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Introduction to Gauge/Gravity Duality
Lecture 4

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Abstract

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1 The Exact Correspondence

While there were many hints and initial steps towards the discovery of the gauge/gravity duality in for instance [1, 2], the significance of the seminal paper by Maldacena [3] cannot be understated. In this work Maldacena proposed quite precisely how the bulk theory of IIB string theory on $AdS_5 \times S^5$ and the boundary $\mathcal{N} = 4$ SYM theory were really two different expansion points of the same underlying theory and it was proposed how this basic duality could be applied to a host of other examples arise in string/M-theory. Now roughly speaking, the data specifying a given quantum field theory is the spectrum of operators $\mathcal{O}^i(x)$ and their correlators and a proposal for how this data is mapped between the bulk and boundary theory was given in [4, 5]. Taken together these papers laid out the entire framework of gauge/gravity duality with enough generality that it could be applied to scenarios far from those initially considered and the mountain of subsequent work can be considered as essentially checking the validity of the proposed duality and its consequences. Before moving on to using the duality to compute some sample correlators, it is worthwhile to state the duality as precisely as we can.

In general the bulk theory is that of interacting strings. A key insight of Maldacena [3] was to determine the precise limit of the boundary theory which would correspond to the low energy or *supergravity* limit of string theory in the bulk. We have not had time in these lectures to describe the 't Hooft limit of Yang-Mills theory [6] but I believe other lecturers at this school will have covered this. Suffice it to say that Yang Mills theory in four dimensions has two particularly useful dimensionless parameters (g_{YM}, N_c), the gauge coupling and the rank of the gauge group. The insight of 't Hooft was that in the limit

$$\left.\begin{array}{ccc} g_{YM} & \to & 0 \\ N_c & \to & \infty \\ \lambda = g_{YM}^2 N_c & << & 1 \end{array}\right\} \text{'t Hooft limit} \tag{1}$$

the Yang-Mills theory simplifies and admits a perturbative *fatgraph* expansion. The Maldacena limit, where the string theory dual reduces to a supergravity dual is then

$$\left.\begin{array}{ccc} g_{YM} & \to & 0 \\ N_c & \to & \infty \\ \lambda = g_{YM}^2 N_c & >> & 1 \end{array}\right\} \text{Maldacena limit}\,. \tag{2}$$

In quantum field theory, the set of all correlators is summarized in terms of a generating functional $W(J_i)$:

$$e^{-W(J_i)} = \langle e^{\int d^4x J_i(x)\mathcal{O}^i(x)} \rangle\,. \tag{3}$$

Note that this expression includes sources for renormalizable and non-renormizable operator deformations. The expression for the correlators is

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \frac{\delta}{\delta J_1(x_1)}\cdots\frac{\delta}{\delta J_n(x_n)} e^{-W(J_i)}\Big|_{J_i=0} \tag{4}$$

and in the limit (1) we can compute these correlators in perturbation theory around $\lambda = 0$. To compute these correlators in the limit (2), we can use gauge/gravity duality. From the set of all correlation functions one can reconstruct $W(J_i)$.

The precise(ish) statement of the duality [5] is

$$\boxed{e^{-S_{string}} = e^{-W}}\,. \tag{5}$$

The LHS of this relation requires further explanation. For starters, we have just seen how the RHS is a function of all the sources $J_i(x)$ and so clearly the LHS must also be a function of such parameters. Once one provides a map between bulk fields and boundary operators

$$\phi^i(x,z) \overset{\sim}{\leftrightarrow} \mathcal{O}^i(x)\,, \tag{6}$$

then the prescription for the LHS of (5) is to extremize the string action as a function of the AdS-boundary values $\phi_0^i(x)$ of the bulk fields. In general this is far too difficult, string theory on AdS space of arbitrary radius is a notoriously hard problem. However in the supergravity limit of string theory, this is a much more tractable problem. The action of supergravity can be worked out to several orders of perturbation theory in a Kaluza-Klein reduction around $AdS_5 \times S^5$.

