Introduction to Integrability in AdS/CFT:
Lecture 2

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Introduction

Recall:

- $\mathcal{N}$=4 SYM is a (super)conformal field theory
- In planar limit, has 1 free parameter λ
- We want to determine $\Delta(\lambda)$ for all (local, gauge-invariant, single-trace) operators, for all λ
- 1-loop (weak coupling) mixing matrix for scalars

SU(2) subsector: $\operatorname{tr} X(x)^M Z(x)^{L-M} + \ldots$

$$\boxed{\Gamma = \frac{\lambda}{8\pi^2} H \,, \quad H = \sum_{l=1}^{L} (1 - \mathcal{P}_{l,l+1})}$$

quantum
spin chain
Hamiltonian

Problem: to determine eigenvectors & eigenvalues

Solved exactly!

[Bethe 31]

Approach used by Bethe is now known as
“coordinate” Bethe ansatz

A different approach was developed later, called
Quantum Inverse Scattering Method (QISM)
& “algebraic” Bethe ansatz

[Yang, Gaudin, Baxter, Zamolodchikov^2, Faddeev, Kulish, Sklyanin, ...]

- Each approach has its advantages/disadvantages
- It is essential to learn both for AdS/CFT!

(also for applications in
statistical mechanics, condensed matter,...)

Plan

- Today: quantum integrability "toolkit":
    - quantum spin chains
    - Yang-Baxter equations
    - quantum inverse scattering method
    - algebraic Bethe ansatz
    - analytical Bethe ansatz
- Subsequent: coordinate Bethe ansatz, & application to $\mathcal{N}$=4 SYM

Quantum spin chains

Example: system of L fixed particles with spin 1/2

L=1: The Hilbert space is $V = \mathcal{C}^2$ 2 dims

with elements $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x_i \in \mathcal{C}$

The observables are the Pauli matrices $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$

For L>1, need tensor product

For vectors:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(\begin{array}{c} x_1 y_1 \\ x_1 y_2 \\ \hline x_2 y_1 \\ x_2 y_2 \end{array} \right)$$

Permutation matrix

$$\mathcal{P}_{12} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{P}_{12} \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right] = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

check:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{c} x_1 y_1 \\ x_1 y_2 \\ \hline x_2 y_1 \\ x_2 y_2 \end{array} \right) = \left(\begin{array}{c} x_1 y_1 \\ x_2 y_1 \\ \hline x_1 y_2 \\ x_2 y_2 \end{array} \right) \quad \checkmark$$

Tensor product of matrices:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \otimes \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} x_{11}y_{11} & x_{11}y_{12} & x_{12}y_{11} & x_{12}y_{12} \\ x_{11}y_{21} & x_{11}y_{22} & x_{12}y_{21} & x_{12}y_{22} \\ \hline x_{21}y_{11} & x_{21}y_{12} & x_{22}y_{11} & x_{22}y_{12} \\ x_{21}y_{21} & x_{21}y_{22} & x_{22}y_{21} & x_{22}y_{22} \end{array}\right)$$

L=2: The Hilbert space is $\underset{\substack{\uparrow \\ 1}}{V} \otimes \underset{\substack{\uparrow \\ 2}}{V}$ 2^2 dims

The observables are

$$\vec{\sigma}_1 \equiv \vec{\sigma} \otimes I\,, \quad \vec{\sigma}_2 \equiv I \otimes \vec{\sigma} \qquad\qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Related by permutation matrix

$$\vec{\sigma}_2 = \mathcal{P}_{12}\; \vec{\sigma}_1\; \mathcal{P}_{12}$$

$$\vec{\sigma}_1 = \mathcal{P}_{12}\; \vec{\sigma}_2\; \mathcal{P}_{12}$$

Subscript denotes the vector space on which the operator acts nontrivially!

general L: The Hilbert space is $V \otimes \cdots \otimes V$ 2^L dims

The observables are

$$\vec{\sigma}_n = \underset{\substack{\uparrow \\ 1}}{I} \otimes \cdots I \otimes \underset{\substack{\uparrow \\ n}}{\vec{\sigma}} \otimes I \otimes \cdots \otimes \underset{\substack{\uparrow \\ L}}{I} \qquad n = 1, \ldots, L$$

Hamiltonian? Many possibilities! We consider here

$$\boxed{H = \frac{1}{2} \sum_{n=1}^{L} (I - \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}) = \sum_{n=1}^{L} (I - \mathcal{P}_{n,n+1})}$$

PBCs $\vec{\sigma}_{L+1} \equiv \vec{\sigma}_1$

“Heisenberg (XXX) quantum spin chain”

- 1-dim model of ferromagnetism
- 1-loop mixing matrix in SU(2) subsector of $\mathcal{N}$=4 SYM

Basic problem: $H|\psi\rangle = E|\psi\rangle$ (*)

H is $2^L \times 2^L$ matrix $\therefore$

Brute-force diagonalization is not an option for L $\gtrsim$ 10

Fortunately, as we shall see, this model is integrable; so there ARE other options!

