Lecture 4. Finite-size effects

Plan

1. Wrapping effect

2. Thermodynamic Bethe ansatz method

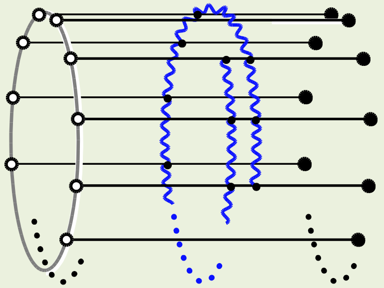
3. Luscher corrections

4. Y-systems

Wrapping problem

- High-order Feynman diagrams connect operators farther away
- When the length of a composite operator is shorter than the order of the perturbative expansion: “wrapping” interactions appear

  → BAE is valid only when the length is infinite
- The length of spin-chain is another important parameter



Three-loop su(2) Konishi

- su(2) Konishi $\mathrm{Tr}\,[ZZXX]\,, \qquad \mathrm{Tr}\,[ZXZX]$

- BAE : $p_1 = -p_2 = p, \qquad \sigma \approx 1 + \mathcal{O}(g^6)$

$$e^{i4p} = \frac{2u+i}{2u-i}, \qquad u = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}$$

- Perturbative solutions $p = \frac{2\pi}{3} - \sqrt{3}g^2 + \frac{9\sqrt{3}}{2}g^4 + \ldots$

- One gets

$$\triangle = 4 + \overbrace{\underset{\overset{\uparrow}{\frac{3}{4\pi^2}\lambda}}{12g^2} - 48g^4 + 336g^6}^{\text{Match with perturbative SYM}} + \mathcal{O}(g^8)$$

Four-loop

- BAE : $p_1 = -p_2 = p$

$$e^{i4p} = e^{-i72\sqrt{3}\zeta(3)g^6} \cdot \frac{2u+i}{2u-i}, \qquad u = \frac{1}{2}\cot\frac{p}{2}\sqrt{1+16g^2\sin^2\frac{p}{2}}$$

$$\uparrow$$

$$\sigma^2(u,v) \approx 1 + 256\zeta(3)g^6\frac{(u-v)(4uv-1)}{(1+4u^2)^2(1+4v^2)^2}, \qquad u = -v = \frac{1}{2\sqrt{3}}$$

- Perturbative solutions

$$p = \frac{2\pi}{3} - \sqrt{3}g^2 + \frac{9\sqrt{3}}{2}g^4 - \frac{72\sqrt{3}+72\sqrt{3}\zeta(3)}{3}g^6 + \dots$$

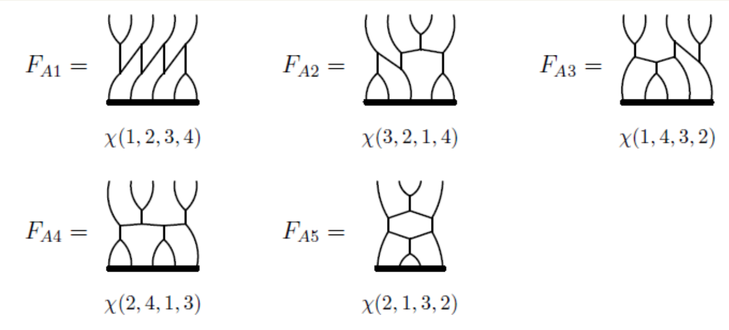
- BAE result:

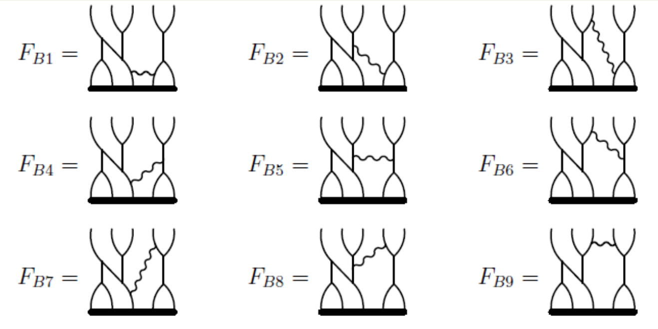
$$\Delta_{\mathrm{BAE}} = 4 + 12g^2 - 48g^4 + 336g^6 - (2820 + 288\zeta(3))g^8 + \mathcal{O}(g^{10})$$

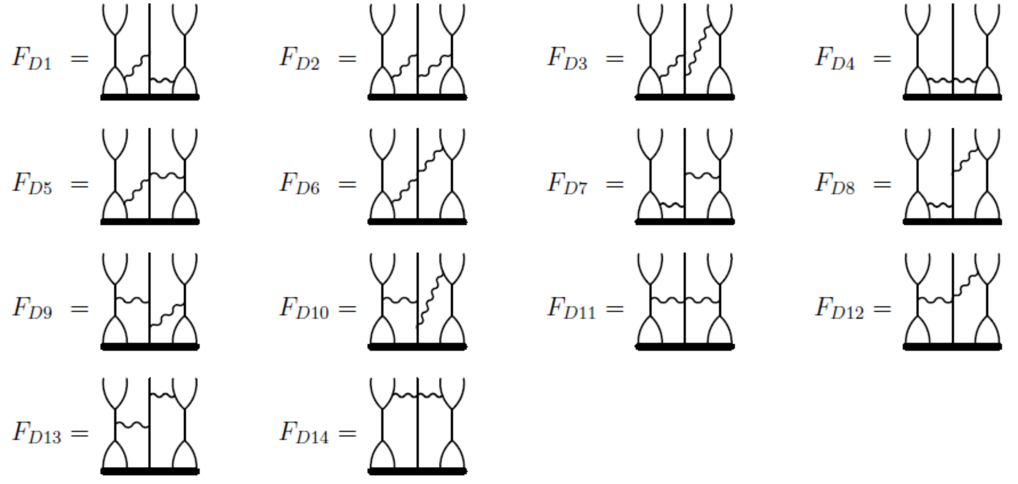
Perturbation theory (Feynman diagrams)

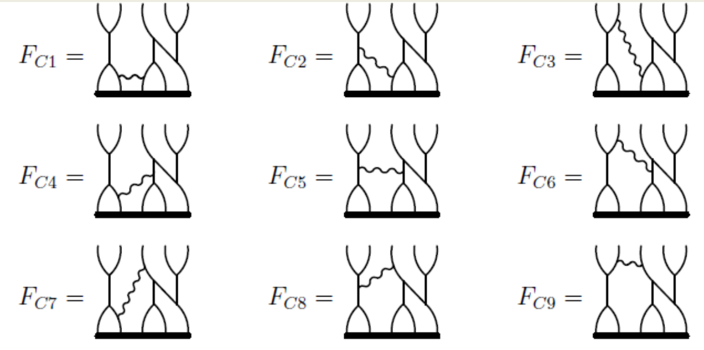
[Fiamberti, Santambrogio, Sieg, Zanon (2008)]