I have put a $\sim$ above the arrow in (6) because it is quite imprecise. More precise is something like

$$\phi^i(x,z)|_{z=0} = J_i(x) \tag{7}$$

but even that is not quite good enough. Recall from lecture 2 that a scalar field in AdS has two linearly independant solutions

$$\phi(x,z) = e^{ik\cdot x} z^{d/2}\Big[c_1 I_\lambda(|k|z) + c_2 K_\lambda(|k|z)\Big]\,. \tag{8}$$

with

$$\lambda = \sqrt{m^2 + \frac{d^2}{4}}. \tag{9}$$

We need to look at the boundary behaviour of these functions

$$I_\lambda(|k|z) \;=\; \frac{|k|^\lambda}{2^\lambda \Gamma(\lambda+1)} z^\lambda + \mathcal{O}(z^{\lambda+2})\,, \tag{10}$$

$$K_\lambda(|k|z) \;=\; \frac{2^{\lambda-1}}{|k|^\lambda \Gamma(\lambda)} z^{-\lambda} + \mathcal{O}(z^{\lambda+2})\,. \tag{11}$$

It is conventional in the literature to define

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{m^2 + \frac{d^2}{4}} \tag{12}$$

then we have near the boundary ($z = 0$)

$$z^{d/2} I_\lambda(|k|z) \;\sim\; z^{\Delta_+} \tag{13}$$

$$z^{d/2} K_\lambda(|k|z) \;\sim\; z^{d-\Delta_+}\,. \tag{14}$$

The basic idea is that one the most divergent mode actually deforms the boundary theory, thus it should correspond to the source for the dual operator $\mathcal{O}(x)$ while the less divergent mode corresponds to the vev for $\mathcal{O}(x)$ (see section III.A of [7] for a lucid summary of these salient points).

So now we see that the relation (7) is clearly imprecise since $\phi^i(x,z)$ will diverge near $z = 0$. The solution to this problem is to renormalize the operator $\phi(x)$ in the boundary theory and (7) becomes

$$\phi^i(x,\epsilon) \overset{\epsilon\to 0}{\sim} \epsilon^{d-\Delta_+} J_i(x)\,. \tag{15}$$

Now since $\phi(x,z)$ is dimensionless, J_i has dimension $[\text{length}]^{\Delta_+ - d}$ and since it couples as

$$\int d^d x J_i(x) \mathcal{O}^i(x) \tag{16}$$

we see that $\mathcal{O}^i(x)$ has dimension Δ_+. This is a very heuristic argument, it would be better to obtain the strong coupling dimension of a given operator by actually computing a two-point function from the bulk theory.

2 Computing Two-Point Functions From the Bulk

By now we should know enough about gauge/gravity duality to actually compute some two-point functions, or equivalently, the dimension of certain (scalar) fields in the Maldacena limit (2). The action for a free scalar field is

$$S_\phi = \frac{1}{2}\int d^5 x \sqrt{g}\Big((\nabla\phi)^2 + m^2\phi^2\Big)\,. \tag{17}$$

We would ideally like to isolate a particular scalar field arising from dimensional reduction of IIB supergravity on S^5, and this can indeed be achieved. For our purposes we will treat this as a "toy" example, we are fixing the overall normalization and a simple quadratic potential. In fact the dilaton-axion sector of the reduction on S^5 is essentially of this form.

We will need the Green's function for a scalar field on AdS_5 which we calculated above and impose a cut-off at the boundary:

$$\phi(x,z)\big|_{z=\epsilon} = \phi_0(x)\,. \tag{18}$$

The solution which achieves this is

$$\phi_\epsilon(x,z) = \int d^4k\, e^{ik\cdot x}\phi_0(k)\left(\frac{z}{\epsilon}\right)^2 \frac{K_\lambda(|k|z)}{K_\lambda(|k|\epsilon)}\,. \tag{19}$$