Hint of integrability: H commutes with

$$\sum_{n=1}^{L} \vec{\sigma}_n \cdot (\vec{\sigma}_{n+1} \times \vec{\sigma}_{n+2})$$

There is a beautiful, systematic way of constructing such conserved quantities & solving (*)

To explain, we must digress...

Yang-Baxter equation
(YBE)

Consider “R-matrix”:

$$\boxed{R(u) \equiv uI \otimes I + i\mathcal{P}} = \left(\begin{array}{cc|cc} u+i & & & \\ & u & i & \\ \hline & i & u & \\ & & & u+i \end{array}\right) = \left(\begin{array}{cc|cc} a & & & \\ & b & c & \\ \hline & c & b & \\ & & & a \end{array}\right)$$

$$a = u + i\,, \quad b = u\,, \quad c = i$$

u: “spectral parameter”

[eventually, parameter of the generating function for conserved quantities]

We regard R(u) as an operator on $V \otimes V$

Let’s now use R(u) to construct operators on $V \otimes V \otimes V$

Operators on $\underset{\overset{\uparrow}{1}}{V} \otimes \underset{\overset{\uparrow}{2}}{V} \otimes \underset{\overset{\uparrow}{3}}{V}$:

$$R_{12}(u) \equiv R(u) \otimes I = \left(\begin{array}{cc|cc} a & & & \\ & b & c & \\ \hline & c & b & \\ & & & a \end{array} \right) \otimes \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = \left(\begin{array}{cc|cc|cc|cc} a & & & & & & & \\ & a & & & & & & \\ \hline & & b & & c & & & \\ & & & b & & c & & \\ \hline & & c & & b & & & \\ & & & c & & b & & \\ \hline & & & & & & a & \\ & & & & & & & a \end{array} \right)$$

$$R_{23}(u) \equiv I \otimes R(u) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes \left(\begin{array}{cc|cc} a & & & \\ & b & c & \\ \hline & c & b & \\ & & & a \end{array} \right) = \left(\begin{array}{cc|cc|cc|cc} a & & & & & & & \\ & b & c & & & & & \\ \hline & c & b & & & & & \\ & & & a & & & & \\ \hline & & & & a & & & \\ & & & & & b & c & \\ \hline & & & & & c & b & \\ & & & & & & & a \end{array} \right)$$

