2nd Asia-Pacific Summer School in Mathematical Physics

22nd Canberra International Physics Summer School

# CFT, AdS/CFT & Integrability

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# Introduction to Integrability in AdS/CFT: Lecture 1

Rafael Nepomechie

University of Miami

# Introduction

Goal: To introduce you to integrability in AdS/CFT

Why should we (integrable community) care?

- "Magic" in CFTs in D>2 !

D=2: Vladimir Bazhanov

- Uses every nice trick we know (BA, TBA, NLIE, Hubbard, ...), and some new ones!

"Integrability Paradise"

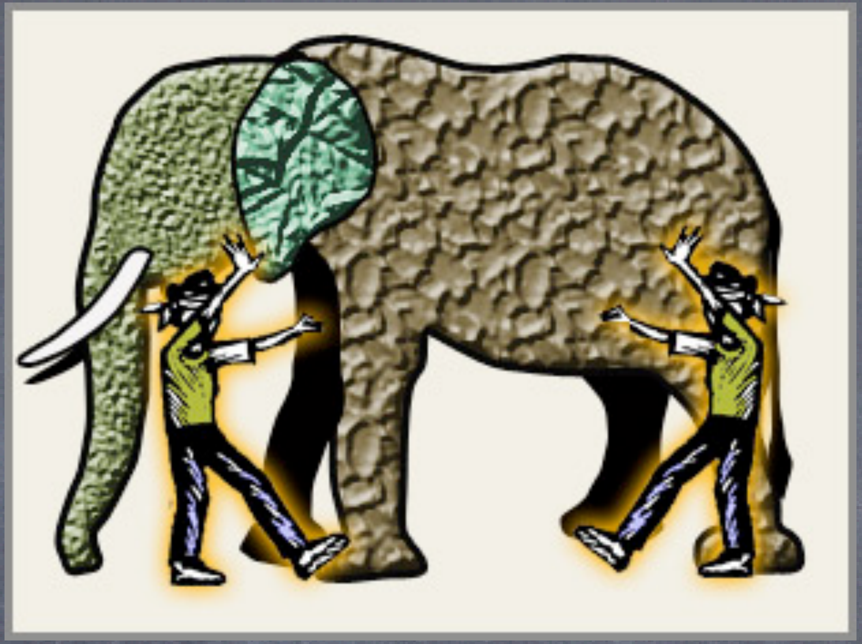
Why does the high-energy community care?

- May lead to solution of a 4D gauge theory
- String theory in a curved background

# So, what model are we talking about?

2 descriptions: blind men & elephant

strong coupling:
classical
type IIB string
on $AdS_5 \times S^5$



weak coupling:
perturbative
$\mathcal{N}=4$ SYM
$CFT_4$

But it is one and the same theory! $\equiv$AdS/CFT [Maldacena 97]

Nick Halmagyi

And - at least in the planar limit - it seems to be integrable!

# Plan

- Today: describe CFT "side"
  - action, symmetries
  - planar limit
  - anomalous dimensions
  - 1-loop mixing matrix
- Subsequent: integrability & how to exploit it

string "side": Changrim Ahn

# $\mathcal{N}$=4 Super Yang-Mills

## Field content:

| | | global SU(4) ≃ SO(6) "R" symmetry |
|---|---|---|
| gauge bosons | $A_\mu(x)\,, \quad \mu = 0,1,2,3$ | 1 |
| 6 massless real scalars | $\phi^I(x)\,, \quad I = 1,\ldots,6$ | 6 |
| 4 chiral fermions | $\psi^a_\alpha(x)\,, \quad a = 1,\ldots,4\,, \quad \alpha = 3,4$ | 4 |
| 4 anti-chiral fermions | $\overline{\psi}_{\dot\alpha\, a}(x)\,, \quad a = 1,\ldots,4\,, \quad \dot\alpha = \dot3,\dot4$ | $\bar{4}$ |

