Introduction to Integrability in AdS/CFT: Lecture 4

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Introduction

Recall:

- 1-loop dilatation operator for all single-trace operators in $\mathcal{N}$=4 SYM is integrable

$\Rightarrow$

- 1-loop anomalous dimensions are given by a set of Bethe equations

Higher loops?

Plan

- Today:
  - dilatation operator at higher loops
  - all-loop asymptotic S-matrix
  - all-loop asymptotic Bethe equations
- Final: bound states & their S-matrices

dilatation operator at higher loops

Dilatation operator has been computed perturbatively in various sectors at small (~2,3) loop order

[Beisert, Kristjansen & Staudacher 03, ...]

$$\mathcal{D} = \sum_{n=0}^{\infty} \left(\frac{\lambda}{16\pi^2} \right)^n \mathcal{D}^{(n)}$$

n: loop order

SU(2) sector:

	# coupled nearest neighbors
$\mathcal{D}^{(1)} = 2\sum_{k=1}^{L} (I - \mathcal{P}_{k,k+1})$	2
$\mathcal{D}^{(2)} = 2\sum_{k=1}^{L} (-4I + 6\mathcal{P}_{k,k+1} - \mathcal{P}_{k,k+1}\mathcal{P}_{k+1,k+2} - \mathcal{P}_{k+1,k+2}\mathcal{P}_{k,k+1})$	3
$\vdots$	
$\mathcal{D}^{(n)}$	n+1

Interaction range grows with loop order!

Seem to have "perturbative integrability":

higher charges also constructed perturbatively

$$Q_k = \sum_{n=1}^{\infty} \lambda^n Q_k^{(n)}$$

$$Q_1 = \mathcal{D} \qquad\qquad [Q_k\,, Q_l] = 0$$

i.e.,

$$0 = \left[\left(\lambda Q_k^{(1)} + \lambda^2 Q_k^{(2)} + \cdots\right), \left(\lambda Q_l^{(1)} + \lambda^2 Q_l^{(2)} + \cdots\right)\right]$$

$$= \lambda^2 \underbrace{\left[Q_k^{(1)}, Q_l^{(1)}\right]}_{\equiv 0} + \lambda^3 \underbrace{\left(\left[Q_k^{(1)}, Q_l^{(2)}\right] + \left[Q_k^{(2)}, Q_l^{(1)}\right]\right)}_{\equiv 0} + \cdots$$

- Does not seem to come from an R-matrix
- New kind of integrability !?
- Finding $\mathcal{D}$ to all loops seems (at least for now) hopeless...

all-loop asymptotic
S-matrix

R-matrix does not seem to work... Try S-matrix approach!

long-range interaction $\Rightarrow$

S-matrix is "asymptotic":
valid only for widely-separated particles

$$\Psi^{(12)}(x_1,x_2) \sim e^{i(p_1x_1+p_2x_2)} + S(p_2,p_1)e^{i(p_2x_1+p_1x_2)} \qquad x_1 \ll x_2$$

Can compute S-matrix perturbatively, from known $\mathcal{D}$,
by introducing "fudge" functions: [Staudacher 04]

$$\Psi^{(12)}(x_1,x_2) = e^{i(p_1x_1+p_2x_2)} f(x_2-x_1,p_1,p_2) + S(p_2,p_1)e^{i(p_2x_1+p_1x_2)} g(x_2-x_1,p_1,p_2)$$

$$f(x,p_1,p_2) = 1 + \sum_{n=0}^{\infty} \lambda^{n+x} f^{(n)}(p_1,p_2)$$

$$\Rightarrow \qquad S(p_1,p_2) = \sum_{n=0}^{\infty} \lambda^n S^{(n)}(p_1,p_2)$$

But will not take us far,
since we hardly know $\mathcal{D}$....