$$R_{13}(u) \equiv \mathcal{P}_{23}\ R_{12}(u)\ \mathcal{P}_{23}$$

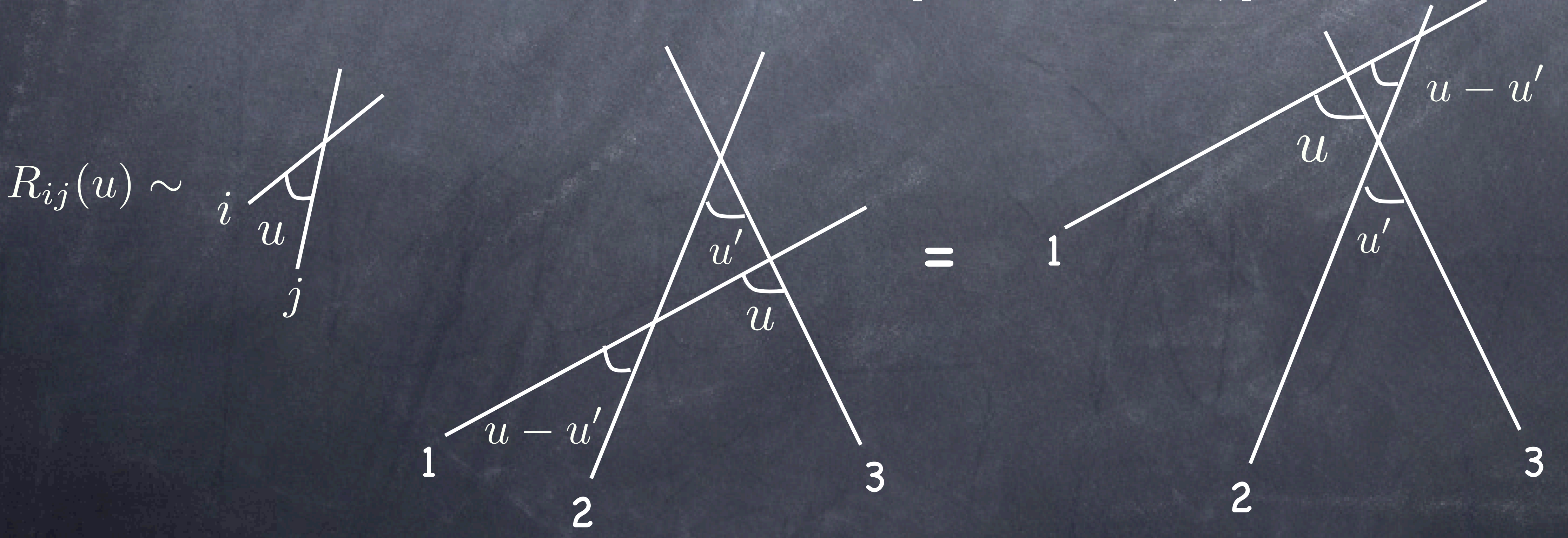
$$\mathcal{P}_{23} \equiv I \otimes \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \left(\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right) = \left(\begin{array}{cc|cc|cc|cc} 1 & & & & & & & \\ & & 1 & & & & & \\ \hline & 1 & & & & & & \\ & & & 1 & & & & \\ \hline & & & & 1 & & & \\ & & & & & & 1 & \\ \hline & & & & & 1 & & \\ & & & & & & & 1 \end{array}\right)$$

$$R_{13}(u) = \left(\begin{array}{cc|cc|cc|cc} a & & & & & & & \\ & b & & & c & & & \\ \hline & & a & & & & & \\ & & & b & & & c & \\ \hline & c & & & b & & & \\ & & & & & a & & \\ \hline & & & c & & & b & \\ & & & & & & & a \end{array}\right)$$

Can now easily check that

$$\boxed{R_{12}(u-u')\ R_{13}(u)\ R_{23}(u') = R_{23}(u')\ R_{13}(u)\ R_{12}(u-u')}$$

- This is the famous YBE!
- Can regard as an equation to be solved for R(u)
- Many families of solutions known
- We are considering here just the simplest, SU(2)-invariant, solution $[g\otimes g\,, R(u)] = 0\,, \quad g\in \mathrm{SU}(2)$



Question: Why should we care about this?

Answer: As we shall now see,
for each regular $(R(0) \propto \mathcal{P})$ solution of YBE,
we can construct a local integrable spin chain!

Quantum Inverse Scattering Method (QISM)

Basic idea: Use R-matrix to construct the Hamiltonian and higher local conserved quantities

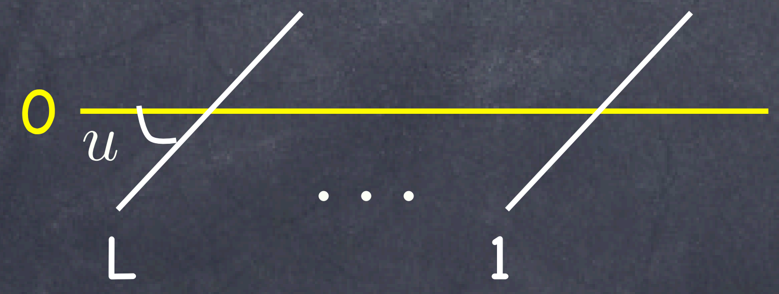
key step: introduce an additional copy of vector space V "auxiliary" space

$$\underset{\underset{0}{\uparrow}}{V} \otimes \underset{\underset{1}{\uparrow}}{V} \otimes \cdots \otimes \underset{\underset{L}{\uparrow}}{V}$$

$$\boxed{T_0(u) \equiv R_{0L}(u) \cdots R_{01}(u)}$$

"monodromy matrix"

$\sim$



“Fundamental Relation” (FR):

$$\boxed{R_{00'}(u-u')\ T_0(u)\ T_{0'}(u') = T_{0'}(u')\ T_0(u)\ R_{00'}(u-u')}$$