We now note that evaluating the action on-shell is easier if we integrate by parts

$$S_\phi = \frac{1}{2}\int_\Sigma d^4x\sqrt{g}\,\phi\, n^\mu\partial_\mu\phi + \frac{1}{2}\int d^5x\sqrt{g}\,\phi\big(-\Box\phi + m^2\phi\big) \tag{20}$$

where n^μ is a vector field orthogonal to the boundary Σ of AdS. We then perform the integral over x

$$\begin{aligned} S_\phi &= \int_\Sigma d^4x \int d^4k_1 d^4k_2 e^{i(k_1+k_2)\cdot x}\frac{\phi_0(k_1)\phi_0(k_2)}{z^5}\left(\frac{z}{\epsilon}\right)^2 \frac{K_\lambda(|k_1|z)}{K_\lambda(|k_1|\epsilon)} z\partial_z\Big[\left(\frac{z}{\epsilon}\right)^2 \frac{K_\lambda(|k_2|z)}{K_\lambda(|k_2|\epsilon)}\Big] \\ &= \int d^4k_1 d^4k_2 \delta(k_1+k_2)\frac{\phi_0(k_1)\phi_0(k_2)}{z^5}\left(\frac{z}{\epsilon}\right)^2 \frac{K_\lambda(|k_1|z)}{K_\lambda(|k_1|\epsilon)} z\partial_z\Big[\left(\frac{z}{\epsilon}\right)^2 \frac{K_\lambda(|k_2|z)}{K_\lambda(|k_2|\epsilon)}\Big] \end{aligned}$$

Expanding the above expression gives

$$G_2(k_1,k_2) = \langle\phi(k_1)\phi(k_2)\rangle \sim \delta(k_1+k_2)\epsilon^{2\Delta_+-8}k^{2\Delta_+}\log k \tag{21}$$

The Fourier transform of this requires some care and gives

$$G_2(x_1,x_2) \sim \frac{1}{(x_1-x_2)^{2\Delta_+}}\,. \tag{22}$$

The procedure just outlined is quite cumbersome and is not the most convenient method available. In [5], Witten uses a more convenient form of the Green's function

$$K(z,\vec{x};\vec{x}') = \frac{z^{\Delta_+}}{(z^2+(\vec{x}-\vec{x}')^2)^{\Delta_+}}\,. \tag{23}$$

A solution to the wave equation with boundary value $\phi_0(x)$

$$\phi(x) = \int d^4x' K(z,\vec{x};\vec{x}')\phi_0(x') \tag{24}$$

The two point function is then

$$\begin{aligned} G_2(x_1,x_2) &= \int d^5x \frac{K(z,x;x_1)z\partial_z K(z,x;x_2)}{z^5}\bigg|_{z=\epsilon} \\ &\sim \frac{\Delta_+}{(x_1-x_2)^{2\Delta_+}} \end{aligned} \tag{25}$$

The normalization of this requires some care and is only important when computing three point functions.

3 Higher Point Functions from the Bulk

To compute higher point functions from the bulk we need to expand the bulk action to higher orders in ϕ:

$$S_{\phi,n} = \frac{1}{2}\int d^5x\sqrt{g}\Big((\nabla\phi_i)^2 + m^2\phi_i^2 + \sum \lambda^{i_1\ldots i_k}\phi_{i_1}\ldots\phi_{i_k}\Big) \tag{26}$$

where we have a set of scalar fields ϕ_i.

The equation of motion is now non-linear and the Green's function cannot be solved exactly but we proceed by perturbation theory. Suppose we just take a cubic theory of one scalar field, then

$$\phi^{(0)} = \int d^4x' K(z,\vec{x};\vec{x}')\phi_0(x') \tag{27}$$

$$\phi^{(1)} = \lambda\int d^4x'dz' G(z,x;z',x')\int d^4x_1 d^4x_2 K(z',\vec{x}';\vec{x}_1)K(z',\vec{x}';\vec{x}_2)\phi_0(x_1)\phi_0(x_2) \tag{28}$$

$$\tag{29}$$

where $G(z,x;z',x')$ is the bulk-bulk propogator which solves

$$(\Box - m^2)G(z,x;z',x') = \frac{1}{\sqrt{g}}\delta(z-z')\delta(\vec{x}-\vec{x}') \tag{30}$$

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