N=1 supergraphs

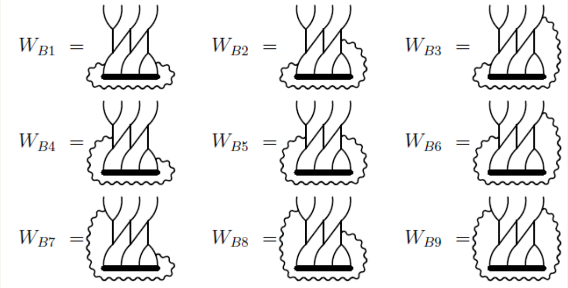


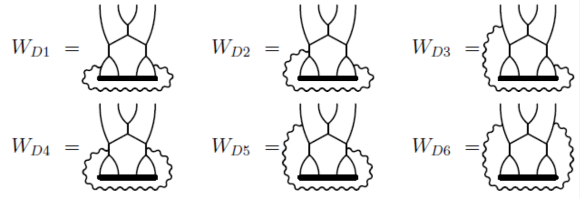


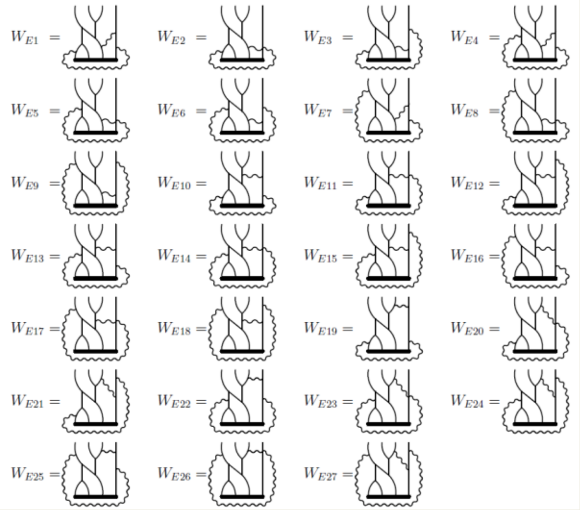


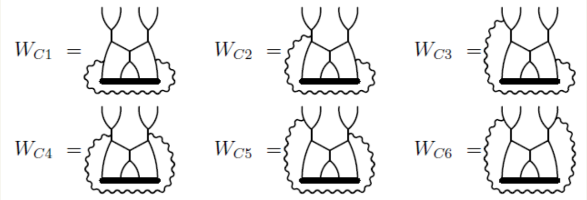


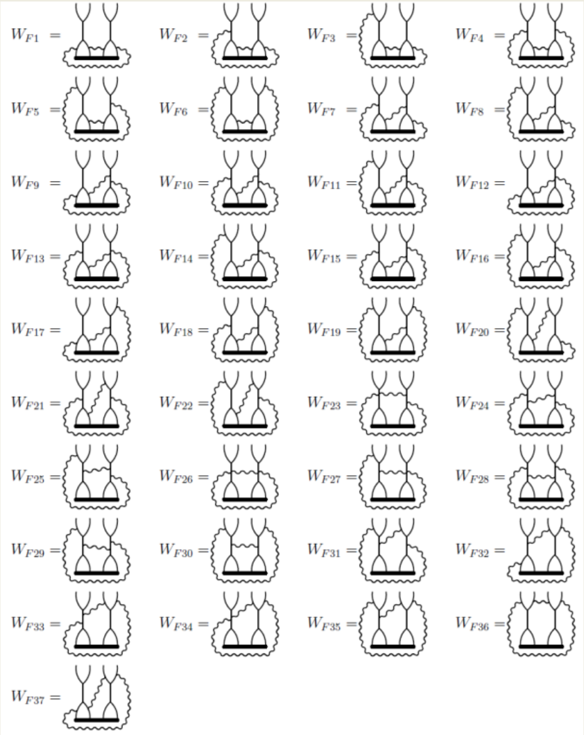
Wrapping diagrams

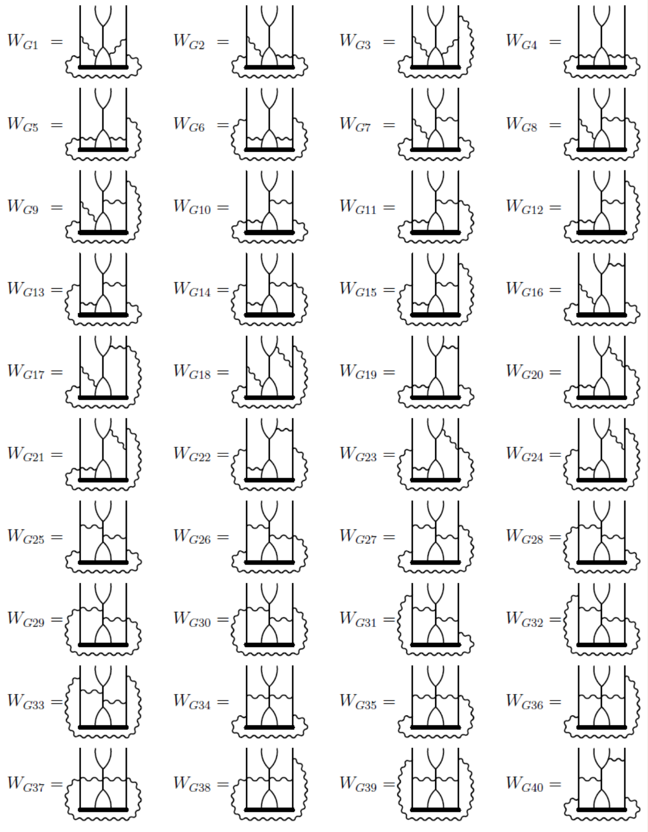












- Perturbative SYM calculation

$$\triangle_{\text{Pert.}} = 4+12g^2-48g^4+336g^6-\boxed{(2496-576\zeta(3)+1440\zeta(5))}g^8+\mathcal{O}(g^{10})$$

- (cf) BAE result:

$$\triangle_{\text{BAE}} = 4+12g^2-48g^4+336g^6-\boxed{(2820+288\zeta(3))}g^8+\mathcal{O}(g^{10})$$

- BAE is wrong at the 4-loop level

$$\delta\triangle = \triangle_{\text{Pert.}}-\triangle_{\text{BAE}} = (324+864\zeta(3)-1440\zeta(5))g^8+\mathcal{O}(g^{10})$$