Proof (L=2):

$$LHS = R_{00'}(u-u')\ R_{02}(u)\ R_{01}(u)\ R_{0'2}(u')\ R_{0'1}(u')$$

All spaces different

$$= \underbrace{R_{00'}(u-u')\ R_{02}(u)\ R_{0'2}(u')}\ R_{01}(u)\ R_{0'1}(u')$$

YBE

$$= R_{0'2}(u')\ R_{02}(u)\ \underbrace{R_{00'}(u-u')\ R_{01}(u)\ R_{0'1}(u')}$$

YBE

$$= R_{0'2}(u')\ R_{02}(u)\ R_{0'1}(u')\ R_{01}(u)\ R_{00'}(u-u')$$

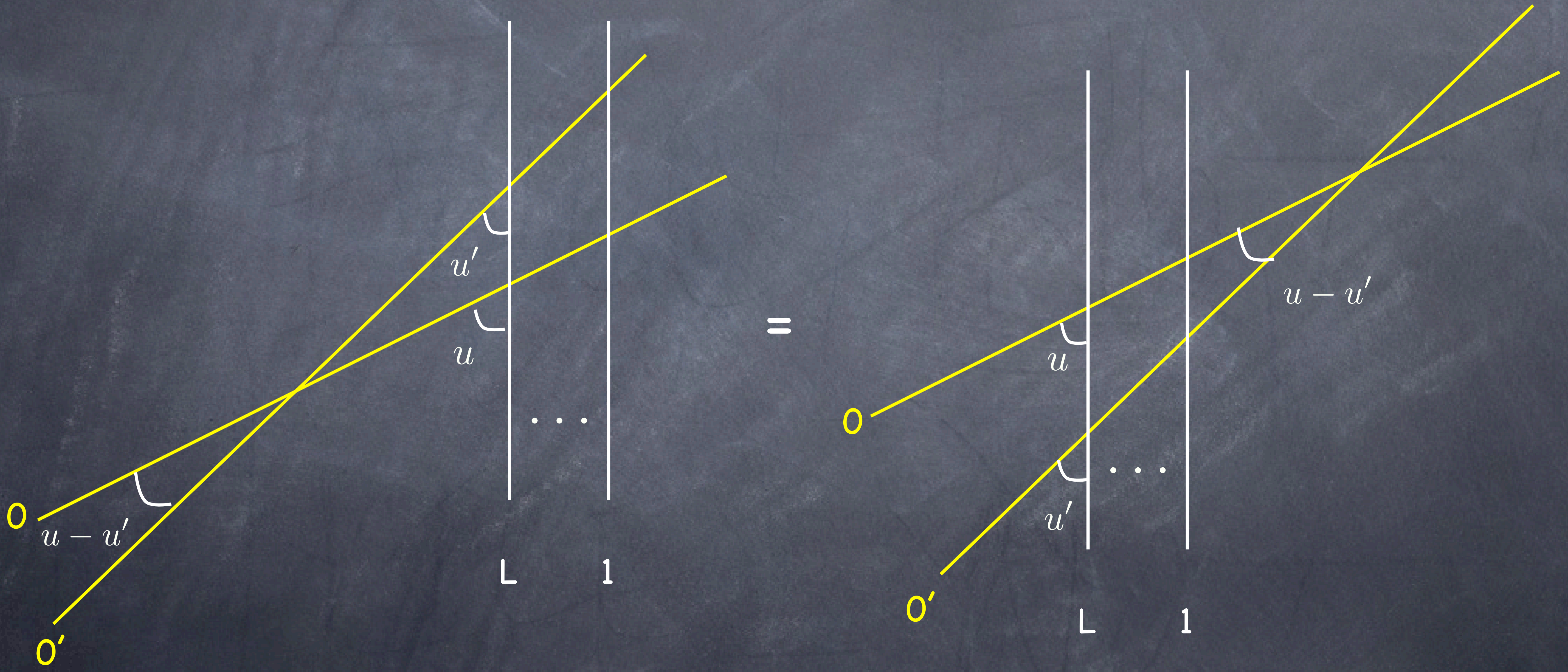
All spaces different

$$= R_{0'2}(u')\ R_{0'1}(u')\ R_{02}(u)\ R_{01}(u)\ R_{00'}(u-u') = RHS$$

□

Graphical proof:

$$R_{00'}(u-u')\ T_0(u)\ T_{0'}(u') = T_{0'}(u')\ T_0(u)\ R_{00'}u-u')$$



$$\boxed{t(u) = \mathrm{tr}_0\, T_0(u)}$$

"transfer matrix"

Acts on $\underset{\underset{1}{\uparrow}}{V} \otimes \cdots \otimes \underset{\underset{L}{\uparrow}}{V}$ (same as spin-chain Hamiltonian!)