All fields transform in the adjoint rep of SU(N) gauge group

e.g. $\phi^I(x) = \sum_{a=1}^{N^2-1} \phi^{I\,a}(x) T^a$ $\qquad (T^a)_i^{\;j}$ N x N traceless Hermitian matrices

$$\phi^I(x) \to U(x)\,\phi^I(x)\,U(x)^\dagger\,, \quad U(x) \in \mathrm{SU(N)}$$

Action:

[Gliozzi, Scherk & Olive 77; Brink, Schwarz & Scherk 77]

$$S = \frac{1}{g_{YM}^2} \int d^4x \ \mathrm{tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \phi^I)^2 - \frac{1}{2} \left[ \phi^I \, , \phi^J \right]^2 + \text{fermions} \right\}$$

symmetries: local SU(N) gauge

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i \left[ A_\mu \, , A_\nu \right]$$

$$D_\mu = \partial_\mu - i \left[ A_\mu \, , \ \right]$$

Lorentz SO(1,3) ≃ SU(2) x SU(2)

scale $x^\mu \to a x^\mu \, , \quad A_\mu \to \frac{1}{a} A_\mu \, , \quad \phi^I \to \frac{1}{a} \phi^I \, , \quad \psi \to \frac{1}{a^{3/2}} \psi$

Aside (important later):

$\Longrightarrow$ "bare" scaling dimensions $\Phi \to \frac{1}{a^{\Delta_0}} \Phi$

$$\Delta_0(A_\mu) = \Delta_0(\phi^I) = 1 \, , \quad \Delta_0(\psi) = 3/2$$

Action:

[Gliozzi, Scherk & Olive 77; Brink, Schwarz & Scherk 77]

$$S = \frac{1}{g_{YM}^2} \int d^4x \; \mathrm{tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \phi^I)^2 - \frac{1}{2} \left[ \phi^I \, , \phi^J \right]^2 + \text{fermions} \right\}$$

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conformal SO(2,4) ≃ SU(2,2)

R-symmetry SO(6) ≃ SU(4)

$\mathcal{N}$=4 supersymmetry

$\mathcal{N}$=4 superconformal PSU(2,2|4) ~SU(2,2|4)/U(1)

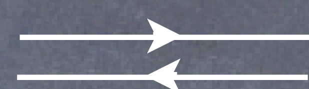
Unbroken by quantum effects! [Mandelstam 83; Brink et al. 83, ... ]

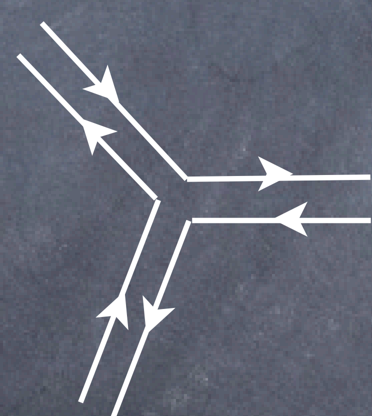
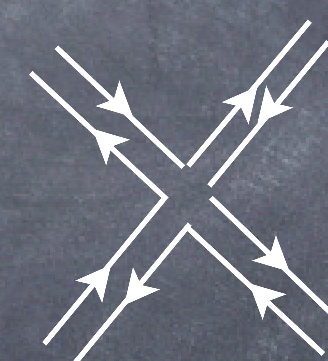
# Planar limit

$N \to \infty\,, \quad g_{YM} \to 0\,, \quad \lambda \equiv g_{YM}^2 N$ fixed ['t Hooft 74]

