}} \exp\left[x(\omega\lambda t + s) - (1 - \lambda^2 + \omega^2 \lambda^2) t^2/2 - s^2/2 - \tau x^2/2 \right] \mathrm{d}\gamma(x), \end{aligned}$$

where the limit is uniform in compact sets of $\mathbb{C}$ in the variables s and t. Using the following identity

$$\int_{\mathbb{R}} e^{-A(x-B)^2}\mathrm{d}x = \sqrt{\frac{\pi}{A}},$$

which holds for every $A, B \in \mathbb{C}$ with $\operatorname{Re} A > 0$ we conclude that

$$F(s,t) = \exp\left[-at^2/2 - \frac{\tau}{\tau+1}s^2/2 + bts\right],$$

with a and b given by (5.17). We can now use the generating function (6.19) to obtain

$$\begin{aligned} F(s,t) &= \left(\sum_{i\geq 0} \frac{a^{i/2}t^i}{i!}H_i(0)\right)\left(\sum_{j\geq 0}\left(\frac{\tau}{\tau+1}\right)^{j/2}\frac{s^j}{j!}H_j(0)\right)\left(\sum_{k\geq 0}\frac{(bts)^k}{k!}\right) \\ &= \sum_{i,j,k\geq 0} \frac{t^{i+k}s^{j+k}}{i!j!k!}a^{i/2}\left(\frac{\tau}{\tau+1}\right)^{j/2} b^k H_i(0)H_j(0) \\ &= \sum_{\ell,m\geq 0} t^\ell s^m \sum_{n=0}^{\ell\wedge m} \frac{\left(\frac{\tau}{\tau+1}\right)^{\frac{\ell\vee m-\ell+n}{2}} a^{\frac{\ell\vee m-m+n}{2}} b^{\ell\wedge m-n}}{n!(|m-\ell|+n)!(\ell\wedge m-n)!} H_n(0)H_{|\ell-m|+n}(0), \end{aligned}$$

where in the last identity we made the following change of variables: $\ell = i+k$ and $m = j+k$.

Using identity (6.18) in conjunction with the fact that $F_N(s,t)$ converges uniformly in compact sets to $F(s,t)$ we deduce that the coefficients of their Taylor series must match and thus the representation (5.16) follows for $\operatorname{Re}\tau \geq 0$. However, expressions (5.16) and (5.15) clearly define analytic functions in the variable τ for $\operatorname{Re}\tau > -1$. Thus, by analytic continuation, (5.16) also holds for $\operatorname{Re}\tau > -1$.

Step 2. It remains to prove the claim stated in (5.19) for $\operatorname{Re}\tau \geq 0$. Let $t_0 > 0$ and assume that $|t| \leq t_0$. Using Lemma 5.2.2 and Jensen's inequality we obtain

$$\left\|\sqrt{1+\tau}\exp\left[x\omega\lambda t-\tfrac{1-\lambda^2+\omega^2\lambda^2}{2}t^2-\tfrac{\tau}{2}x^2\right]-\sum_{\ell=0}^{L}\frac{t^\ell}{\ell!}M_\tau T_\omega S_\lambda(H_\ell)(x)\right\|_{L^q(\mathrm{d}\gamma(x))}$$
$$\leq |1+\tau|^{1/2}B\left\{\frac{B^{L+1}}{(L+1)!}\left\||x+iy|^{L+1}e^{B|x|}\right\|_{L^q(\mathrm{d}\gamma(x)\times\mathrm{d}\gamma(y))}\right.$$
$$+\frac{B^{\lfloor (L-L')/2\rfloor+1}}{(\lfloor (L-L')/2\rfloor+1)!}\left\|e^{B(|y|+|x|)}\right\|_{L^q(\mathrm{d}\gamma(x)\times\mathrm{d}\gamma(y))}$$
$$\left.+\frac{B^{L'+1}}{(L'+1)!}\left\||x+iy|^{L'+1}e^{B|x|}\right\|_{L^q(\mathrm{d}\gamma(x)\times\mathrm{d}\gamma(y))}\right\} \tag{5.20}$$

for all $L' < L$, where B is a constant which depends only on $|\lambda|, |\omega|$ and t_0. Choosing $L' = \lfloor L/2\rfloor$ and using item (1) of Lemma 5.2.1 one can easily see that the right–hand side of (5.20) converges to zero when $L \to \infty$. This finishes the proof. □

5.3 Proof of Theorem 5.1.1

The main ingredient of the proof is the multiplication formula (6.21). By the fact the any polynomial can be uniquely written as a linear combination of Hermite polynomials and by Lemma 5.2.3, it is sufficient to prove that

$$c_{\ell,m}(N) := \int_{\mathbb{R}^N} K_N([H_\ell]_+)(\boldsymbol{x})[H_m]_+(\boldsymbol{x})\mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}) \to c_{\ell,m}, \qquad N\to\infty$$

where $c_{\ell,m}$ is given by (5.16) with the parameters τ, ω, λ given by equations $(i), (ii)$ and (iii) in Theorem 5.1.1. Applying identity (6.21) we obtain

$$c_{\ell,m}(N) = \frac{\ell!m!}{N^{\frac{\ell+m}{2}}}\sum_{\substack{\ell_1+\ldots+\ell_N=\ell\\ m_1+\ldots+m_N=m}}\frac{K_{\ell_1,m_1}}{\ell_1!m_1!}\cdots\frac{K_{\ell_N,m_N}}{\ell_N!m_N!}.$$

By doing a change of variables that counts the number of appearances of each term $\frac{K_{i,j}}{i!j!}$ we obtain that

$$c_{\ell,m}(N) = \frac{\ell! m!}{N^{\frac{\ell+m}{2}}} \sum_{[P_{i,j}]} \frac{N!}{\prod_{i,j} P_{i,j}!} \prod_{i,j} \left(\frac{K_{i,j}}{i!j!} \right)^{P_{i,j}},$$

where the last sum is over the subset of matrices $[P_{i,j}]$, $i = 0, \ldots, \ell$ and $j = 0, \ldots, m$ with non-negative integer entries satisfying the conditions below:

(I) $\sum_{i,j} i P_{i,j} = \ell$,

(II) $\sum_{i,j} j P_{i,j} = m$,

(III) $\sum_{i,j} P_{i,j} = N$.