1-parameter family of commuting operators:

$$\boxed{[t(u)\,, t(u')] = 0}$$

Proof: $R_{00'}(u-u')\; T_0(u)\; T_{0'}(u') = T_{0'}(u')\; T_0(u)\; R_{00'}(u-u')$ FR

$$R_{00'}(u-u')\; T_0(u)\; T_{0'}(u')\; R_{00'}(u-u')^{-1} = T_{0'}(u')\; T_0(u)$$

$$\boxed{t(u) = \mathrm{tr}_0\, T_0(u)}$$ “transfer matrix”

Acts on $\underset{\substack{\uparrow \\ 1}}{V} \otimes \cdots \otimes \underset{\substack{\uparrow \\ L}}{V}$ (same as spin-chain Hamiltonian!)

1-parameter family of commuting operators:

$$\boxed{[t(u)\,, t(u')] = 0}$$

Proof:

$$R_{00'}(u-u')\; T_0(u)\; T_{0'}(u') = T_{0'}(u')\; T_0(u)\; R_{00'}(u-u') \qquad \text{FR}$$

trace

$$\mathrm{tr}_{00'}\, \cancel{R_{00'}}(u-u')\; T_0(u)\; T_{0'}(u')\; \cancel{R_{00'}}(u-u')^{-1} = \mathrm{tr}_{00'}\, T_{0'}(u')\; T_0(u)$$

cyclic property of trace

$$\mathrm{tr}_{00'}\, T_0(u)\; T_{0'}(u') = \mathrm{tr}_{00'}\, T_{0'}(u')\; T_0(u)$$

$$t(u)\; t(u') = t(u')\; t(u) \qquad \square$$

The transfer matrix is a generating function for local conserved quantities:

$$\boxed{\ln t(u) = \sum_{n=0}^{\infty} \frac{u^n}{n!} H_n}$$

Can show that H_1 is the Heisenberg Hamiltonian, H_2 is the next conserved charge, etc.

$$[t(u)\,, t(u')] = 0 \quad \Rightarrow \quad [H_n, H_m] = 0$$

Infinitely many conserved commuting local quantities

integrable!

Starting from other regular R-matrices, obtain corresponding local integrable spin-chain Hamiltonians

- interpretation of H_0

$$t(0) = i^L U \, , \quad U = \mathcal{P}_{12}\mathcal{P}_{23} \cdots \mathcal{P}_{L-1,L}$$

$$U A_n U^\dagger = A_{n+1}$$

U: 1-site shift operator

$$U = e^{iP}$$

P: “momentum”

$$H_0 = \ln t(0) \sim P$$

- eigenvalues of conserved charges

$$[t(u)\,,t(u')]=0$$

$\Rightarrow$ there exist eigenstates of transfer matrix that do not depend on spectral parameter

$$t(u)|\Lambda\rangle=\Lambda(u)|\Lambda\rangle$$

If we can determine $\Lambda(u)$,
then we can get eigenvalues h_n of all charges H_n :

$$h_n=\frac{d^n}{du^n}\ln\Lambda(u)\Big|_{u=0}$$

Algebraic Bethe ansatz

So now we know that the Heisenberg model is integrable.

Question: But are we any closer to solving the model?

(i.e., finding eigenstates & eigenvalues of transfer matrix)

Answer: Yes!

We shall now identify certain creation operators.

Acting with them on the vacuum state

$$\boxed{|0\rangle \equiv \underbrace{\begin{pmatrix}1\\0\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix}}_{L}}$$

all spins up

we can construct the eigenstates! (~ harmonic oscillator)

Recall that the monodromy matrix acts on

$$\underset{\substack{\uparrow \\ 0}}{V} \otimes \underset{\substack{\uparrow \\ 1}}{V} \otimes \cdots \otimes \underset{\substack{\uparrow \\ L}}{V}$$