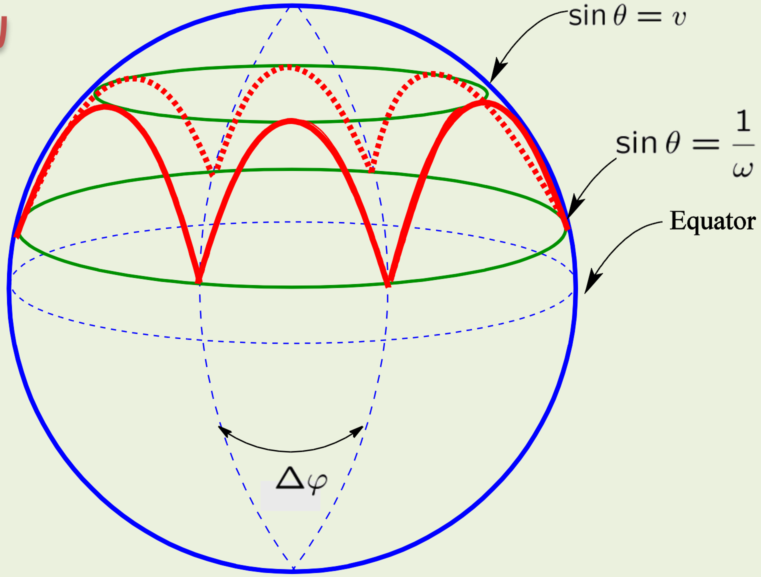
- WHY? Asymptotic BAE is valid only when infinite L !

- Need new formalism which works for a finite-size L

Finite size effect in strong coupling limit

- Dispersion relation for giant magnon

$$E-J \approx 4g\sin\frac{p}{2}\underbrace{-16g\sin^3\frac{p}{2}\ \exp\left[-\left(\frac{J}{2g\sin\frac{p}{2}}+2\right)\right]}_{\text{Finite-size effect}}+\dots$$



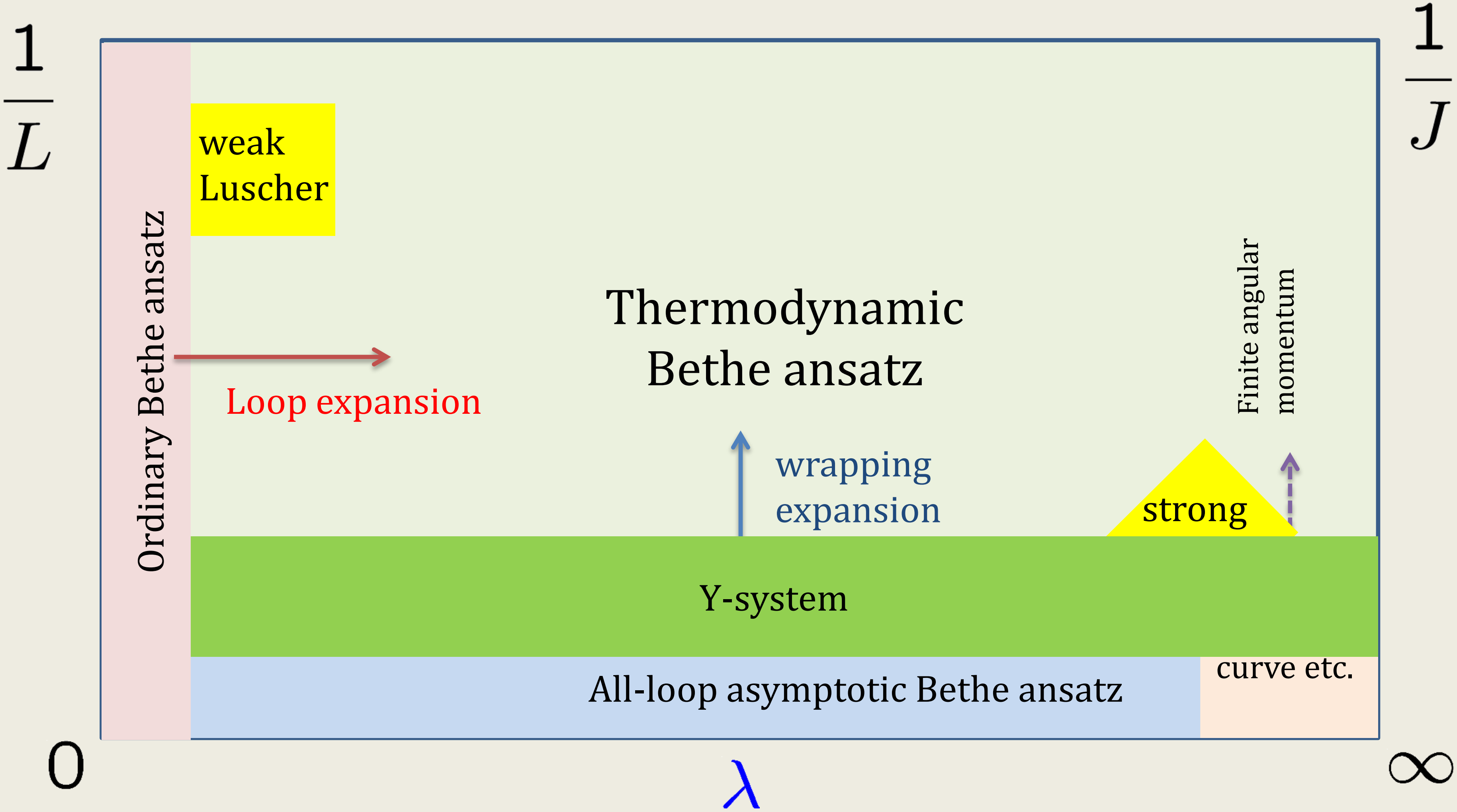
- How to compute this for general g ?