Only planar Feynman graphs survive:

double-line notation

propagators  $\sim g_{YM}^2$ closed curve: $N$

vertices  $\sim \frac{1}{g_{YM}^2}$  $\sim \frac{1}{g_{YM}^2}$

$$\sim (g_{YM}^2)^{3-2} N^3 = (g_{YM}^2 N) N^2$$

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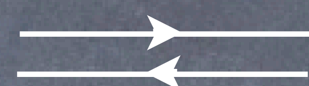
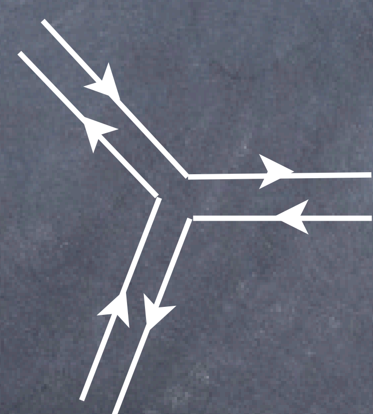
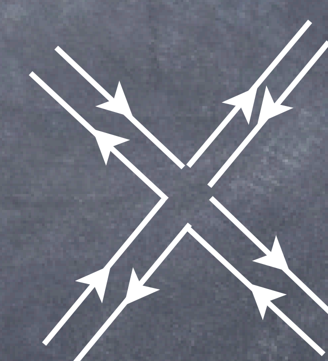
propagators  $\sim g_{YM}^2$ closed curve: $N$

vertices  $\sim \frac{1}{g_{YM}^2}$  $\sim \frac{1}{g_{YM}^2}$

 $\sim (g_{YM}^2)^{6-4} N^4 = (g_{YM}^2 N)^2 N^2$

$N \to \infty\,, \quad g_{YM} \to 0\,, \quad \lambda \equiv g_{YM}^2 N$ fixed ['t Hooft 74]

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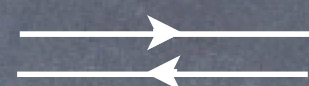
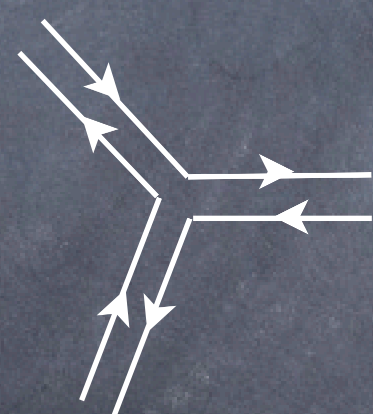
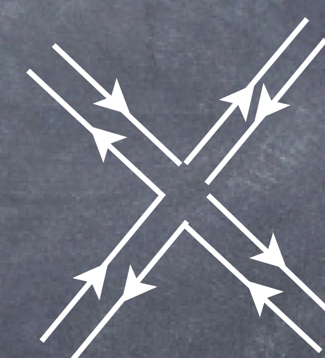
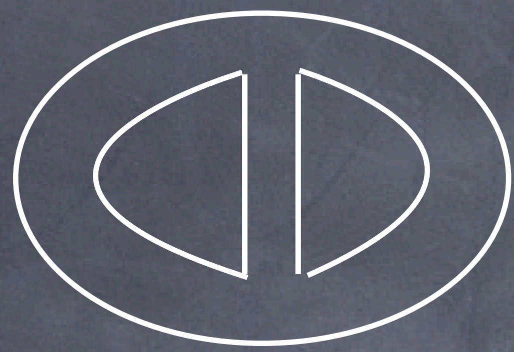
propagators  $\sim g_{YM}^2$ closed curve: $N$

vertices  $\sim \dfrac{1}{g_{YM}^2}$  $\sim \dfrac{1}{g_{YM}^2}$

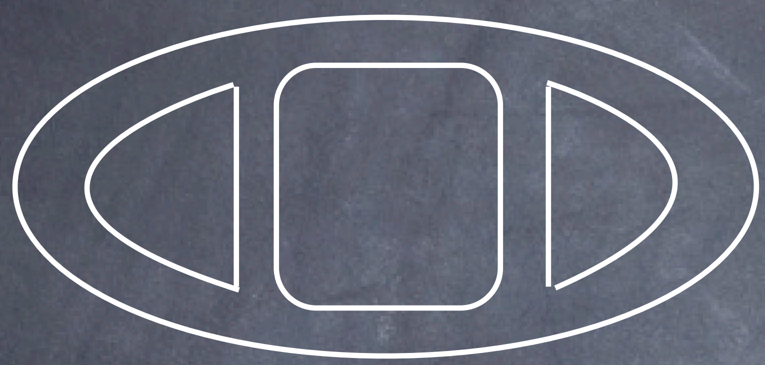
 $\sim (g_{YM}^2)^{6-4} N^2 = (g_{YM}^2 N)^2$