These conditions imply that $P_{i,j} \leq \max\{\ell, m\}$ if $(i,j) \neq (0,0)$ and

$$N \geq P_{0,0} \geq N - \max\{\ell, m\}[(\ell+1)(m+1) - 1].$$

Thus, the subset of matrices determined by (I)–(III) is finite and the number of elements does not depend on N. Also, since $K_{0,0} = 1$ we obtain

$$c_{\ell,m}(N) = \frac{\ell! m!}{N^{\frac{\ell+m}{2}}} \sum_{[P_{i,j}]} \left\{ \frac{N!}{(N - \sum'_{i,j} P_{i,j})! \prod'_{i,j} P_{i,j}!} \prod_{i,j}{}' \left(\frac{K_{i,j}}{i!j!} \right)^{P_{i,j}} \right\}, \quad (5.21)$$

where the symbols $\prod'$ and $\sum'$ mean that the term $(i,j) = (0,0)$ is excluded. We also obtain that for every $[P_{i,j}]$ satisfying (I)–(III) we have

$$\frac{N!}{(N - \sum'_{i,j} P_{i,j})! \prod'_{i,j} P_{i,j}!} \sim \frac{N^{\sum'_{i,j} P_{i,j}}}{\prod'_{i,j} P_{i,j}!}$$

when $N \to \infty$ (the symbol $\sim$ means that the quotient goes to 1 when $N \to \infty$).

We now investigate the possible values for $\sum'_{i,j} P_{i,j}$. Notice that if $P_{0,1}$ or $P_{1,0}$ is not zero then the quantity in the brackets at (5.21) is zero because

$K_{0,1} = K_{1,0} = 0$. If $P_{0,1} = P_{1,0} = 0$, then by equations (I) and (II) we conclude that

$$\frac{\ell+m}{2} = {\sum_{i,j}}' \frac{i+j}{2} P_{i,j} \geq {\sum_{i,j}}' P_{i,j}$$

with equality occurring if and only if $\ell + m = 2(P_{0,2} + P_{2,0} + P_{1,1})$, $\ell + m$ is even and $P_{i,j} = 0$ if $(i,j) \notin \{(0,2),(2,0),(1,1),(0,0)\}$.

We conclude that the limit of (5.21) when $N \to \infty$ is zero if $\ell + m$ is odd and is equal to

$$\ell! m! \sum \frac{K_{0,2}^{P_{0,2}} K_{2,0}^{P_{2,0}} K_{1,1}^{P_{1,1}}}{2^{P_{0,2}+P_{2,0}} P_{0,2}! P_{2,0}! P_{1,1}!} \tag{5.22}$$

if $\ell + m$ is even, where the above sum is over the set of non-negative integers $P_{0,2}, P_{2,0}, P_{1,1}$ satisfying:

(IV) $2P_{2,0} + P_{1,1} = \ell$,

(V) $2P_{0,2} + P_{1,1} = m$.

Depending on whether ℓ is greater than m or not, one can do an appropriate change of variables (for instance if $m \geq \ell$ choose $n = 2P_{2,0}$) to deduce that (5.22) equals to

$$\ell! m! \sum_{\substack{n=0 \\ n \text{ even}}}^{\ell \wedge m} \frac{K_{1,1}^{\ell\wedge m - n} K_{2,0}^{\frac{\ell \vee m - m + n}{2}} K_{0,2}^{\frac{\ell \vee m - \ell + n}{2}}}{2^{\frac{|\ell - m|}{2} + n} (\ell \wedge m - n)! (n/2)! \left(\frac{|\ell-m|+n}{2}\right)!}.$$

Finally, we can apply Lemma 5.2.3 to conclude that the above quantity equals to

$$\int_{\mathbb{R}} \mathcal{C}(H_\ell)(x) H_m(x) \mathrm{d}\gamma(x)$$

if $K_{0,2} = -\tau/(1+\tau)$, $K_{2,0} = \lambda^2 - 1 - \lambda^2\omega^2\tau/(\tau+1)$ and $K_{1,1} = \lambda\omega/(1+\tau)$. This finishes the proof.

5.4 Proof of Theorem 5.1.2

Step 1. First, we claim that for every $N > 0$ the operator K_N defined in Section 5.1.3 satisfies

$$\|K_N(f)(\boldsymbol{x})\|_{L^q(\mathbb{R}^N, \mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}))} \leq \|f(\boldsymbol{x})\|_{L^p(\mathbb{R}^N, \mathrm{d}\beta_N(\sqrt{N}\boldsymbol{x}))} \tag{5.23}$$

for every polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ (recall the notation in Section 5.1.1). Denoting by

$$g(x_1, \dots, x_{N-1}, y_N) = K_{y_1} K_{y_2} \dots K_{y_{N-1}} \left[f(\boldsymbol{y}/\sqrt{N}) \right] (\sqrt{N}x_1, \dots, \sqrt{N}x_{n-1}),$$

where K_{y_j} denotes the restriction to the y_j variable of the operator K, we conclude that

$$K_N(f)(x_1, \dots, x_N) = K_{y_N}[g(x_1, \dots, x_N, y_N/\sqrt{N})](\sqrt{N}x_N).$$

We obtain

$$\begin{aligned}
&\|K_N f(\boldsymbol{x})\|_{L^q(\mathrm{d}\alpha(\sqrt{N}x_1) \times \dots \times \mathrm{d}\alpha(\sqrt{N}x_N))} \\
&= \| \| \|K_{y_N}[g(x_1, \dots, x_{N-1}, \tfrac{y_N}{\sqrt{N}})](\sqrt{N}x_N)\|_{L^q(\mathrm{d}\alpha(\sqrt{N}x_N))} \|_{L^q(\mathrm{d}\alpha(\sqrt{N}x_1) \times \dots \times \mathrm{d}\alpha(\sqrt{N}x_{N-1}))} \\
&\leq \| \| \|g(x_1, \dots, x_{N-1}, y_N)]\|_{L^p(\mathrm{d}\beta(\sqrt{N}y_N))} \|_{L^q(\mathrm{d}\alpha(\sqrt{N}x_1) \times \dots \times \mathrm{d}\alpha(\sqrt{N}x_{N-1}))} \\
&\leq \| \| \|g(x_1, \dots, x_{N-1}, y_N)] \|_{L^q(\mathrm{d}\alpha(\sqrt{N}x_1) \times \dots \times \mathrm{d}\alpha(\sqrt{N}x_{N-1}))} \|_{L^p(\mathrm{d}\beta(\sqrt{N}y_N))},
\end{aligned}$$

where the second inequality is Minkowski's inequality since $q \geq p$. We now can apply the same argument to estimate the quantity

$$\|g(x_1, \dots, x_{N-1}, y_N) \|_{L^q(\mathbb{R}^{N-1}, \mathrm{d}\alpha(\sqrt{N}x_1) \times \dots \times \mathrm{d}\alpha(\sqrt{N}x_{N-1}))}$$

for fixed y_N and conclude by induction that (5.23) is valid (see also [4, Lemma 2]).

