Introduction to Integrability in AdS/CFT: Lecture 5

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Introduction

Recall:

- Anomalous dimensions of "long" operators in $\mathcal{N}$=4 SYM are given by a set of Bethe equations!
- Key: all-loop S-matrix
- Based on su(2|2) symmetry
- To compute "finite-size" corrections for "short" operators, need also all-loop S-matrices for bound states
- su(2|2) symmetry is not enough; need also Yangian symmetry

Plan

- Bound states
- Yangian symmetry
- Topics not covered
- Conclusion & outlook

Bound states

Recall: 1-loop SU(2) sector
(a.k.a. Heisenberg ferromagnet)

2-particle state, L large:

$$|\psi\rangle = \sum_{x,\, r} e^{ipx} \left(\cancel{A_{XX}(12)e^{-ikr}} + A_{XX}(21)e^{ikr} \right) |Z \cdots \overbrace{\underset{\uparrow \atop x_1}{X} Z \cdots Z \underset{\uparrow \atop x_2}{X}}^{r} \cdots Z\rangle$$

$\rightarrow 0$ for $r \rightarrow \infty$

$$x = \frac{x_1 + x_2}{2}, \quad r = x_2 - x_1 > 0, \quad p = p_1 + p_2, \quad k = \frac{p_1 - p_2}{2}$$

bound state: $\begin{cases} k = iq\,, \quad q > 0 \\ A_{XX}(12) = 0 \end{cases}$

$$A_{XX}(21) = S(p_2\,, p_1)\, \cancel{A_{XX}(12)} \Rightarrow$$ pole in $S(p_2\,, p_1) = \dfrac{u_2 - u_1 + i}{u_2 - u_1 - i}$

$$\boxed{u_1 = u - \frac{i}{2}\,, \quad u_2 = u + \frac{i}{2}}$$

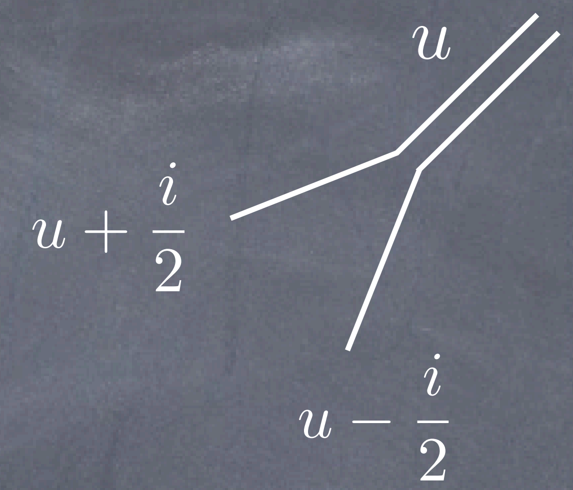
"2-string"

Found already for L=4:

M	$\{u_k\}$	P	E	s	degeneracy (2s+1)
0	-	0	0	2	5
1	1/2	$\pi/2$	2	1	3
1	-1/2	$-\pi/2$	2	1	3
1	0	π	4	1	3
2	i/2,-i/2	π	2	0	1
2	$1/(2\sqrt{3}), -1/(2\sqrt{3})$	0	6	0	1

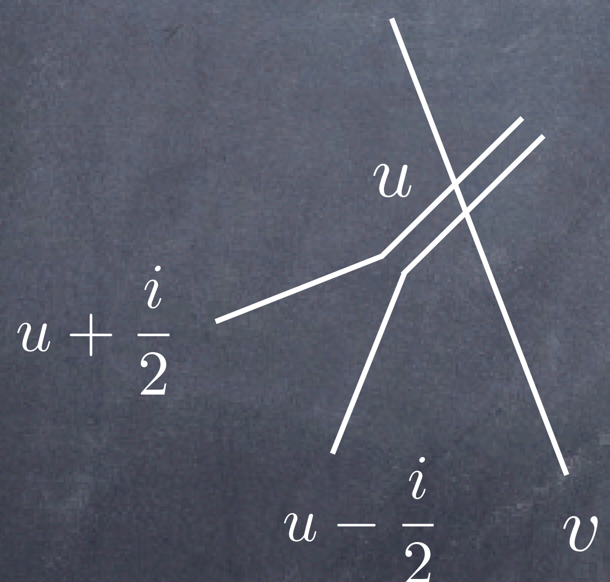
"fusion"