Summary: # loops



$\sim (g_{YM}^2 N) N^2$ 2

planar



$\sim (g_{YM}^2 N)^2 N^2$ 3

$\lambda = g_{YM}^2 N$ power of $\lambda \sim$ # loops

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$\sim (g_{YM}^2 N)^2$ non-planar

suppressed $\sim \dfrac{1}{N^2}$

- Integrability in $\mathcal{N}$=4 SYM has so far appeared only in planar limit
- We shall henceforth work exclusively in this limit
- $\mathcal{N}$=4 SYM reduced to having just 1 free parameter $\lambda$

# Anomalous dimensions

conformal primary operators $\mathcal{O}(x)$

2-point function:

$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle \sim \frac{1}{|x-y|^{2\Delta}}$$

$\Delta$ : ("conformal" or "scaling") dimension

$$\Delta = \Delta(\lambda) = \underset{\substack{\uparrow \\ \text{bare}}}{\Delta_0} + \underset{\substack{\uparrow \\ \text{anomalous}}}{\gamma(\lambda)}$$

Main problem:
Determine $\Delta(\lambda)$ for all operators & for all $\lambda$

For $\lambda$ small ("weak coupling"), can use loop expansion

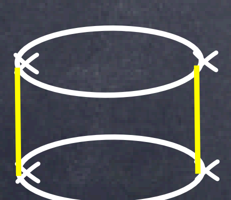
Example: Konishi $\quad \mathcal{O}(x) = \sum_{I=1}^{6} \mathrm{tr}\, \phi^I(x)\phi^I(x)$

- local
- gauge-invariant
- single trace

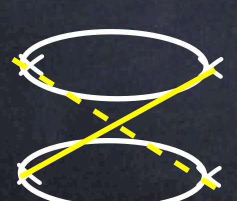
Recall $\quad \Delta_0(\phi^I) = 1$

$$\Delta_0(\mathcal{O}) = 1 + 1 = 2$$

"tree" level (no loops):



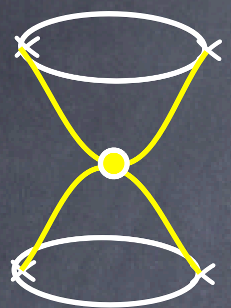
$$\langle \mathcal{O}(x)\, \mathcal{O}(0) \rangle = \frac{3\lambda^2}{16\pi^4 |x|^4} \sim \frac{1}{|x|^{2\Delta_0}} \quad ✓$$



$$\langle \phi_i^{I\ j}(x)\, \phi_k^{J\ l}(y) \rangle_0 = \delta^{IJ} \delta_i^l \delta_k^j \frac{g_{YM}^2}{8\pi^2 |x-y|^2}$$

# 1-loop level: log divergent; need UV cutoff $\Lambda$

$$\langle \mathcal{O}(x)\,\mathcal{O}(0)\rangle = \frac{3\lambda^2}{16\pi^4|x|^4}\left[1 - \frac{3\lambda}{2\pi^2}\log\left(\Lambda|x|\right)\right]$$



+...

$$\approx e^{-\frac{3\lambda}{2\pi^2}\log(\Lambda|x|)} = (\Lambda|x|)^{-\frac{3\lambda}{2\pi^2}}$$

$$\sim \frac{\Lambda^{-\frac{3\lambda}{2\pi^2}}}{|x|^{4+\frac{3\lambda}{2\pi^2}}}$$

**Renormalize:** $\mathcal{O}_R(x) = Z\,\mathcal{O}(x)\,, \quad Z = \Lambda^{\frac{3\lambda}{4\pi^2}}$

$$\langle \mathcal{O}_R(x)\,\mathcal{O}_R(0)\rangle \sim \frac{1}{|x|^{2\Delta}}\,, \quad \Delta = 2 + \boxed{\frac{3\lambda}{4\pi^2}}$$