Set

$$\boxed{T_0(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}}$$

$A(u), \ldots, D(u)$ act on

$$\underset{\substack{\uparrow \\ 1}}{V} \otimes \cdots \otimes \underset{\substack{\uparrow \\ L}}{V}$$

$$t(u) = \mathrm{tr}_0\, T_0(u) = A(u) + D(u)$$

$B(u)|0\rangle \neq 0$ creation

$C(u)|0\rangle = 0$ annihilation

Assume that the eigenstates of $t(u)$ are given by

$$\boxed{|u_1\,, \ldots\,, u_M\rangle \equiv B(u_1) \cdots B(u_M)\ |0\rangle}$$

To compute eigenvalues, must move $t(u) = A(u) + D(u)$

past each of the B's

FR $\Rightarrow$ commutation relations:

$$A(u)\ B(u') = \left(\frac{u - u' - i}{u - u'}\right) B(u')\ A(u) - \frac{i}{u - u'} B(u)\ A(u')$$

$$D(u)\ B(u') = \left(\frac{u - u' + i}{u - u'}\right) B(u')\ D(u) - \frac{i}{u - u'} B(u)\ D(u')$$

Using only first terms, get

$$A(u)|u_1\,, \ldots\,, u_M\rangle = \prod_{k=1}^{M} \left(\frac{u - u_k - i}{u - u_k}\right) B(u_1) \cdots B(u_M)\ \underbrace{A(u)|0\rangle}_{(u+i)^L|0\rangle}$$

$$D(u)|u_1\,, \ldots\,, u_M\rangle = \prod_{k=1}^{M} \left(\frac{u - u_k + i}{u - u_k}\right) B(u_1) \cdots B(u_M)\ \underbrace{D(u)|0\rangle}_{u^L|0\rangle}$$

$$\therefore \qquad t(u)|u_1, \ldots, u_M\rangle = \Lambda(u)|u_1, \ldots, u_M\rangle + \text{“unwanted”}$$

$$\boxed{\Lambda(u) = (u+i)^L \prod_{k=1}^{M} \left(\frac{u - u_k - i}{u - u_k} \right) + u^L \prod_{k=1}^{M} \left(\frac{u - u_k + i}{u - u_k} \right)}$$

So far, $\{u_1, \ldots, u_M\}$ are arbitrary.

Can show that the “unwanted” terms cancel if $\{u_1, \ldots, u_M\}$ satisfy the “Bethe equations” (BEs):

$$\left(\frac{u_j + i}{u_j} \right)^L = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}, \quad j = 1, \cdots, M$$

$$u_j \mapsto u_j - \frac{i}{2}$$

$$\boxed{\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}, \quad j = 1, \cdots, M}$$

In principle, can solve BEs for $\{u_1, \ldots, u_M\}$

& therefore obtain transfer matrix eigenvalues $\Lambda(u)$

$$P \sim \ln t(0) \qquad \Rightarrow \qquad P \sim \ln \Lambda(0) = \boxed{\frac{1}{i} \sum_{k=1}^{M} \ln \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)} \pmod{2\pi}$$

$$H \sim \frac{d}{du} \ln t(u) \Big|_{u=0} \qquad \Rightarrow \qquad E \sim \frac{d}{du} \ln \Lambda(u) \Big|_{u=0} = \boxed{\sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}}}$$

Note: $\{u_1, \ldots, u_M\}$ must be distinct su(2) symmetry: $\left[\vec{S}, t(u)\right] = 0$ $\qquad \vec{S} = \frac{1}{2}\sum_{n=1}^{L} \vec{\sigma}_n$

$\therefore$ can simultaneously diagonalize $t(u), \vec{S}^2, S^z$

$$\vec{S}^2 |u_1 , \ldots , u_M\rangle = s(s+1)|u_1 , \ldots , u_M\rangle$$

$$S^z |u_1 , \ldots , u_M\rangle = m|u_1 , \ldots , u_M\rangle$$

Bethe states are su(2) highest-weight states:

$$S^+ |u_1 , \ldots , u_M\rangle = 0$$

$$\Rightarrow \qquad \boxed{s = m = \frac{L}{2} - M} \qquad [S^z , B(u)] = -B(u) \qquad S^z|0\rangle = \frac{L}{2}|0\rangle$$

$$s \geq 0 \qquad \Rightarrow \qquad \boxed{M \leq \frac{L}{2}}$$

The lower-weight states can be obtained by acting with S⁻

Example: L=4 $M \leq \frac{L}{2}$ $\therefore$ $M = 0, 1, 2$

$$s = \frac{L}{2} - M = 2 - M$$

M	$\{u_k\}$	P	E	s	degeneracy (2s+1)
0	-	0	0	2	5
1	1/2	$\pi/2$	2	1	3
1	-1/2	$-\pi/2$	2	1	3
1	0	π	4	1	3
2	i/2,-i/2	π	2	0	1
2	$1/(2\sqrt{3}), -1/(2\sqrt{3})$	0	6	0	1

total: 16 = 2^4 ✓

Matches with direct diagonalization of H ✓

Hypothesis: For any L, Bethe ansatz gives complete set of (highest-weight) states