Audacious idea: "guess" the exact S-matrix! [Beisert 05]

(analogy: sine-Gordon [Zamolodchikov² 79])

Guiding principle: symmetry

Global symmetry algebra is psu(2,2|4)

But it is partially broken by the vacuum $|0\rangle = |Z^L\rangle$:

$$so(6) \to so(4) = su(2) \times su(2)$$

$$psu(2,2|4) \to psu(2|2) \times psu(2|2) \ltimes \mathbb{R}$$

$\uparrow$

common central charge

$$\mathbb{H} \equiv \mathcal{D} - R_{56}$$

$$R_{56}|0\rangle = L|0\rangle \Rightarrow \quad \mathbb{H}|0\rangle = 0 \quad ✓$$

Elementary excitations:

$$\sum_{x=1}^{L} e^{ipx} | \overset{1}{\overset{\downarrow}{Z}} \cdots \overset{x}{\overset{\downarrow}{\chi}} \cdots \overset{L}{\overset{\downarrow}{Z}} \rangle$$

$$\chi \in \{\phi_{a\dot{a}}\,, D_{\alpha\dot{\alpha}} Z\,; \psi_{\alpha\dot{a}}\,, \bar{\psi}_{a\dot{\alpha}}\}$$

$$a = 1, 2, \quad \dot{a} = \dot{1}, \dot{2}$$

$$\alpha = 3, 4, \quad \dot{\alpha} = \dot{3}, \dot{4}$$

$\boxslash \qquad \boxslash$

$$(2|2) \times (2|2) = (4 + 4|4 + 4) = (8|8)$$

Fundamental reps $\quad su(2|2) \times su(2|2)$

ZF operators: $\quad A^{\dagger}_{k\dot{k}}(p) = A^{\dagger}_{k}(p) \otimes \dot{A}^{\dagger}_{\dot{k}}(p)$

$$k = 1, \ldots, 4, \quad \dot{k} = \dot{1}, \ldots, \dot{4}$$

Focus on single copy of $su(2|2)$

Generators:

bosonic $\mathbb{L}_a{}^b\,,\quad \mathbb{R}_\alpha{}^\beta$ $\qquad a,b=1,2,\quad \alpha,\beta=3,4$

SUSY $\mathbb{Q}_\alpha{}^a,\quad \mathbb{Q}_a^{\dagger\alpha}$

central charges $\mathbb{H}\,,\quad \mathbb{C}\,,\quad \mathbb{C}^\dagger$

Algebra:

$$\left[\mathbb{L}_a{}^b\,,\mathbb{J}_c\right]=\delta_c^b\mathbb{J}_a-\frac{1}{2}\delta_a^b\mathbb{J}_c\,,\qquad \left[\mathbb{R}_\alpha{}^\beta\,,\mathbb{J}_\gamma\right]=\delta_\gamma^\beta\mathbb{J}_\alpha-\frac{1}{2}\delta_\alpha^\beta\mathbb{J}_\gamma$$

$$\left[\mathbb{L}_a{}^b\,,\mathbb{J}^c\right]=-\delta_a^c\mathbb{J}^b+\frac{1}{2}\delta_a^b\mathbb{J}^c\,,\qquad \left[\mathbb{R}_\alpha{}^\beta\,,\mathbb{J}^\gamma\right]=-\delta_\alpha^\gamma\mathbb{J}^\beta+\frac{1}{2}\delta_\alpha^\beta\mathbb{J}^\gamma$$

$$\left\{\mathbb{Q}_\alpha{}^a\,,\mathbb{Q}_\beta{}^b\right\}=\epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}\,,\qquad \left\{\mathbb{Q}_a^{\dagger\alpha}\,,\mathbb{Q}_b^{\dagger\beta}\right\}=\epsilon^{\alpha\beta}\epsilon_{ab}\mathbb{C}^\dagger$$

$$\left\{\mathbb{Q}_\alpha{}^a\,,\mathbb{Q}_b^{\dagger\beta}\right\}=\delta_b^a\mathbb{R}_\alpha{}^\beta+\delta_\alpha^\beta\mathbb{L}_b{}^a+\frac{1}{2}\delta_b^a\delta_\alpha^\beta\mathbb{H}$$

Action of symmetry generators on ZF operators:

bosonic:

$$[\mathbb{L}_a{}^b\,, A_c^\dagger(p)] = (\delta_c^b \delta_a^d - \frac{1}{2}\delta_a^b \delta_c^d) A_d^\dagger(p)\,, \quad [\mathbb{L}_a{}^b\,, A_\gamma^\dagger(p)] = 0$$

$$[\mathbb{R}_\alpha{}^\beta\,, A_\gamma^\dagger(p)] = (\delta_\gamma^\beta \delta_\alpha^\delta - \frac{1}{2}\delta_\alpha^\beta \delta_\gamma^\delta) A_\delta^\dagger(p)\,, \quad [\mathbb{R}_\alpha{}^\beta\,, A_c^\dagger(p)] = 0$$