1-loop anomalous dimension

function of $\lambda$

For general conformal primary operator $\mathcal{O}(x)$

$$\langle \mathcal{O}(x)\, \mathcal{O}(0) \rangle \sim \frac{1}{|x|^{2\Delta_0}} \left[ 1 - 2\gamma \log\left( \Lambda |x| \right) + \ldots \right]$$

$$\sim \frac{\Lambda^{-2\gamma}}{|x|^{2\Delta_0 + 2\gamma}}$$

Renormalize: $\mathcal{O}_R(x) = Z\, \mathcal{O}(x)\,, \quad Z = \Lambda^{\gamma}$

$$\langle \mathcal{O}_R(x)\, \mathcal{O}_R(0) \rangle \sim \frac{1}{|x|^{2\Delta}}\,, \quad \Delta = \Delta_0 + \gamma$$

For general conformal primary operator $\mathcal{O}(x)$

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Renormalize: $\mathcal{O}_R(x) = Z\, \mathcal{O}(x)\,,$ $\boxed{Z = \Lambda^{\gamma}}$ $\log Z = \gamma \log \Lambda$

$$\langle \mathcal{O}_R(x)\, \mathcal{O}_R(0) \rangle \sim \frac{1}{|x|^{2\Delta}}\,, \quad \Delta = \Delta_0 + \gamma$$

Note: $$\gamma = \frac{d \log Z}{d \log \Lambda} = Z^{-1} \frac{dZ}{d \log \Lambda}$$

- chiral primary operators

superconformal symmetry $\Rightarrow$

certain conformal primary operators ("chiral" or "BPS")
have 0 anomalous dimensions,
to all orders in $\lambda$ !

[review: Minahan 10]

→ Changrim Ahn

# 1-loop mixing matrix

How about $\Delta$ for $\mathcal{O}^{I_1 \ldots I_L}(x) = \mathrm{tr}\, \phi^{I_1}(x) \ldots \phi^{I_L}(x)$ ?

(Also local, gauge-invariant, single trace)

NOT a conformal primary operator!

Only certain linear combinations are conformal primaries

$\psi_{I_1 \ldots I_L} \mathcal{O}^{I_1 \ldots I_L}(x)$ "mixing problem"

- Already saw this for Konishi (L=2)

$$\sum_{I=1}^{6} \mathcal{O}^{I\,I}(x)$$

- At 1 loop, scalars mix only among themselves
- Cyclicity of trace $\Rightarrow$ $\mathcal{O}^{I_1 \ldots I_L}(x) = \mathcal{O}^{I_2 \ldots I_L I_1}(x)$ etc.

In general:

$$\mathcal{O}_R^a = Z_b^a \mathcal{O}^b$$ can compute Z perturbatively

$$\Gamma \equiv Z^{-1} \frac{dZ}{d\log\Lambda}$$ mixing matrix

$$\Gamma \circ \mathcal{O}_n = \gamma_n \mathcal{O}_n$$ eigenvectors: conformal primaries

eigenvalues: anomalous dimensions

$$\Delta_n = \Delta_0 + \gamma_n$$ conformal dimensions

$$\mathcal{D} = \Delta_0 + \Gamma$$ dilatation operator

1-loop mixing matrix for $\mathcal{O}[\psi] \equiv \psi_{I_1 \dots I_L} \, \mathrm{tr}\, \phi^{I_1}(x) \dots \phi^{I_L}(x)$

$$\Gamma = \frac{\lambda}{8\pi^2} H \,, \quad H = \frac{1}{2} \sum_{l=1}^{L} \left( K_{l,l+1} + 2 - 2\mathcal{P}_{l,l+1} \right)$$