Thermodynamic Bethe ansatz

- From S-matrix to the finite-size effect

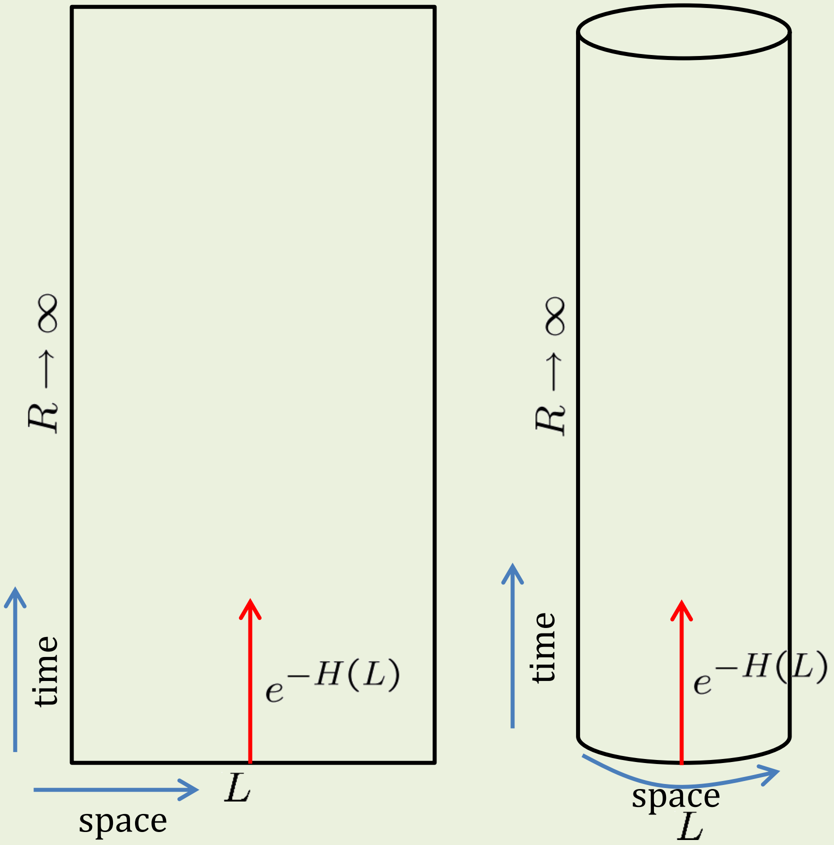
- Al. B. Zamolodchikov (1990)

Phase diagram of integrable methods

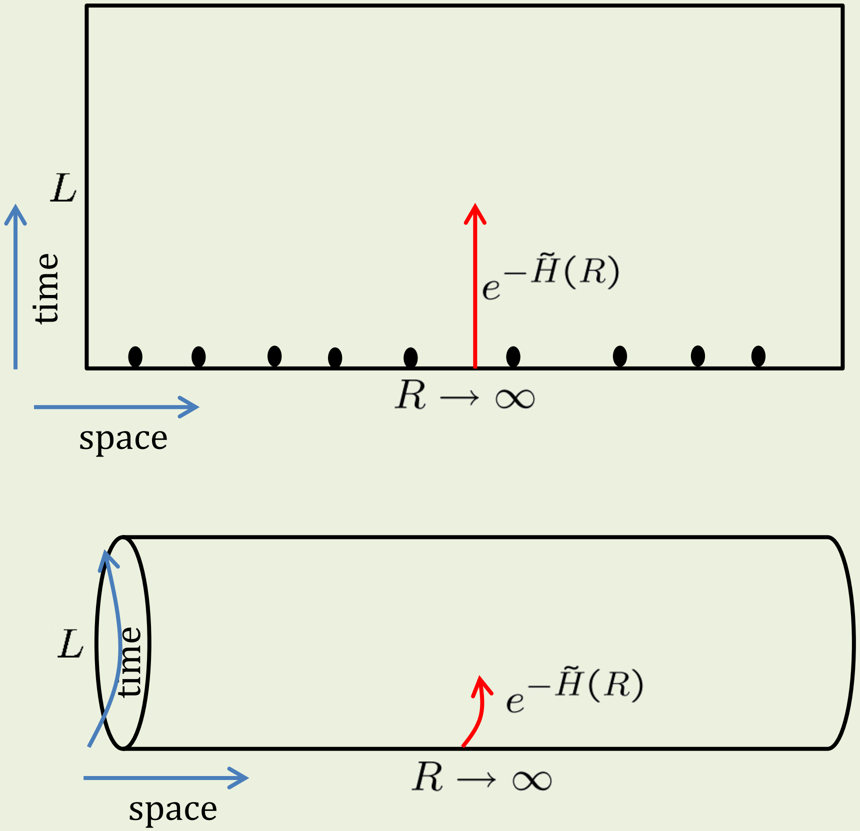


2d Euclidean geometry with PBC

Physical space



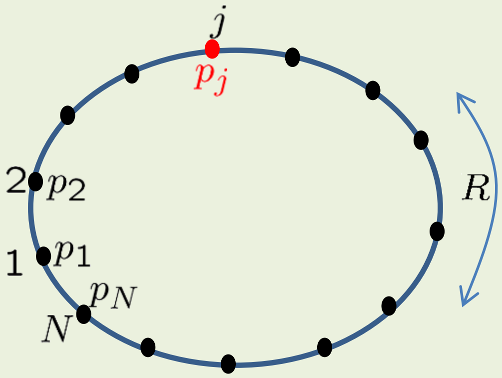
Mirror space



Channel duality

- Mirror channel
  - Particles with a dispersion relation $(\tilde{e}(u), \tilde{p}(u))$
  - S-matrix and scattering are valid only when $R \to \infty$
  - N-particles in a box of length R
  - Bethe-Yang equation $e^{i\tilde{p}(u_j)R} \prod_{k \neq j,1}^{N} S(u_j, u_k) = 1$
  - Partition function

$$\tilde{Z}(R, L) = \mathrm{Tr}\left[e^{-L\tilde{H}(R)}\right]$$



- Physical channel
  - Dispersion relation $(e, p) = (-i\tilde{p}, -i\tilde{e})$
  - Partition function $Z(L, R) = \mathrm{Tr}\left[e^{-RH(L)}\right] \approx e^{-RE_0(L)}$ as $R \to \infty$

$$\tilde{Z}(R, L) = Z(L, R) \quad \to \quad E_0(L) = -\frac{1}{R} \ln \tilde{Z}(R, L) = \frac{L}{R} \tilde{\mathcal{F}}(L)$$

Free energy with temperature

$$T = \frac{1}{L}$$

  - Computing free energy in the mirror space

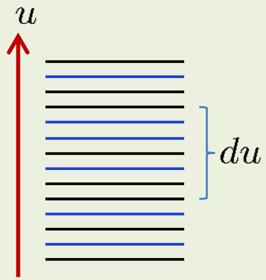
$$\tilde{\mathcal{F}}(L) = \tilde{E} - T\mathcal{S}$$

- Mirror free energy with $N, \ R \rightarrow \infty$
  - Log of Bethe-Yang equation :

any integer

$$\tilde{p}(u_j) - \frac{i}{R}\sum_{k\neq j,1}^{N} \ln S(u_j,u_k) = 2\pi\frac{n_j}{R} \ \rightarrow \ \tilde{p}(u_j) + \int u'\rho(u')\frac{1}{i}\ln S(u_j,u') = 2\pi\frac{n_j}{R}$$

$$\rightarrow \ \frac{d\tilde{p}}{du} + \int du'\rho(u')\frac{1}{i}\frac{\partial}{\partial u}\ln S(u,u') = 2\pi[\rho_h(u)+\rho(u)]$$