Step 2. Now, let $\mathcal{C} \in \mathfrak{C}$ be the Centered Gaussian operator associated with K and $f \in \mathbb{C}[x]$ with $\|f\|_{L^p(\mathbb{R},\mathrm{d}\gamma)} = 1$. Since $\mathbb{C}[x]$ is dense in $L^q(\mathbb{R}, \mathrm{d}\gamma)$, for every $\varepsilon > 0$ we can find $g \in \mathbb{C}[x]$ with $\|g\|_{L^{q'}(\mathbb{R},\mathrm{d}\gamma)} = 1$ such that

$$\|\mathcal{C}(f)\|_{L^q(\mathbb{R},\mathrm{d}\gamma)} \leq \left| \int_{\mathbb{R}} \mathcal{C}(f)(x)g(x)\mathrm{d}\gamma(x) \right| + \varepsilon.$$

However, for N sufficiently large we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \mathcal{C}(f)(x)g(x)\mathrm{d}\gamma(x) \right| \\
&\qquad \leq \left| \int_{\mathbb{R}^N} K_N(f_+)(\boldsymbol{x})g_+(\boldsymbol{x})\mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}) \right| + \varepsilon \\
&\qquad \leq \|K_N(f_+)(\boldsymbol{x})\|_{L^q(\mathbb{R}^N,\mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}))} \, \|g_+(\boldsymbol{x})\|_{L^{q'}(\mathbb{R}^N,\mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}))} + \varepsilon \\
&\qquad \leq \|f_+(\boldsymbol{x})\|_{L^p(\mathbb{R}^N,\mathrm{d}\beta_N(\sqrt{N}\boldsymbol{x}))} \, \|g_+(\boldsymbol{x})\|_{L^{q'}(\mathbb{R}^N,\mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}))} + \varepsilon \\
&\qquad = \|f(x)\|_{L^p(\mathbb{R},\mathrm{d}\beta^N(\sqrt{N}x))} \, \|g(x)\|_{L^{q'}(\mathbb{R},\mathrm{d}\alpha^N(\sqrt{N}x))} + \varepsilon,
\end{aligned} \tag{5.24}$$

where the second inequality is Hölder's inequality and the third one is due to (5.23).

Since $\mathrm{d}\alpha$ is a standardized probability measure, Fatou's lemma for weakly convergent probabilities [31, Theorem 1.1] together with the convergence of the absolute moments in the Central Limit Theorem [74, Theorem 2] imply that

$$\lim_{N\to\infty} \int_{\mathbb{R}} h(x)\mathrm{d}\alpha^N(\sqrt{N}x) = \int_{\mathbb{R}} h(x)\mathrm{d}\gamma(x) \tag{5.25}$$

for every continuous function $h(x)$ satisfying an estimate of the form $|h(x)| \leq A(1 + |x|^A)$, for some $A > 0$. Thus, we conclude that the right–hand side of (5.24) converges to $1 + \varepsilon$ when $N \to \infty$. By the arbitrariness of $\varepsilon > 0$ we conclude that $\|\mathcal{C}(f)\|_{L^q(\mathrm{d}\gamma)} \leq 1$. By the density of $\mathbb{C}[x]$ in $L^p(\mathbb{R}, \mathrm{d}\gamma)$ for finite $p \geq 1$, we conclude that $\mathcal{C}$ extends to a bounded linear operator of norm not greater than one.

Step 3. Now, observe that since $K_{0,0} = 1$ we have

$$\int_{\mathbb{R}^N} K_N(1)(\boldsymbol{x}) \mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}) = 1$$

for every $N > 0$. Thus, we obtain

$$1 = \int_{\mathbb{R}} \mathcal{C}(1)(x) \mathrm{d}\gamma(x) \leq \|\mathcal{C}(1)\|_{L^q(\mathbb{R},\mathrm{d}\gamma)} \leq 1.$$

This implies that $|\mathcal{C}(1)(x)| = 1$ for every real x. We conclude that $\tau = 0$ or, equivalently, $\mathcal{C}(1) = 1$.

Now, let $f \in \mathbb{C}[x]$ and let $g(x)$ be a continuous function satisfying an estimate of the form $|g(x)| \leq A(1+|x|^A)$. Given $\varepsilon > 0$, take $h \in \mathbb{C}[x]$ such that $\|g - h\|_{L^{q'}(\mathbb{R},\mathrm{d}\gamma)} < \varepsilon$. By estimate (5.23) and Holder's inequality we conclude that

$$\begin{aligned}\Big| \int_{\mathbb{R}} \mathcal{C}(f)(x) g(x) \mathrm{d}\gamma(x) - \int_{\mathbb{R}^N} K_N(f_+)(\boldsymbol{x}) g_+(\boldsymbol{x}) \mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}) \Big| \\ \leq \varepsilon \|\mathcal{C}(f)\|_{L^q(\mathrm{d}\gamma)} + \|f(x)\|_{L^p(\mathrm{d}\alpha^N(\sqrt{N}x))} \|g(x) - h(x)\|_{L^{q'}(\mathrm{d}\alpha^N(\sqrt{N}x))} \\ + \Big| \int_{\mathbb{R}} \mathcal{C}(f)(x) h(x) \mathrm{d}\gamma(x) - \int_{\mathbb{R}^N} K_N(f_+)(\boldsymbol{x}) h_+(\boldsymbol{x}) \mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}) \Big|.\end{aligned}$$

We can now use the Central Limit Theorem as stated in (5.25) to obtain that

$$\begin{aligned}\limsup_{N\to\infty} \Big| \int_{\mathbb{R}} \mathcal{C}(f)(x) g(x) \mathrm{d}\gamma(x) - \int_{\mathbb{R}^N} K_N(f_+)(\boldsymbol{x}) g_+(\boldsymbol{x}) \mathrm{d}\alpha_N(\sqrt{N}\boldsymbol{x}) \Big| \\ \leq \varepsilon \left(\|\mathcal{C}(f)\|_{L^q(\mathrm{d}\gamma)} + \|f(x)\|_{L^p(\mathrm{d}\gamma)} \right).\end{aligned}$$

The proof is complete once we let $\varepsilon \to 0$.

Chapter 6

Appendix

6.1 L^p de Branges Spaces

De Branges spaces are closely related to Hardy spaces in the upper half–plane $\mathbb{C}^+ = \{\operatorname{Im} z > 0\}$. For a given $p \in [1, \infty)$ the Hardy space $H^p(\mathbb{C}^+)$ is defined as the space of holomorphic functions $F : \mathbb{C}^+ \to \mathbb{C}$ such that

$$\sup_{y>0} \|F(x+iy)\|_{L^p} < \infty, \tag{6.1}$$

where $\|\cdot\|_{L^p}$ denotes the standard L^p-norm in the variable x. This space endowed with the norm (6.1) defines a Banach space of holomorphic functions on the upper half–plane.