$$A^{(2)\,\dagger}(u) = A^{\dagger}(u+\frac{i}{2})A^{\dagger}(u-\frac{i}{2})$$



S-matrix:

$$A^{(2)\,\dagger}(u)\,A^{\dagger}(v) = S^{(2,1)}(u,v)\,A^{\dagger}(v)\,A^{(2)\,\dagger}(u)$$



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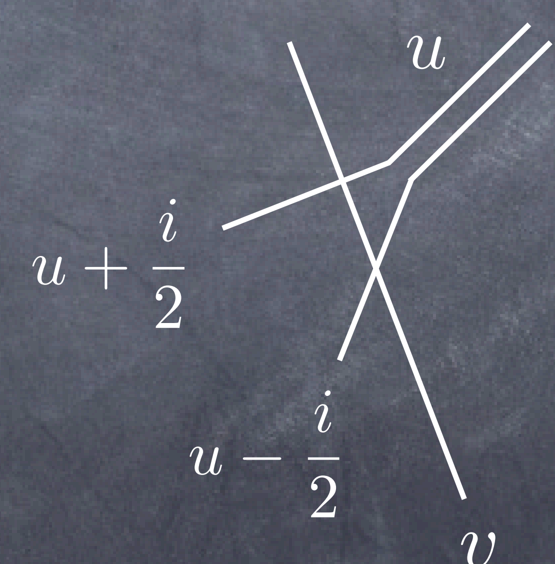

$$S^{(2,1)}(u,v) = S(u-\frac{i}{2},v)S(u+\frac{i}{2},v)$$

“String” hypothesis: for L→ ∞, Bethe roots form Q-strings

$$u_j^{(Q)} = u + i\,\frac{2j - Q - 1}{2}, \quad j = 1, \ldots, Q$$

↑

real “center”

imaginary parts differ by i

$$p_Q = \sum_{j=1}^{Q} \frac{1}{i} \ln e_1(u_j^{(Q)}) = \frac{1}{i} \ln e_Q(u)$$

$$\epsilon_Q = \sum_{j=1}^{Q} \frac{1}{u_j^{(Q)\,2} + \frac{1}{4}} = \frac{Q}{u^2 + \frac{Q^2}{4}} = \frac{4}{Q} \sin^2 \frac{p_Q}{2}$$

bound states of Q particles

S-matrices:

$$S^{(Q_1,Q_2)}(u,v) = \prod_{j_1=1}^{Q_1} \prod_{j_2=1}^{Q_2} S(u_{j_1}^{(Q_1)}, v_{j_2}^{(Q_2)})$$

all-loop SU(2) sector:

$$S(p_1,p_2)=\frac{x_1^+-x_2^-}{x_1^--x_2^+}\,\frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\,\sigma(p_1,p_2)^2$$

pole at $x_1^-=x_2^+$

Similar hypothesis: Q-particle bound states [Dorey, ... 06]

$$x_1^-=x_2^+,\quad x_2^-=x_3^+,\quad \cdots,\quad x_{Q-1}^-=x_Q^+$$

$$X^+\equiv x_1^+,\quad X^-\equiv x_Q^- \qquad X^++\frac{1}{X^+}-X^--\frac{1}{X^-}=\frac{iQ}{g}$$