  - *n* = # of particles, $\rho(u) = \frac{1}{R}\frac{dn}{du}, \quad \rho_h(u) = \frac{1}{R}\frac{dn}{du}$
  - *dn* = # of particles with u-values between *u* and *u+du*
  - *n* = # of unoccupied ('holes') states

  - Energy $\tilde{E} = \sum_{j=1}^{N} \tilde{e}(u_j) = R\int du \ \rho(u) \ \tilde{e}(u)$

  - Entropy : log of # of cases $\mathcal{S} = R\int du\,[(\rho_h+\rho)\ln(\rho_h+\rho) - \rho_h\ln\rho_h - \rho\ln\rho]$

  - Free energy: $\boxed{L\tilde{F}(L) = R\int du\,\{L\tilde{e}(u)\rho(u) - [(\rho_h+\rho)\ln(\rho_h+\rho) - \rho_h\ln\rho_h - \rho\ln\rho]\}}$

  - Minimize free energy with the constraint of PBC

- Lagrange multiplier

$$F[\rho_h,\rho] = R\int du \left\{ L\tilde{e}(u)\rho(u) - [(\rho_h+\rho)\ln(\rho_h+\rho) - \rho_h\ln\rho_h - \rho\ln\rho] - \lambda(u)\left[\rho_h(u)+\rho(u) - \int\frac{du'}{2\pi}K(u,u')\rho(u')\right]\right\}$$

$$K(u,u') \equiv \frac{1}{i}\frac{\partial}{\partial u}\ln S(u,u')$$

$$\frac{\delta}{\delta\rho_h(u)}F[\rho_h,\rho] = \frac{\delta}{\delta\rho(u)}F[\rho_h,\rho] = 0 \quad\longrightarrow\quad \begin{cases} \ln\rho_h - [\ln(\rho_h+\rho)] - \lambda(u) = 0 \\ L\tilde{E}(u) - [\ln(\rho_h+\rho) - \ln\rho] - \lambda(u) + \int\frac{du'}{2\pi}K(u',u)\lambda(u') = 0 \end{cases}$$

$$\epsilon(u) \equiv \ln[\rho_h/\rho]$$

- TBA eq.

$$\boxed{\epsilon(u) = L\tilde{e}(u) - \int\frac{du'}{2\pi}K(u',u)\ln\left[1+e^{-\epsilon(u')}\right]}$$

- Minimized free energy : plug into *F* and use TBA and partial integrate

$$\boxed{E_0(L) = -\int\frac{du}{2\pi}\,\tilde{p}'(u)\;\ln\left[1+e^{-\epsilon(u)}\right]}$$

- Generalizations needed
  - Multi-species
  - Excited states
  - Non-diagonal S-matrix

- Multi-species : with dispersion relations $(\tilde{e}_n(u),\ \tilde{p}_n(u))\,, \quad n = 1, \ldots, M$
- S-matrix : $S_{n,m}(u, u')$
- TBA eq. $$\boxed{\epsilon_n(u) = L\tilde{e}_n(u) - \sum_{m=1}^{M} \int \frac{du'}{2\pi} K_{nm}(u', u) \ln\left[1 + e^{-\epsilon_m(u')}\right]}$$ $K_{nm}(u, u') \equiv \frac{1}{i}\frac{\partial}{\partial u} \ln S_{nm}(u, u')$
- Ground-state energy : $$\boxed{E_0(L) = -\sum_{n=1}^{M} \int \frac{du}{2\pi}\ \tilde{p}'_n(u)\ \ln\left[1 + e^{-\epsilon_n(u)}\right]}$$
- Excited states for single species: partial integrate

$$E_0(L) = \int \frac{du}{2\pi}\ \tilde{p}(u)\ \partial_u \ln\left[1 + e^{-\epsilon(u)}\right] \qquad \epsilon(u) = L\tilde{e}(u) + \int \frac{du'}{2\pi i} \ln S(u', u) \partial_{u'} \ln\left[1 + e^{-\epsilon(u')}\right]$$

- If $\ln\left[1 + e^{-\epsilon(u_j)}\right] = 0$, deform the integral contour and residue integrate

Dorey,Tateo (1996)

Mirror momentum

Physical energy

$$E(L) = -\sum_j i\tilde{p}(u_j) + \int \frac{du}{2\pi}\ \tilde{p}(u)\ \partial_u \ln\left[1 + e^{-\epsilon(u)}\right] = \sum_j e(u_j) - \int \frac{du}{2\pi}\ \tilde{p}'(u)\ \ln\left[1 + e^{-\epsilon(u)}\right]$$

$$\epsilon(u) = L\tilde{e}(u) + \sum_i \ln S(u_i, u) - \int \frac{du'}{2\pi} K(u', u) \ln\left[1 + e^{-\epsilon(u')}\right]$$

- Multi-species excited states :

$$E(L) = \sum_i e_{n_i}(u_i) - \sum_{n=1}^{M} \int \frac{du}{2\pi}\, \tilde{p}'_n(u)\ \ln\left[1 + e^{-\epsilon_n(u)}\right]$$

$$\epsilon_n(u) = L\tilde{e}_n(u) + \sum_i \ln S_{n_i,n}(u_i,u) - \sum_{m=1}^{M} \int \frac{du'}{2\pi} K_{nm}(u',u) \ln\left[1 + e^{-\epsilon_m(u')}\right]$$

- Non-diagonal S-matrix :
- Diagonalize the transfer matrix to derive “Bethe-Yang” or “asymptotic Bethe” equations
- Interpret these as PBC conditions :
  - “physical” (momentum carrying) : Bethe-Yang equation
  - “magnonic” (no momentum) particles : Bethe ansatz equations
- Read off “effective” diagonal S-matrices
- Apply TBA equations derived already
- (ex) su(2) S-matrix

$$e^{ip(\theta_j)L} \prod_{n=1}^{N} \underset{\overset{\uparrow}{S\text{pp}}}{a(\theta_j - \theta_n)} \prod_{k=1}^{M} \frac{a(u_k - \theta_j)}{\underset{\overset{\uparrow}{S\text{pm}}}{b(u_k - \theta_j)}} = 1 \qquad \prod_{n=1}^{N} \frac{b(u_k - \theta_n)}{\underset{\overset{\uparrow}{S\text{mp}}}{a(u_k - \theta_n)}} \prod_{j\neq k, j=1}^{M} \frac{u_k - u_j + i}{\underset{\overset{\uparrow}{S\text{mm}}}{u_k - u_j - i}} = 1$$