It can be proven that for every $F \in H^p(\mathbb{C}^+)$ the limit

$$F(x) = \lim_{y \to 0} F(x+iy)$$

exists for almost every real x and defines a function in $L^p(\mathbb{R})$. Moreover, we have the following Poisson representation

$$\operatorname{Re} F(x+iy) = \frac{y}{\pi} \int_{\mathbb{R}} \frac{\operatorname{Re} F(s)}{(x-s)^2 + y^2} \mathrm{d}s \qquad \text{if} \quad y > 0, \tag{6.2}$$

and the following Cauchy integral formula:

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{F(s)}{s-z} \mathrm{d}s = F(z), \qquad \text{if} \quad z \in \mathbb{C}^+ \tag{6.3}$$

and

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{F(s)}{s-z} \mathrm{d}s = 0, \qquad \text{if} \quad \overline{z} \in \mathbb{C}^+. \tag{6.4}$$

Using the Poisson representation (6.2) and Young's inequality for convolutions, one can deduce that

$$\sup_{y>0} \|F(\cdot + iy)\|_p = \|F\|_p. \tag{6.5}$$

Using (6.3) one can show that $H^p(\mathbb{C}^+)$ is indeed a Banach space. All these facts are contained in [58].

Proposition 6.1.1. *Let $F(z)$ be an holomorphic function in $\mathbb{C}^+$ that has a continuous extension to the closed upper half–plane. The following are equivalent:*

(1) $\sup_{y>0} \|F(\cdot + iy)\|_p < \infty$.

(2) *$F(z)$ is of bounded type in $\mathbb{C}^+$, $v(F) \leq 0$ and*

$$\|F\|_p < \infty.$$

Proof. First we prove that (2) implies (1). Since $F(z)$ is of bounded type with non–positive mean type we have (see [6, Problem 27])

$$\log|F(z)| \leq \frac{y}{\pi} \int_{\mathbb{R}} \frac{\log|F(t)|}{(x-t)^2 + y^2} \mathrm{d}t.$$

Jensen's inequality implies that

$$|F(z)| \leq \frac{y}{\pi} \int_{\mathbb{R}} \frac{|F(t)|}{(x-t)^2 + y^2} \mathrm{d}t.$$

Applying Young's inequality for convolutions and Fatou's lemma we conclude that

$$\sup_{y>0} \|F(\cdot + iy)\|_p = \|F\|_p.$$

Now we show that (1) implies (2). Write $\operatorname{Re} F(t) = g(t) - h(t)$, where $g(t) = \max\{\operatorname{Re} F(t), 0\}$ and $h(t) = \max\{-\operatorname{Re} F(t), 0\}$. Let $G(z)$ and $H(z)$ be holomorphic functions in $\mathbb{C}^+$ such that

$$\operatorname{Re} G(z) = \frac{y}{\pi}\int_{\mathbb{R}} \frac{g(t)}{(x-t)^2+y^2}\mathrm{d}t$$

and

$$\operatorname{Re} H(z) = \frac{y}{\pi}\int_{\mathbb{R}} \frac{h(t)}{(x-t)^2+y^2}\mathrm{d}t$$

for all $z \in \mathbb{C}^+$.

Since $\operatorname{Re} H(z) > 0$ and $\operatorname{Re} G(z) > 0$ in $\mathbb{C}^+$, we conclude that $G(z)$ and $H(z)$ are of bounded type with non–positive mean type (see [6, Problem 20]). By representation (6.2) $F(z)$ differs from $G(z) - H(z)$ by a constant, and we deduce that $F(z)$ is of bounded type with non–positive mean type. □

The de Branges space $\mathcal{H}^p(E)$ is defined as the space of entire functions $F(z)$ such that $F(z)/E(z)$ and $F^*(z)/E(z)$ are of bounded type with non–positive mean type and $F(x)/E(x) \in L^p(\mathbb{R})$ when restricted to the real axis.

The above proposition shows that $F \in \mathcal{H}^p(E)$ if and only if

$$\sup_{y\in\mathbb{R}} \|F(\cdot + iy)/E(\cdot + i|y|)\|_p < \infty,$$

or equivalently, if F/E and F^*/E belong to $H^p(\mathbb{C}^+)$. Using the Cauchy representation (6.3) and (6.4) one can deduce that

$$F(w) = \int_{\mathbb{R}} \frac{F(x)\overline{K(w,x)}}{|E(x)|^2}\mathrm{d}x \tag{6.6}$$

for all $w \in \mathbb{C}^+$, where $K(w, z)$ is the kernel defined in (2.6). Using the reproducing kernel property (6.6) together with the completeness of Hardy spaces it can be proven that the spaces $\mathcal{H}^p(E)$ are indeed Banach spaces with norm given by

$$\|F\|_{E,p} = \left(\int_{\mathbb{R}} \left| \frac{F(x)}{E(x)} \right|^p \mathrm{d}x \right)^{1/p}.$$

Evidently $\|K(w,.)\|_{E,q} < \infty$ for every $1 < q \leq \infty$ and $w \in \mathbb{C}$. Using Hölder's inequality we obtain an important estimate

$$|F(w)| \leq \|F\|_{E,p} \|K(w,\cdot)\|_{E,p'}, \tag{6.7}$$

where p' is the conjugate exponent of p. Using the known fact that the space $H^{p'}(\mathbb{C}^+)$ can be identified with the dual space of $H^p(\mathbb{C}^+)$ for $p \in (1,\infty)$ one can deduce that

$$\mathcal{H}^p(E)' = \mathcal{H}^{p'}(E) \qquad \text{for} \quad p \in (1,\infty). \tag{6.8}$$

That is, if Λ is a bounded functional over $\mathcal{H}^p(E)$ then there exists a function $\Lambda \in \mathcal{H}^{p'}(E)$ such that

$$\langle \Lambda,\, F \rangle = \int_{\mathbb{R}} \frac{F(x)\overline{\Lambda(x)}}{|E(x)|^2} \mathrm{d}x$$

for all $F \in \mathcal{H}^p(E)$. The proof of this duality result deals with model spaces for $\mathcal{H}^p(\mathbb{C}^+)$ which diverges from the purposes of this section. For the interested reader we refer to [2, Proposition 1.1] and [24, Lemma 4.2].

Another important result in this theory is a theorem of Krein [50].

Theorem 6.1.2 (Krein)**.** *Let $F(z)$ be an entire function. The following are equivalent:*

(1) *$F(z)$ is of exponential type and*

$$\int_{\mathbb{R}} \frac{\log^+ |F(x)|}{1+x^2} \mathrm{d}x < \infty.$$

(2) *$F(z)$ and $F^*(z)$ are of bounded type in $\mathbb{C}^+$.*

In this situation we have $\tau(F) = \max\{v(F), v(F^*)\}$.