Analytical Bethe ansatz

Fact: $\Lambda(u)$ are polynomials in u, of degree L

Proof: Recall

$$t(u) = \mathrm{tr}_0\, R_{0L}(u) \cdots R_{01}(u)\,, \quad R(u) = uI + i\mathcal{P}$$

$\Rightarrow$

$$t(u) = \sum_{n=0}^{L} t_n u^n$$

t_n: u-independent matrices

$$[t(u)\,, t(u')] = 0 \quad \Rightarrow \quad [t_n\,, t_m] = 0$$

can diagonalize simultaneously!

$$t_n|\Lambda\rangle = \Lambda_n|\Lambda\rangle$$

$\therefore$

$$\Lambda(u) = \sum_{n=0}^{L} \Lambda_n u^n$$

polynomial in u, of degree L ■

Corollary: $\Lambda(u)$ are regular (no poles) for finite u

Useful short-cut for finding Λ(u) & BEs:

[Reshetikhin, ...]

Vacuum eigenvalue:

$$t(u)|0\rangle = \Lambda^{(0)}(u)|0\rangle$$

$$\Lambda^{(0)}(u) = (u+i)^L + u^L$$

Assume general eigenvalue is "dressed" vacuum eigenvalue:

$$\Lambda(u) = (u+i)^L \frac{Q(u-i)}{Q(u)} + u^L \frac{Q(u+i)}{Q(u)}$$

$$Q(u) = \prod_{j=1}^{M} (u - u_j)$$ zeros u_j still to be determined

$\Lambda(u)$ must not have pole at u_j $\Rightarrow$

$$(u_j+i)^L Q(u_j - i) + u_j^L Q(u_j + i) = 0$$ Bethe equations!

Assumed only simple poles - i.e., distinct Bethe roots

Higher-order poles $\Rightarrow$ spurious equations

Epilogue

Returning to $\mathcal{N}$=4 SYM...

In SU(2) subsector $\quad \mathrm{tr}\, X(x)^M Z(x)^{L-M} + \ldots$

1-loop anomalous dimensions: $\quad \gamma = \frac{\lambda}{8\pi^2} \sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}}$

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L = \prod_{\substack{k=1 \\ k\neq j}}^{M} \frac{u_j - u_k + i}{u_j - u_k - i}, \quad j = 1, \cdots, M$$

cyclicity $\quad \Rightarrow \quad P = \frac{1}{i} \sum_{k=1}^{M} \ln \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right) = 0$

Example: L=4

M	$\{u_k\}$	P	E	s	degeneracy (2s+1)
0	-	0	0	2	5
~~1~~	~~1/2~~	~~$\pi/2$~~	~~2~~	~~1~~	~~3~~
~~1~~	~~-1/2~~	~~$\pi/2$~~	~~2~~	~~1~~	~~3~~
~~1~~	~~0~~	~~π~~	~~4~~	~~1~~	~~3~~
~~2~~	~~i/2,-i/2~~	~~π~~	~~2~~	~~0~~	~~1~~
2	$1/(2\sqrt{3}), -1/(2\sqrt{3})$	0	6	0	1

Returning to $\mathcal{N}$=4 SYM...

In SU(2) subsector $\quad \mathrm{tr}\, X(x)^M Z(x)^{L-M} + \ldots$

1-loop anomalous dimensions: $\quad \gamma = \frac{\lambda}{8\pi^2} \sum_{k=1}^{M} \frac{1}{u_k^2 + \frac{1}{4}}$

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} \right)^L = \prod_{\substack{k=1 \\ k \neq j}}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \,, \qquad j = 1\,, \cdots\,, M$$

cyclicity $\Rightarrow$ $\quad P = \frac{1}{i} \sum_{k=1}^{M} \ln \left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right) = 0$

Many questions remain:

- other operators?
- higher loops?

Stay tuned!