Proof:

$$x_1^++\frac{1}{x_1^+}-\cancel{x_1^-}-\cancel{\frac{1}{x_1^-}}=\frac{i}{g}$$

$$\cancel{x_2^+}+\cancel{\frac{1}{x_2^+}}-\cancel{x_2^-}-\cancel{\frac{1}{x_2^-}}=\frac{i}{g}$$

$$\vdots$$

$$\cancel{x_Q^+}+\cancel{\frac{1}{x_Q^+}}-x_Q^--\frac{1}{x_Q^-}=\frac{i}{g}$$

Add

all-loop SU(2) sector:

$$S(p_1,p_2)=\frac{x_1^+-x_2^-}{x_1^--x_2^+}\frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\sigma(p_1,p_2)^2$$

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$$X^+\equiv x_1^+,\quad X^-\equiv x_Q^- \qquad X^++\frac{1}{X^+}-X^--\frac{1}{X^-}=\frac{iQ}{g}$$

$$e^{ip_Q}=\frac{X^+}{X^-}$$

$$\mathbb{H}=-ig\left(X^+-\frac{1}{X^+}-X^-+\frac{1}{X^-}\right)=\sqrt{Q^2+16g^2\sin^2\frac{p_Q}{2}}$$

all-loop full SU(2|2):

So far, we have been considering bound states of just 1 type of particle:

$$A^\dagger, A^\dagger A^\dagger, A^\dagger A^\dagger A^\dagger, \ldots$$

But there are in fact 4 types of particles:
2 bosons ($A_1^\dagger, A_2^\dagger$) and 2 fermions ($A_3^\dagger, A_4^\dagger$) !

Example: Q=2

4 bosons:

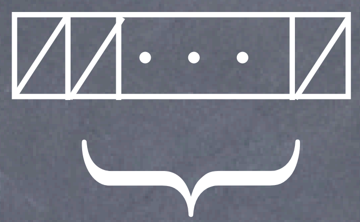
$$A_1^\dagger A_1^\dagger$$
$$A_1^\dagger A_2^\dagger + (1 \leftrightarrow 2)$$
$$A_2^\dagger A_2^\dagger$$
$$A_3^\dagger A_4^\dagger - (3 \leftrightarrow 4)$$

4 fermions:

$$A_1^\dagger A_3^\dagger + (1 \leftrightarrow 3)$$
$$A_1^\dagger A_4^\dagger + (1 \leftrightarrow 4)$$
$$A_2^\dagger A_3^\dagger + (2 \leftrightarrow 3)$$
$$A_2^\dagger A_4^\dagger + (2 \leftrightarrow 4)$$

$\therefore$ 8-dim rep of su(2|2)

Q-particle bound states form
4Q-dimensional totally symmetric reps of su(2|2)



Q boxes

2Q bosons:

Q+1: $A^\dagger_{a_1} \cdots A^\dagger_{a_Q} + \ldots$ $\qquad a_i = 1, 2, \quad \alpha_i = 3, 4$

Q-1: $A^\dagger_{a_1} \cdots A^\dagger_{a_{Q-2}} A^\dagger_{\alpha_1} A^\dagger_{\alpha_2} + \ldots$

2Q fermions:

$$A^\dagger_{a_1} \cdots A^\dagger_{a_{Q-1}} A^\dagger_{\alpha} + \ldots$$

ZF operators: $A_J^{(Q)\,\dagger}(p), \quad J = 1, \ldots, 4Q$

Want all-loop S-matrices

$$A_I^{(Q_1)\,\dagger}(p_1)\,A_J^{(Q_2)\,\dagger}(p_2) = S^{(Q_1,Q_2)\;I'\,J'}_{\;\;\;\;\;\;\;\;\;\;\;\;I\;J}(p_1,p_2)\,A_{J'}^{(Q_2)\,\dagger}(p_2)\,A_{I'}^{(Q_1)\,\dagger}(p_1)$$

- Fusion procedure does not seem to work [Arutyunov & Frolov 08]

- su(2|2) symmetry is not enough

Example: $Q_1=Q_2=2$

- Can find action of su(2|2) generators on ZF operators

- Demanding that su(2|2) generators commute with 2-particle scattering does not determine all amplitudes - need 1 additional relation!