Remark. An application of this theorem can show that if a Hermite–Biehler function $E(z)$ is also of bounded type in $\mathbb{C}^+$ then $E(z)$ is of exponential type and $F \in \mathcal{H}^p(E)$ if and only if $\tau(F) \leq \tau(E)$ and $F/E \in L^p(\mathbb{R})$.

6.2 Homogeneous Spaces

In what follows we briefly review the construction of a special family of de Branges spaces called *homogeneous spaces* which were introduced by de Branges (see [6, Section 50] and [47]).

Let $\alpha > -1$ be a parameter and consider the real entire functions $A_\alpha(z)$ and $B_\alpha(z)$ given by

$$A_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n}}{n!(\alpha+1)(\alpha+2)\ldots(\alpha+n)} = \Gamma(\alpha+1)\left(\tfrac{1}{2}z\right)^{-\alpha} J_\alpha(z) \tag{6.9}$$

and

$$B_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+1}}{n!(\alpha+1)(\alpha+2)\ldots(\alpha+n+1)} = \Gamma(\alpha+1)\left(\tfrac{1}{2}z\right)^{-\alpha} J_{\alpha+1}(z), \tag{6.10}$$

where $J_\alpha(z)$ denotes the classical Bessel function of the first kind given by

$$J_\alpha(z) = \sum_{n\geq 0} \frac{(-1)^n(\frac{1}{2}z)^{2n+\alpha}}{n!\,\Gamma(\alpha+n+1)}.$$

If we write $z = x + iy$ then, for every $\alpha > -1$, we have

$$J_\alpha(z) = \sqrt{\frac{2}{\pi z}}\left(\cos(z - \alpha\pi/2 - \pi/4) + e^{|y|}O(1/|z|)\right) \tag{6.11}$$

for $x > 0$. This estimate can be found in [75, Section 7.21].

If we write

$$E_\alpha(z) = A_\alpha(z) - iB_\alpha(z),$$

then the function $E_\alpha(z)$ is a Hermite–Biehler function with no real zeros. Moreover, it is of bounded type in $\mathbb{C}^+$ and of exponential type in $\mathbb{C}$, with $v(E_\alpha) = \tau(E_\alpha) = 1$. Observe that when $\alpha = -1/2$ we have simply $A_{-1/2}(z) = \cos z$ and $B_{-1/2}(z) = \sin z$.

These special functions also satisfy the following differential equations

$$\begin{aligned} A'_\alpha(z) &= -B_\alpha(z) \\ B'_\alpha(z) &= A_\alpha(z) - (2\alpha+1)B_\alpha(z)/z. \end{aligned} \tag{6.12}$$

By (6.9), (6.10) and (6.11) we have

$$|E_\alpha(x)|^{-2} \simeq_\alpha |x|^{2\alpha+1} \tag{6.13}$$

and

$$|x|^{2\alpha+1}|A_\alpha(x)B_\alpha(x)| = C_\alpha\big(|\sin(2x - \alpha\pi)| + O(1/|x|)\big)$$

for $|x| \geq 1$. We conclude that $A_\alpha B_\alpha \notin \mathcal{H}^2(E_\alpha^2)$, hence $B_\alpha \notin \mathcal{H}^2(E_\alpha)$. Also, by (6.12) we have

$$i\frac{E'_\alpha(z)}{E_\alpha(z)} = 1 - (2\alpha+1)\frac{B_\alpha(z)}{zE_\alpha(z)}. \tag{6.14}$$

for all real $z \in \mathbb{C}^+$. Hence $[E'_\alpha(z)/E_\alpha(z)] \in H^\infty(\mathbb{C}^+)$.

Denoting by $\varphi_\alpha(z)$ the phase function associated with $E_\alpha(z)$ and using the fact that $\varphi'_\alpha(t) = \operatorname{Re}[iE'_\alpha(t)/E_\alpha(t)]$ for all real t, we can use (6.14) to obtain

$$\varphi'_\alpha(t) = 1 - \frac{(2\alpha+1)A_\alpha(t)B_\alpha(t)}{t|E_\alpha(t)|^2}.$$

Hence,

$$\varphi'_\alpha(t) \simeq_\alpha 1 \quad \text{for all real } t. \tag{6.15}$$

For each $F \in \mathcal{H}^2(E_\alpha)$ we have the remarkable identity

$$\int_{\mathbb{R}} |F(x)|^2 \, |E_\alpha(x)|^{-2} \, \mathrm{d}x = c_\alpha \int_{\mathbb{R}} |F(x)|^2 \, |x|^{2\alpha+1} \, \mathrm{d}x \,, \tag{6.16}$$

with $c_\alpha = \pi \, 2^{-2\alpha-1} \, \Gamma(\alpha{+}1)^{-2}$. Using the fact that $E_\alpha(z)$ is of bounded type, we can apply Krein's Theorem 6.1.2 together with (6.13) and (6.16) to conclude that $F \in \mathcal{H}^2(E_\alpha)$ if and only if $F(z)$ has exponential type at most 1 and either side of (6.16) is finite. Again, by Krein's Theorem, for any integer $\nu > 0$ and $p \in [1, \infty)$ we have that $F \in \mathcal{H}^p(E_\alpha^\nu)$ if and only if $F(z)$ has exponential type at most ν and $F/E_\alpha^\nu \in L^p(\mathbb{R})$.

For $\alpha > -1/2$, the *Hankel's integral* for $J_\alpha(z)$ is given by

$$J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(\alpha + 1/2)\sqrt{\pi}} \int_{-1}^{1} e^{isz} (1 - s^2)^{\alpha - \frac{1}{2}} \, \mathrm{d}s.$$

Using (6.9) and (6.10) and an integration by parts, we deduce the following integral representation for $\alpha > -1/2$

$$E_\alpha(z) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1/2)\sqrt{\pi}} \int_{-1}^{1} e^{isz} (1 - s^2)^{\alpha - \frac{1}{2}} (1 - s) \, \mathrm{d}s.$$

By simple estimates, we deduce from the above representation that $v(E_\alpha^*) = v(E_\alpha) = 1$ for $\alpha > -1/2$. Also, it is not known if the zeros of $E_\alpha(z)$ have a positive distance from the real axis. Thus, we cannot directly apply Theorem 3.1.2 for homogeneous spaces in the case $p > 2$. Nevertheless, Theorem 3.4.2 will follow for these homogeneous spaces by verifying that the alternative condition (3.29) holds, as pointed out in the remark in the end of Section 3.3.1.