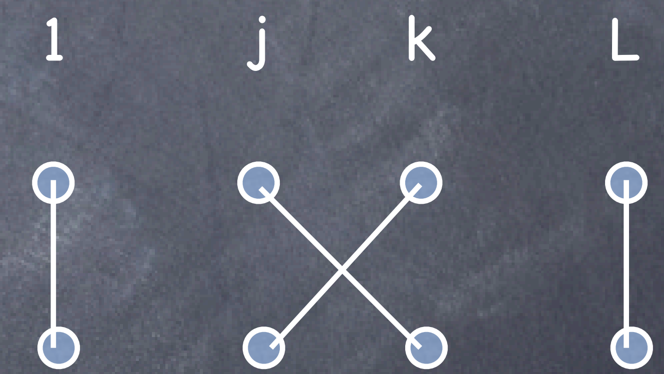
PBCs

$L + 1 \equiv 1$

$$(\Gamma \circ \psi)_{I_1 \cdots I_L} = \Gamma_{I_1 \cdots I_L}^{I_1' \cdots I_L'} \psi_{I_1' \cdots I_L'}$$

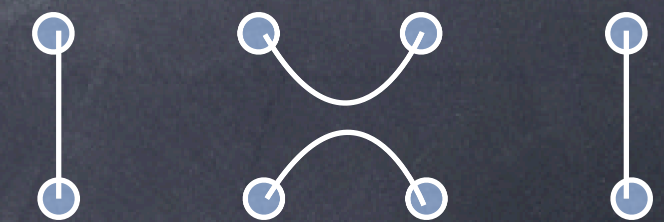
permutation:

$$(\mathcal{P}_{jk})_{I_1 \cdots I_L}^{I_1' \cdots I_L'} = \delta_{I_1}^{I_1'} \cdots \delta_{I_j}^{I_k'} \cdots \delta_{I_k}^{I_j'} \cdots \delta_{I_L}^{I_L'}$$




trace:

$$(K_{jk})_{I_1 \cdots I_L}^{I_1' \cdots I_L'} = \delta_{I_1}^{I_1'} \cdots \delta_{I_j, I_k} \delta^{I_j', I_k'} \cdots \delta_{I_L}^{I_L'}$$



planar limit $\Longrightarrow$ only nearest neighbors l, l+1

# Check 1: chiral primary operators

e.g., $$\psi_{I_1 I_2} = \delta_{I_1,j}\delta_{I_2,k} + \delta_{I_1,k}\delta_{I_2,j} - \frac{1}{3}\delta_{I_1,I_2}\delta_{j,k} \qquad j,k \in \{1,\ldots,6\}$$

symmetric & traceless

$$\Gamma \circ \psi = \frac{2\lambda}{16\pi^2}\left(K_{1,2} + 2 - 2\mathcal{P}_{1,2}\right) \circ \psi$$

$$= \frac{2\lambda}{16\pi^2}\left(0 + 2 - 2\right)\psi$$

$$= 0 \qquad ✓$$

## Check 2: Konishi $\psi_{I_1 I_2} = \delta_{I_1 I_2}$

$$\Gamma \circ \psi = \frac{2\lambda}{16\pi^2} \left(K_{1,2} + 2 - 2\mathcal{P}_{1,2}\right) \circ \psi$$

$$= \frac{2\lambda}{16\pi^2} \left(6 + 2 - 2\right) \psi$$

$$= \frac{3\lambda}{4\pi^2} \psi \qquad ✓$$

# SU(2) subsector

$$X = \phi^1 + i\phi^2\,, \quad Y = \phi^3 + i\phi^4\,, \quad Z = \phi^5 + i\phi^6$$

Consider just Z & X $\quad \mathrm{tr}\, X(x)^M Z(x)^{L-M} + \ldots$

$$\boxed{\Gamma = \frac{\lambda}{8\pi^2} H\,, \quad H = \sum_{l=1}^{L} (1 - \mathcal{P}_{l,l+1})}$$

PBCs

$L + 1 \equiv 1$

Problem: to determine eigenvectors & eigenvalues

Tomorrow: exact solution!