Effective diagonal S-matrix for AdS/CFT

$$1 = \prod_{k=1}^{K_2} \overset{\displaystyle S_{22}\atop\uparrow}{\boxed{\frac{u_{2j}-u_{2k}-i}{u_{2j}-u_{2k}+i}}} \prod_{k=1}^{K_3+K_1} \overset{\displaystyle S_{23}\atop\uparrow}{\boxed{\frac{u_{2j}-u_{3k}+\frac{i}{2}}{u_{2j}-u_{3k}-\frac{i}{2}}}}$$

$$S_{32} \longleftarrow \quad 1 = \prod_{k=1}^{K_2} \boxed{\frac{u_{3j}-u_{2k}+\frac{i}{2}}{u_{3j}-u_{2k}-\frac{i}{2}}} \prod_{k=1}^{K_4} \boxed{\frac{x_{3j}-x^+_{4k}}{x_{3j}-x^-_{4k}}} \longrightarrow S_{34}$$

$$\left(\frac{x^+_{4j}}{x^-_{4j}}\right)^{L'} = \prod_{k=1}^{K_4} \boxed{\sigma^2(x_{4j},x_{4k})\frac{u_{4j}-u_{4k}+i}{u_{4j}-u_{4k}-i}} \longrightarrow S_{44}$$

$$S_{43} \longleftarrow \quad \times \prod_{k=1}^{K_3+K_1} \boxed{\frac{x^-_{4j}-x_{3k}}{x^+_{4j}-x_{3k}}} \prod_{k=1}^{K_5+K_7} \frac{x^-_{4j}-x_{5k}}{x^+_{4j}-x_{5k}}$$

$$1 = \prod_{k=1}^{K_6} \frac{u_{5j}-u_{6k}+\frac{i}{2}}{u_{5j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{5j}-x^+_{4k}}{x_{5j}-x^-_{4k}}$$

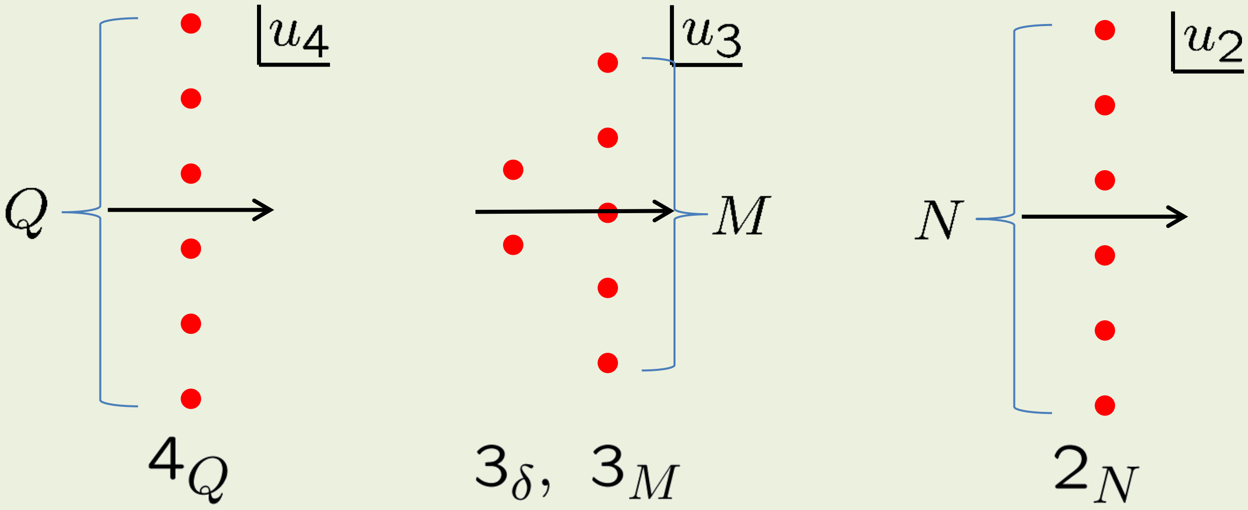
$$1 = \prod_{k=1}^{K_6} \frac{u_{6j}-u_{6k}-i}{u_{6j}-u_{6k}+i} \prod_{k=1}^{K_5+K_7} \frac{u_{6j}-u_{5k}+\frac{i}{2}}{u_{6j}-u_{5k}-\frac{i}{2}}$$

$$\boxed{\begin{aligned}
1 &= \prod_{k=1}^{M_2} S_{22}(x_{2j},x_{2k}) \prod_{k=1}^{M_3} S_{23}(x_{2j},x_{3k}) \\
1 &= \prod_{k=1}^{M_2} S_{32}(x_{3j},x_{2k}) \prod_{k=1}^{M_4} S_{34}(x_{3j},x_{4k}) \\
1 &= e^{ip_jR} \prod_{k=1}^{M_4} S_{44}(x_{4j},x_{4k}) \prod_{k=1}^{M_3} S_{43}(x_{4j},x_{3k}) \prod_{k=1}^{M_5} S_{43}(x_{4j},x_{5k}) \\
1 &= \prod_{k=1}^{M_6} S_{32}(x_{5j},x_{6k}) \prod_{k=1}^{M_4} S_{34}(x_{5j},x_{4k}) \\
1 &= \prod_{k=1}^{M_6} S_{22}(x_{6j},x_{6k}) \prod_{k=1}^{M_5} S_{23}(x_{6j},x_{5k})
\end{aligned}}$$

String hypothesis

Arutyunov-Frolov

- AdS/CFT contains infinite # of bound states and need their ABAEs
- The bound states belong to higher dimensional representation of su(2|2) and their S-matrices can be determined by both su(2|2) and "yangian" symmetry [(cf) Rafael's Lecture]
- Bypassing derivation ABAE for the bound states, one can find the "diagonal" S-matrices by studying the string solutions by a similar logic of su(2) case
- Classes of strings $4_Q,\ 3_\delta,\ 3_M,\ 2_N$



- Effective diagonal S-matrices for the bound states

$$S_{44}^{(QQ')} = \sigma_{QQ'} E_{QQ'}$$

$$S_{43}^{(QM)} = \frac{x(u_{-Q}) - x(v_M)}{x(u_Q) - x(v_M)} \frac{x(u_{-Q}) - x(v_{-M})}{x(u_Q) - x(v_{-M})} \frac{x(u_Q)}{x(u_{-Q})} \prod_{j=1}^{M-1} e_{M-Q-2j}$$

$$S_{43}^{(Q\delta)} = \frac{x(u_{-Q}) - x(v)^\delta}{x(u_Q) - x(v)^\delta} \sqrt{\frac{x(u_Q)}{x(u_{-Q})}}$$

$$S_{33}^{(MM')} = S_{22}^{(MM')^{-1}} = E_{MM'}$$

$$S_{33}^{(M\delta)} = S_{23}^{(M\delta)} = e_M$$

$$e_n(u) \equiv \frac{u + in/2g}{u - in/2g}$$

$$E_{n,m} = e_{|n-m|} e_{|n-m|+2}^2 \cdots e_{n+m-2}^2 e_{n+m}$$

$$x(u_M) + \frac{1}{x(u_M)} = u_M, \quad x^+(u_M) + \frac{1}{x^+(u_M)} - x^-(u_M) - \frac{1}{x^-(u_M)} = \frac{iM}{g}$$