Lemma 6.2.1. *Let $\alpha > -1$ be a real number and $\nu > 0$ be an integer. The space $\mathcal{H}^p(E_\alpha^\nu)$ satisfies the following properties:*

(1) *$\mathcal{H}^p(E_\alpha^\nu) \subset \mathcal{H}^q(E_\alpha^\nu)$ if $1 \le p < q < \infty$.*

(2) $\mathcal{H}^p(E_\alpha^\nu)$ *is closed by differentiation for every* $p \in [1, \infty)$.

(3) *If* $p \in [1, \infty)$ *there exists a constant* $C_{\alpha,p} > 0$ *such that*

$$\sum_{A_\alpha(t)=0} \frac{|F(t)|}{(1+|t|)K_{\nu,\alpha}(t,t)^{1/2}} \leq C_{\alpha,p} \|F\|_{E_\alpha^\nu,p} \quad \textit{for every} \quad F \in \mathcal{H}^p(E_\alpha^\nu),$$

where the function $K_{\nu,\alpha}(w,z)$ *denotes the reproducing kernel of* $\mathcal{H}^2(E_\alpha^\nu)$.

Proof. Item (1). The desired inclusions follow by the previous discussion about the properties of homogeneous spaces and Lemma 3.3.1.

Item (2). Define an auxiliary function $\Psi(z)$ in the following way. If $\alpha < 0$ write $\Psi(z) = E_{-1-\alpha}(z)^\nu$. If $\alpha \geq 0$, let $k \geq 1$ be an integer such that $|1/2 + \alpha - k| \leq 1/2$ and define $\Psi(z) = [E_{-3/4}(z)^{4(k+1)} E_{k-\alpha}(z)]^\nu$. We conclude that $\Psi(z)$ is of exponential type and, by (6.13), $|\Psi(x)| \simeq |x|^{\nu(\alpha+1/2)} \simeq |E_\alpha(x)|^{-\nu}$ for $|x| \geq 1$. By (6.14) and some simple calculations we have $|\Psi'(x)| \ll |\Psi(x)|$ for all real x. Also, by redefining $\widetilde{\Psi}(z) = \Psi(az)$ for some $a > 0$, we can assume that $\Psi(z)$ has exponential type 1. We conclude that $F \in \mathcal{H}^p(E_\alpha^\nu)$ if and only if $F(z)$ if of exponential type at most ν and $F\Psi \in L^p(\mathbb{R}, \mathrm{d}x)$.

If $F \in \mathcal{H}^p(E_\alpha^\nu)$ has a finite number of zeros then a simple calculation would show that $F'(z)$ is a finite combination of functions in $\mathcal{H}^p(E_\alpha^\nu)$. If $F(z)$ has an infinite number of zeros, let $k = \lceil |\nu(\alpha + 1/2)| \rceil + 2$ and $w_1, \ldots, w_k$ be zeros of $F(z)$. Define then $G(z) = F(z)/[(z - w_1) \ldots (z - w_k)]$. Clearly, $G(z)$ if of exponential type at most ν and, by an application of Holder's inequality, $G \in L^1(\mathbb{R}, \mathrm{d}x)$. Since the Paley–Wiener spaces are closed by differentiation we deuce that $G' \in L^1(\mathbb{R}, \mathrm{d}x)$ and has exponential type at most ν. Hence $F'(z)$ has exponential type at most ν. On the other hand, $F\Psi \in L^p(\mathbb{R}, \mathrm{d}x)$ and again this implies that $(F\Psi)' \in L^p(\mathbb{R}, \mathrm{d}x)$. Since $F'\Psi = (F\Psi)' - F\Psi'$ and $|\Psi'(x)| \ll |\Psi(x)|$ for all real x, we conclude that $F'\Psi \in L^p(\mathbb{R}, \mathrm{d}x)$. Hence $F' \in \mathcal{H}^p(E_\alpha^\nu)$.

Item (3). By item (2) it is sufficient to prove that

$$\sum_{A_\alpha(t)=0} \frac{|F(t)|}{(1+|t|)K_{\nu,\alpha}(t,t)^{1/2}} \ll_{p,\alpha,\nu} \|F\|_{E_\alpha^\nu,p}, \quad \text{for every} \quad F \in \mathcal{H}^p(E_\alpha^\nu).$$

By (6.15) we conclude that $K_{\nu,\alpha}(x,x)^{1/2} \simeq |E_\alpha(x)|^\nu$ for all real x and the zeros of $A_\alpha(z)$ are separated. We can use Hölder's inequality to conclude that

$$\sum_{A_\alpha(t)=0} \frac{|F(t)|}{(1+|t|)K_{\nu,\alpha}(t,t)^{1/2}} \ll_{p,\alpha,\nu} \left(\sum_{A_\alpha(t)=0} \left| \frac{F(t)}{E_\alpha(t)^\nu} \right|^p \right)^{1/p}.$$

Hence, we only need to show that

$$\sum_{A_\alpha(t)=0} \left| \frac{F(t)}{E_\alpha(t)^\nu} \right|^p \ll_{p,\alpha,\nu} \int_{\mathbb{R}} \left| \frac{F(t)}{E_\alpha(t)^\nu} \right|^p \mathrm{d}t \tag{6.17}$$

for all $F \in \mathcal{H}^p(E_\alpha^\nu)$. Since $\Psi F \in L^p(\mathbb{R}, \mathrm{d}x)$ we can apply the Plancherel-Pólya Theorem to obtain

$$\sum_{A_\alpha(t)=0} |F(t)\Psi(t)|^p \ll_{p,\alpha,\nu} \int_{\mathbb{R}} |F(t)\Psi(t)|^p \, \mathrm{d}t$$

for every $F \in \mathcal{H}^p(E_\alpha^\nu)$. This implies (6.17) and concludes the proof. □

Remark. The proof of item (2) is inspired in the proof of [16, Theorem 20].

6.3 Hermite Polynomials

The Hermite polynomials $\{H_\ell(x)\}_{\ell\geq 0}$ are the orthogonal polynomial associated with the normal distribution $\mathrm{d}\gamma$. They are recursively defined in the following way: $H_0(x) = 1$, $H_1(x) = x$ and $H_\ell(x)$ is defined as the unique monic polynomial of degree ℓ that is orthogonal to $\{H_0, \ldots, H_{\ell-1}\}$ with respect to the inner product generated by $\mathrm{d}\gamma$, that is,

$$\int_{\mathbb{R}} H_\ell(x) H_m(x) \mathrm{d}\gamma(x) = 0$$

if $\ell > m$. It is known that they form a complete orthogonal basis for $L^2(\mathbb{R}, \mathrm{d}\gamma)$ and are dense in $L^q(\mathbb{R}, \mathrm{d}\gamma)$ for every $q \in [1, \infty)$.