SUSY:

$$\mathbb{Q}_\alpha{}^a\, A_b^\dagger(p) = e^{-ip/2}\left[a(p)\delta_b^a A_\alpha^\dagger(p) + A_b^\dagger(p)\,\mathbb{Q}_\alpha{}^a\right]$$

$$\mathbb{Q}_\alpha{}^a\, A_\beta^\dagger(p) = e^{-ip/2}\left[b(p)\epsilon_{\alpha\beta}\epsilon^{ab} A_b^\dagger(p) - A_\beta^\dagger(p)\,\mathbb{Q}_\alpha{}^a\right]$$

$$\mathbb{Q}_a^{\dagger\alpha}\, A_b^\dagger(p) = e^{ip/2}\left[c(p)\epsilon_{ab}\epsilon^{\alpha\beta} A_\beta^\dagger(p) + A_b^\dagger(p)\,\mathbb{Q}_a^{\dagger\alpha}\right]$$

$$\mathbb{Q}_a^{\dagger\alpha}\, A_\beta^\dagger(p) = e^{ip/2}\left[d(p)\delta_\beta^\alpha A_a^\dagger(p) - A_\beta^\dagger(p)\,\mathbb{Q}_a^{\dagger\alpha}\right]$$

central:

$$\mathbb{C}\, A_i^\dagger(p) = e^{-ip}\left[a(p)b(p)A_i^\dagger(p) + A_i^\dagger(p)\,\mathbb{C}\right]$$

$$\mathbb{C}^\dagger\, A_i^\dagger(p) = e^{ip}\left[c(p)d(p)A_i^\dagger(p) + A_i^\dagger(p)\,\mathbb{C}^\dagger\right]$$

$$\mathbb{H}\, A_i^\dagger(p) = \left[a(p)d(p) + b(p)c(p)\right] A_i^\dagger(p) + A_i^\dagger(p)\,\mathbb{H}$$

Determination of a, b, c, d:

- $A_i^\dagger(p)|0\rangle$ form a rep of algebra $\Rightarrow$ $ad - bc = 1$
- rep unitary $\Rightarrow$ $d = a^*\,, c = b^*$
- $\mathbb{C}$ on 2-particle states $\Rightarrow$ $ab = ig(e^{ip} - 1)$

  g constant

Consistent with

$$a = \sqrt{g}\eta\,, \quad b = \sqrt{g}\frac{i}{\eta}\left(\frac{x^+}{x^-} - 1\right)\,, \quad c = -\sqrt{g}\frac{\eta}{x^+}\,, \quad d = \sqrt{g}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right)$$

where

$$\boxed{x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\,, \qquad \frac{x^+}{x^-} = e^{ip}}\,, \qquad \eta = e^{i\frac{p}{4}}\sqrt{i(x^- - x^+)}$$

For 1-particle states:

$$\mathbb{H} = ig\left(x^- - \frac{1}{x^-} - x^+ + \frac{1}{x^+}\right) = \sqrt{1 + 16g^2\sin^2\frac{p}{2}}$$

Compare with 1 loop to determine g:

$$\begin{aligned}\Delta - J &= (\Delta_0 + \gamma) - J \\ &= 1 + \gamma \\ &= 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} + \cdots\end{aligned} \quad \longleftarrow \quad \boxed{g = \frac{\sqrt{\lambda}}{4\pi}}$$

Consistent with

$$a = \sqrt{g}\eta\,, \quad b = \sqrt{g}\frac{i}{\eta}\left(\frac{x^+}{x^-} - 1\right)\,, \quad c = -\sqrt{g}\frac{\eta}{x^+}\,, \quad d = \sqrt{g}\frac{x^+}{i\eta}\left(1 - \frac{x^-}{x^+}\right)$$

where

$$\boxed{x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}\,, \qquad \frac{x^+}{x^-} = e^{ip}}\,, \qquad \eta = e^{i\frac{p}{4}}\sqrt{i(x^- - x^+)}$$