Yangian symmetry

Yangian

[Drinfeld 85, ...]

Generators $\mathbb{J}^A, \hat{\mathbb{J}}^A, \ldots$

$$[\mathbb{J}^A, \mathbb{J}^B] = f^{AB}_C \mathbb{J}^C, \quad [\mathbb{J}^A, \hat{\mathbb{J}}^B] = f^{AB}_C \hat{\mathbb{J}}^C$$

+ Jacobi + Serre relations

Nontrivial coproduct

$$\Delta(\hat{\mathbb{J}}^A) = \hat{\mathbb{J}}^A \otimes \mathbb{I} + \mathbb{I} \otimes \hat{\mathbb{J}}^A + \frac{\alpha}{2} f^A_{BC} \mathbb{J}^B \mathbb{J}^C$$

Motivation from QISM: g-invariant R-matrix

monodromy matrix $T_a(u) = R_{a1}(u) \cdots R_{aN}(u)$

algebra $R_{ab}(u-v)\, T_a(u)\, T_b(v) = T_b(v)\, T_a(u)\, R_{ab}(u-v)$

coproduct:

1 site $\longrightarrow$ 2 sites: $R_{a1}(u) \to R_{a1}(u) R_{a2}(u)$

i.e., $\Delta(T_a(u)) = T_a(u) \otimes T_a(u)$

large u expansion of monodromy matrix:

$$\ln T_a(u) = -\frac{1}{u} t_A \mathbb{J}^A + \frac{1}{u^2} t_A \hat{\mathbb{J}}^A + \ldots$$

coproduct for $T_a(u)$ $\Rightarrow$ coproduct for $\hat{\mathbb{J}}^A$

For su(2|2): Evaluation representation [Beisert 07]

$$\hat{\mathbb{J}}^A = -\frac{1}{2} igu\, \mathbb{J}^A \qquad u = \frac{1}{2}\left(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-}\right)$$

Nontrivial coproduct, e.g.

$$\Delta(\hat{\mathbb{L}}_2{}^1) = \hat{\mathbb{L}}_2{}^1 \otimes \mathbb{I} + \mathbb{I} \otimes \hat{\mathbb{L}}_2{}^1$$
$$+\frac{1}{2}\mathbb{L}_2{}^c \otimes \mathbb{L}_c{}^1 - \frac{1}{2}\mathbb{L}_c{}^1 \otimes \mathbb{L}_2{}^c - \frac{1}{2}\mathbb{Q}_2^{\dagger\gamma} \otimes \mathbb{Q}_\gamma{}^1 - \frac{1}{2}\mathbb{Q}_\gamma{}^1 \otimes \mathbb{Q}_2^{\dagger\gamma}$$

Action on fundamental ZF operators $A_I^{(1)\dagger}(p)$:

$$\hat{\mathbb{L}}_2{}^1\, A_1^{(1)\dagger}(p) = -\frac{1}{2} igu\, A_2^{(1)\dagger}(p) + A_1^{(1)\dagger}(p)\, \hat{\mathbb{L}}_2{}^1 - \frac{1}{2}A_1^{(1)\dagger}(p)\, \mathbb{L}_2{}^1 + \frac{1}{2}A_2^{(1)\dagger}(p)\, (\mathbb{L}_1{}^1 - \mathbb{L}_2{}^2)$$
$$+\frac{1}{2}cA_4^{(1)\dagger}(p)\, \mathbb{Q}_3{}^1 - \frac{1}{2}cA_3^{(1)\dagger}(p)\, \mathbb{Q}_4{}^1 - \frac{1}{2}aA_3^{(1)\dagger}(p)\, \mathbb{Q}_2^{\dagger 3} - \frac{1}{2}aA_4^{(1)\dagger}(p)\, \mathbb{Q}_2^{\dagger 4}$$

$$\hat{\mathbb{L}}_2{}^1\, A_2^{(1)\dagger}(p) = A_2^{(1)\dagger}(p)\, \hat{\mathbb{L}}_2{}^1 + \frac{1}{2}A_2^{(1)\dagger}(p)\, \mathbb{L}_2{}^1\,, \cdots$$

Yangian generators $\hat{\mathbb{J}}^A$ commute with 2-particle scattering!