They satisfy the following recursion relation

$$H_{\ell+1}(x) = H_1(x)H_\ell(x) - \ell H_{\ell-1}(x)$$

for every $\ell \geq 1$. By an application of this last formula we obtain two useful identities

$$\int_{\mathbb{R}} |H_\ell(x)|^2 \mathrm{d}\gamma(x) = \ell! \qquad \forall \ell \geq 0$$

and

$$H_\ell(0) = \frac{(-1)^{\ell/2}\ell!}{(\ell/2)!2^{\ell/2}} \tag{6.18}$$

if ℓ is even and $H_\ell(0) = 0$ if ℓ is odd. The associated generating function is given by

$$e^{xt - t^2/2} = \sum_{\ell \geq 0} \frac{t^\ell}{\ell!} H_\ell(x), \tag{6.19}$$

where the convergence is uniform for t, x in any fixed compact set of $\mathbb{C}$ (see Lemma 5.2.2). We also have the following integral representation

$$H_\ell(x) = \int_{\mathbb{R}} (x + iy)^\ell \mathrm{d}\gamma(y). \tag{6.20}$$

A very important formula for our purposes is the multiplication formula below

$$\frac{H_\ell(x_1 + \ldots + x_N)}{\ell!} = \frac{1}{N^{\ell/2}} \sum_{\ell_1 + \ldots + \ell_N = \ell} \frac{H_{\ell_1}(\sqrt{N}x_1)}{\ell_1!} \cdots \frac{H_{\ell_N}(\sqrt{N}x_N)}{\ell_N!}, \tag{6.21}$$

which holds for every $(x_1, \ldots, x_N) \in \mathbb{C}^N$. This last formula can be deduced by using formula (6.20) and the fact that $\mathrm{d}\gamma(x) = \mathrm{d}\gamma^N(\sqrt{N}x)$ for every $N > 0$ (see the notation Section 5.1.1).

All these facts about Hermite polynomials can be found in [72, Chapter 5].

Bibliography

[1] A. Baranov, Differentiation in De Branges Spaces and Embedding Theorems, Journal of Mathematical Sciences 101 (2000), no. 2, 2881–2913.

[2] A. Baranov, Estimates of the L^p-Norms of Derivatives in Spaces of Entire Functions, Journal of Mathematical Sciences 129 (2005), no. 4, 3927–2943.

[3] J. Barton, H. Montgomery, and J. Vaaler, Note on a Diophantine Inequality in Several Variables, Proc. Amer. Math. Soc. 129 (2001), no. 2, 337–345.

[4] W. Beckner, Inequalities in Fourier Analysis, Annals of Mathematics 102 (1975), 159–182.

[5] R. Bojanic and R. DeVore, On Polynomials of Best One Sided Approximation, L'Enseignement Mathématique 12 (1966), 139–164.

[6] L. de Branges, *Hilbert Spaces of Entire Functions*, Prentice-Hall Series in Modern Analysis, 1968.

[7] E. Carneiro, Sharp Approximations to the Bernoulli Periodic Functions by Trigonometric Polynomials, J. Approx. Theory 154 (2008), 90–104.

[8] E. Carneiro and V. Chandee, Bounding $\zeta(s)$ in the Critical Strip, Journal of Number Theory 131 (2011), 363–384.

[9] E. Carneiro, V. Chandee, F. Littmann and M. Milinovich, Hilbert Spaces and the Pair Correlation of Zeros of the Riemann Zeta–Function, J. Reine Angew. Math (to appear).

[10] E. Carneiro, V. Chandee and M. Milinovich, Bounding $S(t)$ and $S_1(t)$ on the Riemann Hypothesis, Math. Ann. 356 (2013), 939–968.

[11] E. Carneiro, V. Chandee and M. Milinovich, A Note on the Zeros of Zeta and L–functions, Math. Z. 281 (2015), 315–332.

[12] E. Carneiro and R. Finder, On the Argument of L–functions, Bull. Braz. Math. Soc. 46 (2015), no. 4, 601-620.

[13] E. Carneiro and F. Gonçalves, Extremal Problemas in de Branges Spaces: The Case of Truncated and Odd Functions, Math. Z. 280 (2015), 17–45.

[14] E. Carneiro and F. Littmann, Bandlimited Approximations to the Truncated Gaussian and Applications, Constr. Approx. 38 (2013), 19–57.

[15] E. Carneiro and F. Littmann, Entire Approximations For a Class of Truncated and Odd Functions, J. Fourier Anal. Appl. 19 (2013), 967–996.

[16] E. Carneiro and F. Littmann, Extremal Functions in de Branges and Euclidean Spaces, Adv. Math. 260 (2014), 281–349.

[17] E. Carneiro and F. Littmann, Extremal Functions in de Branges and Euclidean Spaces II, American Journal of Mathematics, (to appear).

[18] E. Carneiro, F. Littmann, and J. Vaaler, Gaussian Subordination for the Beurling–Selberg Extremal Problem, Trans. Amer. Math. Soc. 365 (2013), 3493–3534.

[19] E. Carneiro and J. Vaaler, Some Extremal Functions in Fourier Analysis II, Trans. Amer. Math. Soc. 362 (2010), 5803–5843.

[20] E. Carneiro and J. Vaaler, Some Extremal Functions in Fourier Analysis, III, Constr. Approx. 31, no. 2 (2010), 259–288.

[21] J. Carruth, F. Gonçalves and M. Kelly, On Selberg's Box–Minorant Problem, Preprint (2015).

[22] V. Chandee and K. Soundararajan, Bounding $|\zeta(1/2+it)|$ on the Riemann Hypothesis, Bull. London Math. Soc. 43 (2011), no. 2, 243–250.

[23] K. Chandrasekharan, *Elliptic Functions*, Springer-Verlag, Berlin, 1985.

[24] W. Cohn, Radial Limits and Star–Invariant Subspaces of Bounded Mean Oscillation, Amer. J. Math. 108 (1986), 719–749.

[25] E. Davies, L. Gross and B. Simon, *Hypercontractivity: A Bibliographic Review*, Proceedings of the Hoegh-Krohn Memorial Conference, 370–389.

[26] D. Donoho and B. Logan, Signal Recovery and the Large Sieve, SIAM J. Appl. Math. 52 (1992), no. 2, 577–591.

[27] M. Drmota and R. Tichy, Sequences, Discrepancies and Applications, Lecture Notes in Mathematics 1651, Springer-Verlag, Berlin, 1997

[28] K. Dyakonov, Entire Functions of Exponential Type and Model Subspaces in H^p, Zap. Nauchn. Semin. LOMI 190 (1991), 81–100.

[29] K. Dyakonov, Differentiation in Star-invariant Subspaces I: Boundedness and Compactness, J. Funct. Anal. 192 (2002), 364–386.