For 1-particle states:

$$\mathbb{H} = ig\left(x^- - \frac{1}{x^-} - x^+ + \frac{1}{x^+}\right) = \boxed{\sqrt{1 + 16g^2\sin^2\frac{p}{2}}}$$

- exact!
- $\mathbb{C}, \mathbb{C}^\dagger$ essential

Compare with 1 loop to determine g:

$$\begin{aligned}\Delta - J &= (\Delta_0 + \gamma) - J \\ &= 1 + \gamma \\ &= 1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} + \cdots\end{aligned}$$

$$\boxed{g = \frac{\sqrt{\lambda}}{4\pi}}$$

Useful parametrization in terms of Jacobi elliptic functions:

$$p = 2\,\mathrm{am}\,z\,, \quad x^{\pm} = \frac{1}{4g}\left(\frac{\mathrm{cn}\,z}{\mathrm{sn}\,z} \pm i\right)(1 + \mathrm{dn}\,z)$$

elliptic modulus $k = -16g^2$

periods $2\omega_1$ (real), $2\omega_2$ (imaginary)

[Arutyunov & Frolov 07]

Finally, can determine exact S-matrix:

$$A_i^\dagger(p_1)\, A_j^\dagger(p_2) = S_{i\,j}^{i'\,j'}(p_1, p_2)\, A_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)$$

Demand that the symmetry generators $\mathcal{J}$ commute with 2-particle scattering

i.e, consider $\mathcal{J}\, A_i^\dagger(p_1)\, A_j^\dagger(p_2)|0\rangle$

Can first exchange $A^\dagger$'s, then move $\mathcal{J}$ to right $\mathcal{J}|0\rangle = 0$

Or first move $\mathcal{J}$ to right, then exchange $A^\dagger$'s

Consistency $\Rightarrow$ linear equations for S-matrix elements bosonic generators $\Rightarrow$

$$S^{a\,a}_{a\,a} = A\,, \quad S^{\alpha\,\alpha}_{\alpha\,\alpha} = D\,,$$

$$S^{a\,b}_{a\,b} = \frac{1}{2}(A - B)\,, \quad S^{b\,a}_{a\,b} = \frac{1}{2}(A + B)\,,$$

$$S^{\alpha\,\beta}_{\alpha\,\beta} = \frac{1}{2}(D - E)\,, \quad S^{\beta\,\alpha}_{\alpha\,\beta} = \frac{1}{2}(D + E)\,,$$

$$S^{\alpha\,\beta}_{a\,b} = -\frac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}\,C\,, \quad S^{a\,b}_{\alpha\,\beta} = -\frac{1}{2}\epsilon^{ab}\epsilon_{\alpha\beta}\,F\,,$$

$$S^{a\,\alpha}_{a\,\alpha} = G\,, \quad S^{\alpha\,a}_{a\,\alpha} = H\,, \quad S^{a\,\alpha}_{\alpha\,a} = K\,, \quad S^{\alpha\,a}_{\alpha\,a} = L\,,$$

$a\,, b \in \{1\,, 2\}, \quad a \neq b$ $\qquad$ $\alpha\,, \beta \in \{3\,, 4\}, \quad \alpha \neq \beta$

SUSY generators $\Rightarrow$

$$A = S_0 \frac{x_2^- - x_1^+}{x_2^+ - x_1^-} \frac{\eta_1 \eta_2}{\tilde{\eta}_1 \tilde{\eta}_2} \,,$$

$$B = -S_0 \left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2 \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)} \right] \frac{\eta_1 \eta_2}{\tilde{\eta}_1 \tilde{\eta}_2} \,,$$

$$C = S_0 \frac{2i x_1^- x_2^- (x_1^+ - x_2^+) \eta_1 \eta_2}{x_1^+ x_2^+ (x_1^- - x_2^+)(1 - x_1^- x_2^-)} \,, \qquad D = -S_0 \,,$$

$$E = S_0 \left[1 - 2 \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)} \right] \,,$$

$$F = S_0 \frac{2i (x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-) \tilde{\eta}_1 \tilde{\eta}_2} \,,$$