- Asymptotic BAE for these strings can be constructed straightforwardly since the scatterings are diagonal and TBA can be derived accordingly

TBA for AdS/CFT

- Thermodynamic BAE

Arutyunov,Frolov; Bombardelli,Fioravanti,Tateo; Gromov,Kazakov,Kozak,Vieira (2009)

$$\begin{aligned}
\epsilon_4^{(Q)} &= L\tilde{e}_Q - L_4^{(Q')} \star K_{44}^{(Q'Q)} - L_3^{(M)} \star K_{34}^{(MQ)} - L_3^{(\delta)} \star K_{34}^{(\delta Q)} \\
\epsilon_3^{(M)} &= -L_4^{(Q)} \star K_{43}^{(QM)} - L_3^{(M')} \star K_{33}^{(M'M)} - L_3^{(\delta)} \star K_{33}^{(\delta M)} \\
\epsilon_2^{(N)} &= L_2^{(N')} \star K_{22}^{(N'N)} - L_3^{(\delta)} \star K_{32}^{(\delta N)} \\
\epsilon_3^{(\delta)} &= -L_4^{(Q)} \star K_{43}^{(Q\delta)} - L_3^{(M)} \star K_{33}^{(M\delta)} - L_2^{(N)} \star K_{23}^{(N\delta)}
\end{aligned}$$

$$A \star K(u) = \int \frac{du'}{2\pi} A(u')K(u', u)$$

$$K_{ab}^{(nm)}(u, u') \equiv -i\partial_u \ln S_{ab}^{(nm)}(u, u')$$

- Physical dispersion relation

$$e_n(p) = \sqrt{n^2 + 16g^2 \sin^2 \frac{p}{2}}$$

- Mirror one $(e, p) = (i\tilde{p}, -i\tilde{e})$

$$\tilde{e}_n(\tilde{p}) = 2 \sinh^{-1} \left(\frac{1}{4g} \sqrt{\tilde{p}^2 + n^2} \right)$$

- Finite-size energy

$$E_0(L) = -\sum_{Q=1}^{\infty} \int \frac{du}{2\pi} \, \tilde{p}'_Q \, \ln\left(1 + e^{-\epsilon_4^{(Q)}}\right)$$

- Excited TBA by analytic continuation or by “Y-system”

Universal kernel

- One can introduce kernel which connects only nearest neighbors

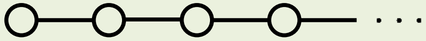
$$s\mathbb{I}_{MN} = \delta_{MN} - (K+1)^{-1}_{MN}, \qquad s(u) = \frac{g}{2\cosh g\pi u}$$

Incidence matrix of a Dynkin diagram

$$K \equiv \frac{1}{i}\frac{\partial}{\partial u}\ln S$$

  - (ex) bound-state S-matrix of su(2)-invariant theory

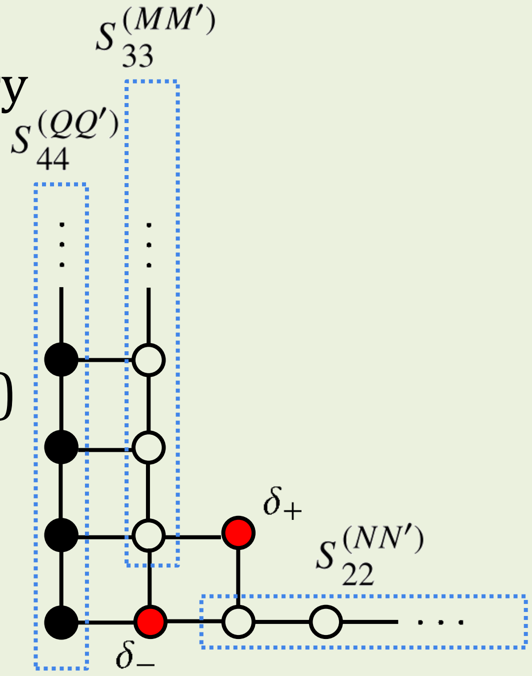
$$S^{(nm)}(u-v) = E_{nm}(u-v) = e_{|n-m|}e^{2}_{|n-m|+2}\cdots e^{2}_{n+m-2}e_{n+m}(u-v)$$



$$\mathbb{I}_{MN} = \delta_{M,N-1} + \delta_{M,N+1}$$

  - universal kernel for AdS/CFT (2d Dynkin diagram)

$$S^{(nm)}_{ab}(u,v) \ \rightarrow\ K^{(nm)}_{ab}$$



- Define Y-functions :

$$Y_{Q,0} = e^{-\epsilon_4^{(Q)}}, \quad Y_{M+1,1} = e^{-\epsilon_3^{(M)}}, \quad Y_{1,N+1} = e^{\epsilon_2^{(N)}}, \quad Y_{1,1} = e^{\epsilon_3^{(\delta=-)}}, \quad Y_{2,2} = e^{\epsilon_3^{(\delta=+)}}$$