Recall: su(2|2) not enough to determine bound-state S-matrices

Imposing that also Yangian generators $\hat{\mathbb{J}}^A$ commute with 2-particle scattering completely determines bound-state S-matrices!

[de Leeuw 08; Arutyunov, de Leeuw & Torrielli 09]

Topics not covered

- AdS (string) “side”
- Finite-size effects

Changrim Ahn

Boundary

Another interesting class of operators:
"determinant-like"

$$\epsilon_{i_1 \cdots i_N} \epsilon^{j_1 \cdots j_N} Y^{i_1}_{j_1} \cdots Y^{i_{N-1}}_{j_{N-1}} (Z \cdots Z \chi Z \cdots Z)^{i_N}_{j_N}$$

- local & gauge-invariant, but no trace
- correspond to open string attached to D-brane
- dilatation operator is an integrable open spin-chain Hamiltonian [Berenstein & Vazquez 05, ...]
- all-loop boundary S-matrix & Bethe equations

[Hofman & Maldacena 07, Galleas 09, ...]

- “Twisted” AdS_5/CFT_4

[Leigh & Strassler 95,
Lunin & Maldacena 05,
Frolov 05, ...]

- 3-parameter deformation of S^5 / SU(4) R-symmetry
  - still integrable!
  - all-loop S-matrix & Bethe equations [Beisert & Roiban 05, Ahn, Bajnok, Bombardelli & N 10]
  - finite-size corrections, ...

- N-point functions & space-time scattering amplitudes

-focus of much of the recent activity
in AdS/CFT integrability

-space-time scattering amplitudes in $\mathcal{N}$=4 SYM
have Yangian symmetry [Drummond et al; Arkani-Hamed, ...]

-mysterious connections with quantum integrability
(TBA, etc.) at strong coupling [Alday & Maldacena, ...]

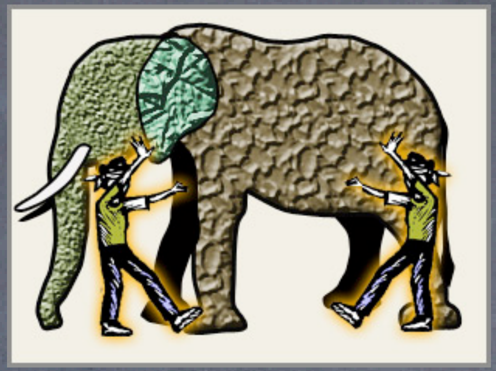
$\vdots$

- AdS_4/CFT_3

Another correspondence, in one dimension lower:

[Aharony, Bergman, Jafferis & Maldacena 08]

strong coupling: classical type IIA string on $AdS_4 \times CP^3$



weak coupling: $\mathcal{N}$=6 Chern-Simons CFT_3

- has planar limit
- 2-loop dilatation operator is integrable [Minahan & Zarembo 08]
- all-loop S-matrix & Bethe equations [Gromov & Vieira 08, Ahn & N 08]

⋮

Conclusions & outlook

- $\mathcal{N}$=4 SYM is a 4D CFT

- The problem of computing anomalous dimensions (2-point functions) of single-trace operators in the planar limit seems to be integrable

- The key to exploiting integrability: all-loop S-matrices

- The key to the key: Yangian symmetry

- However, there is still no understanding (e.g. R-matrix) of this integrability

- There are many indications that this integrability extends much further: boundaries, twisting, N-point functions, scattering amplitudes, AdS_4/CFT_3, ...
- Maybe planar $\mathcal{N}=4$ SYM & $\mathcal{N}=6$ SCS can be "solved" ?
- Much still to be learned about integrability in D>2 CFT!
- Already used every known tool in 2D integrability; may require developing some new tools
- What are you waiting for?