$$G = S_0 \frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)} \frac{\eta_1}{\tilde{\eta}_1} \,, \qquad H = S_0 \frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)} \frac{\eta_1}{\tilde{\eta}_2} \,,$$

$$K = S_0 \frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)} \frac{\eta_2}{\tilde{\eta}_1} \,, \qquad L = S_0 \frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)} \frac{\eta_2}{\tilde{\eta}_2}$$

$$\eta_1 = \eta(p_1) e^{ip_2/2} \,, \quad \eta_2 = \eta(p_2) \,, \quad \tilde{\eta}_1 = \eta(p_1) \,, \quad \tilde{\eta}_2 = \eta(p_2) e^{ip_1/2}$$

- Satisfies YBE
- Does not have "difference" property; can be mapped to R-matrix for Hubbard model [Shastry 86]
- Has Yangian symmetry [more next lecture] [Beisert 07]
- su(2|2) symmetry determines S-matrix only up to overall scalar factor S_0
- Full S-matrix: two copies of su(2|2) S-matrix

$$\mathbb{S} = S \otimes \dot{S}$$

Determination of scalar factor:

- Assume unitarity

$$S_{12}(p_1,p_2)\,S_{21}(p_2,p_1)=\mathbb{I}$$

$\Rightarrow$

$$\boxed{S_0(p_1,p_2)\,S_0(p_2,p_1)=1}$$

- Assume crossing symmetry [Janik 06, ...]

$$\mathcal{C}_1^{-1}\,S_{12}^{t_1}(z_1,z_2)\,\mathcal{C}_1\,S_{12}(z_1+\omega_2,z_2)=\mathbb{I}$$

$\Rightarrow$

$$\boxed{S_0(z_1,z_2)\,S_0(z_1,z_2-\omega_2)=\frac{\left(\frac{1}{x_1^-}-x_2^-\right)(x_1^--x_2^+)}{\left(\frac{1}{x_1^+}-x_2^-\right)(x_1^+-x_2^+)}}$$

- Can solve under some additional physical assumptions

$$S_0(p_1,p_2)^2 = \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\,\frac{1-\frac{1}{x_1^+ x_2^-}}{1-\frac{1}{x_1^- x_2^+}}\,\sigma(p_1,p_2)^2$$

"dressing phase":

$$\sigma(x_1^\pm\,,x_2^\pm) = \frac{R(x_1^+\,,x_2^+)\;R(x_1^-\,,x_2^-)}{R(x_1^+\,,x_2^-)\;R(x_1^-\,,x_2^+)}\,, \qquad R(x_1\,,x_2) = e^{i[\chi(x_1\,,x_2)-\chi(x_2\,,x_1)]}$$

$$\chi(x_1\,,x_2) = -i\oint_{|z_1|=1}\frac{dz_1}{2\pi}\oint_{|z_2|=1}\frac{dz_2}{2\pi}\,\frac{\ln\Gamma\left(1+ig(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2})\right)}{(x_1-z_1)(x_2-z_2)}$$

[BES, BHL, DHM;
review Vieira & Volin 10]

→ Changrim Ahn

all-loop asymptotic
Bethe equations

Recall: Bethe-Yang equations

$$e^{ip_k L} = -\Lambda(p_k; p_1, \ldots, p_M)\,, \quad k = 1, \ldots, M$$

$\Lambda(p; p_1, \ldots, p_M)$ eigenvalues of (inhomogeneous) transfer matrix

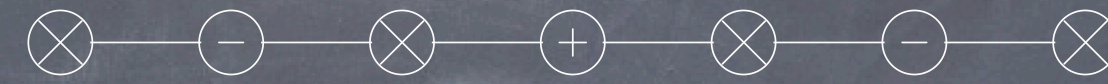
$$t(p; p_1, \ldots, p_M) = \mathrm{tr}_0\, \mathbb{S}_{01}(p, p_1) \cdots \mathbb{S}_{0M}(p, p_M)$$

Can determine using (nested) algebraic Bethe ansatz, etc.