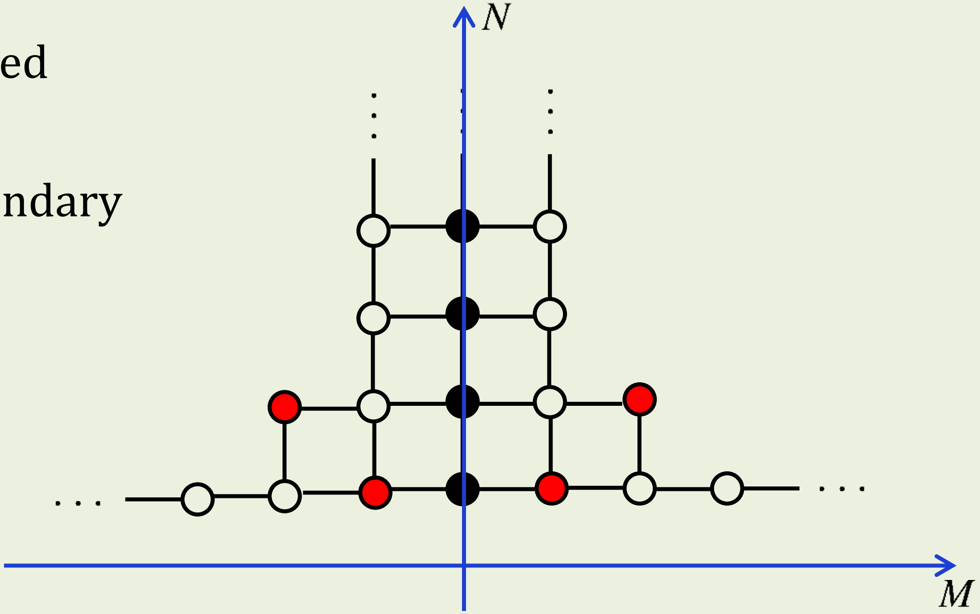
- Add another side of su(2|2) S-matrix

$$Y_{Q,0} = e^{-\epsilon_4^{(Q)}}, \quad Y_{M+1,-1} = e^{-\epsilon_3^{(M)}}, \quad Y_{1,-(N+1)} = e^{\epsilon_2^{(N)}}, \quad Y_{1,-1} = e^{\epsilon_3^{(\delta=-)}}, \quad Y_{2,-2} = e^{\epsilon_3^{(\delta=+)}}$$

- TBA with universal kernel

$$\ln Y_{N,M} = s\star\Big[\ln(1+Y_{N,M+1}) + \ln(1+Y_{N,M-1})\Big] - s\star\Big[\ln(1+Y^{-1}_{N+1,M}) + \ln(1+Y^{-1}_{N-1,M})\Big]$$

  - “deriving term” can be absorbed into boundary condition
  - Excitation states from the boundary conditions of Y’s



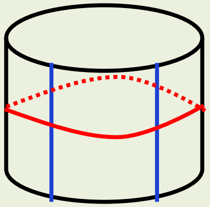
Luscher correction

- Analytic analysis of TBA is possible when $L\tilde{e}_n(u)$ is large
- Consider two-particle excitation for one-particle species theory
  - TBA eq. $\epsilon(u) = L\tilde{e}(u) + \ln S(u_1, u) + \ln S(u_2, u) - \int \frac{du'}{2\pi} K(u', u) \ln\left[1 + e^{-\epsilon(u')}\right]$
  - Constraint eq. $1 + e^{-\epsilon(u_i)} = 0$
  - Energy $E(L) = e(u_1) + e(u_2) - \int \frac{du}{2\pi} \tilde{p}'(u) \ \ln\left[1 + e^{-\epsilon(u)}\right]$
- In the limit of $L\tilde{e}(u) \gg 1$

$$\epsilon(u) \approx L\tilde{e}(u) + \ln S(u_1, u) + \ln S(u_2, u)$$

  - Bethe-Yang eq. $e^{-\epsilon(u_1)} = -e^{-ip(u_1)L} S(u_2, u_1) = -1$
  - Finite-size correction for energy

$$E \ = \ e(u_1) + e(e_2) - \int \frac{dq}{2\pi} e^{-L\tilde{e}(q)} S(u, u_1) S(u, u_2), \quad q \equiv \tilde{p}(u)$$

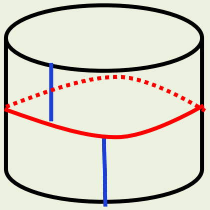


Classical string with finite J

- Energe correction for a single Giant Magnon $J \gg g \gg 1$ **Janik,Lukowski (2007)**

$$\delta E \approx -16 g \sin^3 \frac{p}{2} \; \exp\left[-\left(\frac{J}{2g\sin\frac{p}{2}}+2\right)\right]+\ldots$$

- Luscher formula simplified when S-matrix has a pole



$$\delta E \approx -i\left[1-\frac{e'(p)}{e'(q^*)}\right]\cdot \underset{q=q^*}{\mathrm{res}} \sum_b S_{ba}^{ba}(q,p)\cdot e^{-iLq^*}$$

$$\delta E_{\mathrm{Luscher}} \approx -16 g \sin^3 \frac{p}{2} \; e^{-\frac{J}{2g\sin\frac{p}{2}}-2}$$

Wrapping correction for 4-loop Konishi

- $\delta\triangle = \triangle_{\text{Pert.}} - \triangle_{\text{BAE}} = (324+864\zeta(3)-1440\zeta(5))g^8+\mathcal{O}(g^{10})$ **Bajnok, Janik (2008)**
- Luscher formula : (*L*=4) [*μ* term vanishes]

$$E(L) = \sum_{k=1,2} e_a(p_k) - \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{a_1,a_2} (-1)^F \left[S_{a_1a}^{a_2a}(q,p_1)S_{a_2a}^{a_1a}(q,p_2)\right] e^{-L\tilde{e}_{a_1}(q)}$$

  - Exponential factor using the mirror dispersion relation

$$\tilde{e}_n(\tilde{p}) = 2\sinh^{-1}\left(\frac{1}{4g}\sqrt{\tilde{p}^2+n^2}\right) \;:\; e^{-2L\sinh^{-1}\frac{\sqrt{n^2+q^2}}{4g}} \to \frac{4^L g^{2L}}{(n^2+q^2)^L} \sim \mathcal{O}(g^8)$$

  - All the bound states contribute to the same order and one needs the matrix element and dressing factor for these in the mirror space
  - After some algebras, the integrand becomes

$$-\int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{147456Q^2(3q^3+3Q^2-4)^2}{(q^2+Q^2)^4(9q^4+6[3(Q-2)Q+2]q^2+[3(Q-2)Q+4]^2)} \frac{1}{9q^4+6[3(Q+2)Q+2]q^2+[3(Q+2)Q+4]^2}$$

  - Residue integrals

$$\sum_{Q=1}^{\infty}\left\{-\frac{\text{num}(Q)}{(9Q^4-3Q^2+1)^4(27Q^6-27Q^4+36Q^2+16)}+\frac{864}{Q^3}-\frac{1140}{Q^5}\right\}$$

$$\text{num}(Q) = 7776Q(19683Q^{18}-78732Q^{16}+150903Q^{14}-134865Q^{12}+1458Q^{10}+48357Q^8-13311Q^6-1053Q^4+369Q^2-10)$$

$$\delta E = 324 + 864\zeta(3) - 1140\zeta(5)$$

Summary

- Spectral problem for any operator and any value of coupling constant is “solved”

- Still solve infinite TBA equations

- Numerical solutions need a lot of running time

- Efforts to reduce it to a finite set of equations (NLIE) seem successful [Gromov-Kazakov-Leurent-Volin, Balog-Hegedus]