[30] J. Epperson, The Hypercontractive Approach to Exactly Bounding an Operator with Complex Gaussian Kernel, Journal of Functional Analysis 87 (1989), 1–30.

[31] E. Feinberg, P. Kasyanov, and N. Zadoianchuk, Fatou's Lemma for Weakly Converging Probabilities, Theory Probab. Appl. 58 (2014), no. 4, 683–689.

[32] P. Gallagher, Pair Correlation of Zeros of the Zeta Function, J. Reine Angew. Math. 362 (1985), 72–86.

[33] M. I. Ganzburg and D. S. Lubinsky, Best Approximating Entire Functions to $|x|^\alpha$ in L^2, Contemporary Mathematics 455 (2008), 93–107.

[34] D. A. Goldston and S. M. Gonek, A Note on S(t) and The Zeros of the Riemann Zeta–function, Bull. London Math. Soc. 39 (2007), 482–486.

[35] F. Gonçalves, Interpolation Formulas with Derivatives in de Branges Spaces, Trans. Amer. Math. Soc (to appear).

[36] F. Gonçalves, A Central Limit Theorem for Operators, Preprint (2015).

[37] F. Gonçalves, M. Kelly and J. Madrid, One–Sided Band–Limited Approximations in Euclidean Spaces of Some Radial Functions, Bull. Braz. Math. Soc. 46 (2015), no. 4, 563–599.

[38] F. Gonçalves an F. Littmann, Interpolation Formulas with Derivatives in de Branges Spaces II, Preprint (2015).

[39] S. Gonek and H. Montgomery, Kronecker's Approximation Theorem, Preprint (2015).

[40] S. Graham and J. Vaaler, A Class of Extremal Functions for the Fourier Transform, Transactions of the American Mathematical Society 265 (1985), no. 1, 283–302.

[41] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Applied and Numerical Harmonic Analysis, 2001.

[42] L. Gross, Logarithmic Sobolev Inequalities, American Journal of Mathematics, vol. 97, No. 4, 1061–1083.

[43] G. Harman, Small Fractional Parts of Additive Forms, Philos. Trans. Roy. Soc. London Ser. A 345 (1993), 327-338.

[44] G. Harman, Metric number theory, London Mathematical Society Monographs, New Series 18, vol. 18, New York, 1998.

[45] A. Haynes, M. Kelly and B. Weiss, Equivalence Relations on Separated Nets Arising From Linear Toral Flows, Proc. London Math. Soc. 109 (2014), no.5, 1203–1228.

[46] I. I. Hirschman and D. V. Widder, *The Convolution Transform*, Princeton Univ. Press, 1955.

[47] J. Holt and J. D. Vaaler, The Beurling–Selberg Extremal Functions for a Ball in the Euclidean space, Duke Mathematical Journal 83 (1996), 203–247.

[48] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, 1983.

[49] M. Kaltenbäck and H. Woracek, Pólya Class Theory for Hermite–Biehler Functions of Finite Order, J. London Math. Soc. 68 (2003), no. 2, 338–354.

[50] M. G. Krein, A Contribution to the Theory of Entire Functions of Exponential Type, Bull. Acad. Sci. URSS. Ser. Math. 11 (1947), 309–326.

[51] X. J. Li and J. D. Vaaler, Some Trigonometric Extremal Functions and the Erdös-Turán Type Inequalities, Indiana Univ. Math. J. 48 (1999), no. 1, 183–236.

[52] E. Lieb, Gaussian Kernels Have Only Gaussian Maximizers, Invent. Math. 102 (1990), 179–208.

[53] F. Littmann, Entire Approximations to the Truncated Powers, Constr. Approx. 22 (2005), no. 2, 273–295.

[54] F. Littmann, Quadrature and Extremal Bandlimited Functions, SIAM J. Math. Anal. 45 (2013), no. 2, 732–747.

[55] F. Littmann, Entire majorants via Euler-Maclaurin Summation, Trans. Amer. Math. Soc. 358 (2006), no. 7, 2821–2836.

[56] F. Littmann, Quadrature and Extremal Bandlimited Functions, SIAM J. Math. Anal. 45 (2013), no. 2, 732–747.

[57] F. Littmann and M. Spanier, Extremal Functions with Vanishing Condition, Constructive Approximation 42 (2015), Issue 2, 209–229.

[58] J. Mashreghi, *Representation Theorems in Hardy spaces*, London Mathematical Society Student texts 74, 2009.

[59] M. Kelly, A Variation on Selberg's Approximation Problem, Mathematika 61 (2015), Issue 01, 213–235.

[60] H. Montgomery, The Analytic Principle of the Large Sieve, Bull. Amer. Math. Soc. 84 (1978), no. 4, 547–567.

[61] H. Montgomery, Distribution of Zeros of the Riemann Zeta Function, Proc. Int. Cong. Math. Vancouver (1974), 379–381.

[62] H. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS no. 84, Amer. Math. Soc., Providence, 1994.

[63] H. Montgomery and R. Vaughan, Hilbert's Inequality, J. London Math. Soc. 8 (1974), no. 2, 73–81.

[64] J. Ortega–Cerdà and K. Seip, Fourier Frames, Ann. of Math. 155 (2002), no. 3, 789–806.

[65] M. Plancherel and G. Polya, Fonctions Entiéres et Intégrales de Fourier Multiples (Seconde partie), Comment. Math. Helv. 10 (1938), 110–163.

[66] I. J. Schoenberg, Metric Spaces and Completely Monotone Functions, Ann. of Math. 39 (1938), no. 4, 811–841

[67] A. Selberg, Remarks on Sieves, Proc. 1972 Number Theory Conference, Univ. of Colorado, Boulder, 205–216.

[68] A. Selberg, Lectures on Sieves, *Atle Selberg: Collected Papers*, Vol. II, Springer-Verlag, 1991.

[69] B. Simon, OPUC on One Foot, Bull. Amer. Math. Soc. 42 (2005), no. 4, 431–460.

[70] E. Stein and R. Shakarchi, *Complex Analysis*, Princeton Lectures in Analysis II, 2007

[71] E. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

[72] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications Volume XXIII, Fourth Edition, 1975.

[73] J. Vaaler, Some Extremal Functions in Fourier Analysis, Bull. Amer. Math. Soc. 12 (1985), 183–215.

[74] B. Von Bahr, On the Convergence of Moments in the Central Limit Theorem, Ann. Math. Statist. 36 (1965), no. 3 (1965), 808–818.

[75] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Mathematical Library Edition, 1995.

[76] F. Weissler, Two–Point Inequalities, the Hermite Semigroup, and the Gauss-Weierstrass Semigroup, Journal of Functional Analysis 32 (1979), 102–121.

Index