$$U_j(x_{j,k}) \prod_{\substack{j'=1 \\ (j',k')\neq(j,k)}}^{7} \prod_{k'=1}^{K_{j'}} \frac{u_{j,k} - u_{j',k'} + \frac{i}{2} A_{j,j'}}{u_{j,k} - u_{j',k'} - \frac{i}{2} A_{j,j'}} = 1\,, \quad j = 1, \ldots, 7$$

$$u_{j,k} = g\Big(x_{j,k} + \frac{1}{x_{j,k}}\Big)\,, \quad u_{j,k} \pm \frac{i}{2} = g\Big(x^{\pm}_{j,k} + \frac{1}{x^{\pm}_{j,k}}\Big)$$

Cartan matrix



$$A=\begin{pmatrix} & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & \end{pmatrix}$$

$$U_2=U_6=1\,,\quad U_1(x)=U_3^{-1}(x)=U_5^{-1}(x)=U_7(x)=\prod_{k=1}^{K_4} S_{\rm aux}(x_{4,k},x)$$

$$U_4(x)=U_{\rm s}(x)\left(\frac{x^-}{x^+}\right)^L\prod_{k=1}^{K_1} S_{\rm aux}^{-1}(x,x_{1,k})\prod_{k=1}^{K_3} S_{\rm aux}(x,x_{3,k})\prod_{k=1}^{K_5} S_{\rm aux}(x,x_{5,k})\prod_{k=1}^{K_7} S_{\rm aux}^{-1}(x,x_{7,k})$$

$$S_{\rm aux}(x_1,x_2)=\frac{1-1/x_1^+x_2}{1-1/x_1^-x_2}\,,\qquad U_{\rm s}(x)=\prod_{k=1}^{K_4}\sigma(x,x_{4,k})^2$$

$$\gamma=2ig\sum_{k=1}^{K_4}\left(\frac{1}{x_{4,k}^+}-\frac{1}{x_{4,k}^-}\right)$$

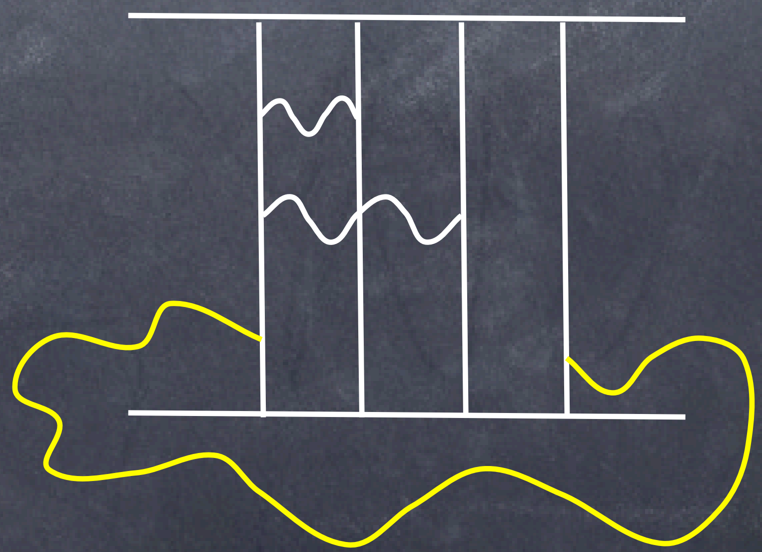
- First conjectured! [Beisert & Staudacher 05]
- Then derived [Beisert 05, Martins & Mello 07]

- In weak-coupling limit $\lambda \to 0$, reduce to 1-loop Bethe equations ✓

- In thermodynamic limit $L, \lambda \to \infty$, reduce to eqs from algebraic curve

- Valid only for "long" operators

- The problem with "short" operators: for length L, "wrapping" corrections (to the anomalous dimensions) appear at loop-order L, due to interactions of range L+1



Epilogue

- We don't know the all-loop dilatation operator for single-trace operators in $\mathcal{N}$=4 SYM
- Nevertheless, we know that the (all-loop) anomalous dimensions of "long" operators are given by a set of BEs!
- Key: all-loop S-matrix
- Based on su(2|2) symmetry
- To compute "finite-size" corrections for "short" operators, need also all-loop S-matrices for bound states
- su(2|2) symmetry is not enough; need also Yangian symmetry

- final lecture!