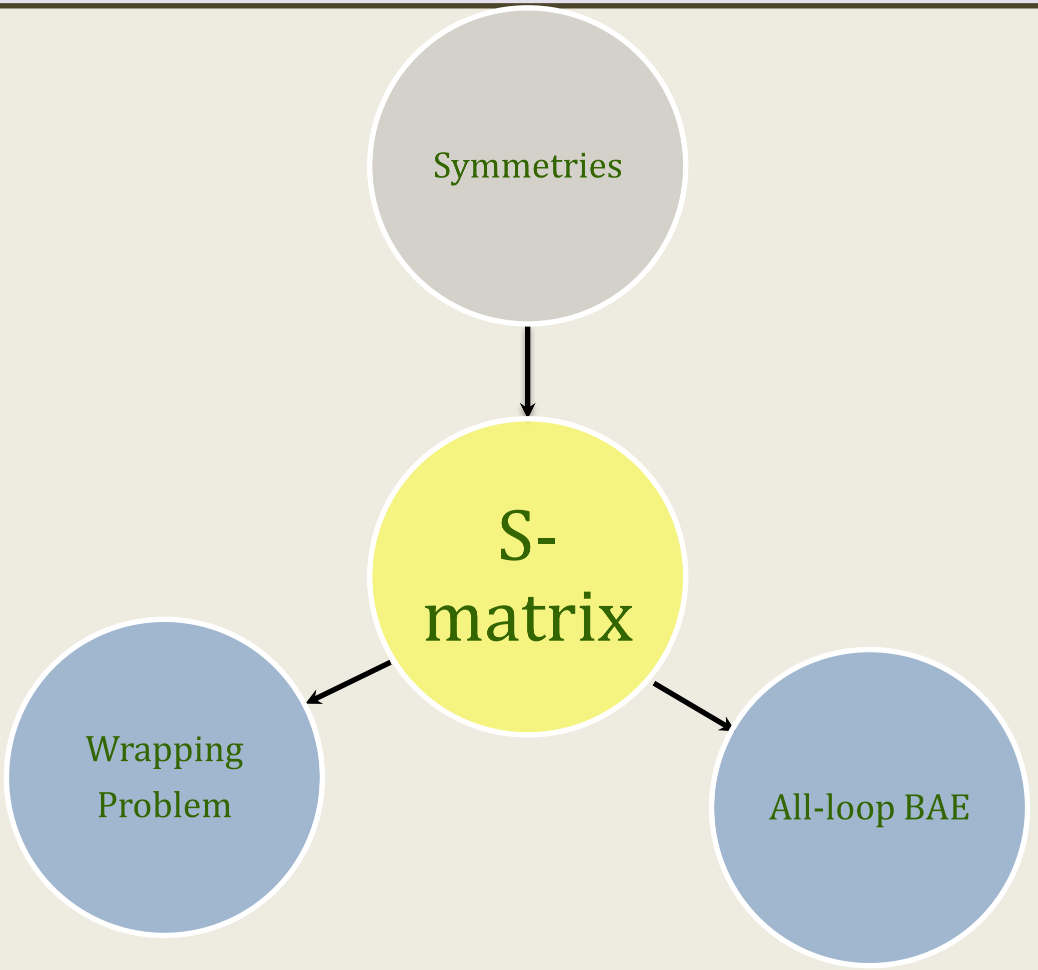
Lecture 3. Nonperturbative integrability S-matrix

S-matrix program



Excitation spectrum

- Excitation over the vacuum

$$\mathrm{Tr}\left[Z\cdots\chi_1 Z\cdots\chi_2 Z\cdots\chi_3 Z\cdots Z\chi_n Z\cdots\right]$$

Any field except Z, $\overline{\mathrm{Z}}$

$$\Phi_{a\dot{a}}\oplus\chi^{a}_{\dot{\alpha}}\oplus\chi^{\dot{a}}_{\alpha}\oplus D_{\alpha\dot{\alpha}}Z$$

$$a,\dot{a}=1,\ 2\ ;\alpha,\dot{\alpha}=3,\ 4$$

  - Each SYM field is a "meson" made of a "quark" and an "anti-quark"

$$\Phi_{a\dot{a}}=\phi_a\phi_{\dot{a}},\quad \chi^{a}_{\dot{\alpha}}=\phi_a\psi_{\dot{\alpha}},\quad \chi^{\dot{a}}_{\alpha}=\psi_\alpha\phi_{\dot{a}},\quad \mathcal{D}_{\alpha\dot{\alpha}}Z=\psi_\alpha\psi_{\dot{\alpha}}$$

  - Fundamental representation of su(2|2)

$$\boxslash=(\phi_a|\psi_\alpha)=(\phi_1,\phi_2|\psi_3,\psi_4)$$

Centrally extended su(2|2) symmetry

- Symmetry of the excitations: su(2|2) x su(2|2) **Beisert (2008)**

$$\left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}_a^{\dagger\beta} & \mathbb{R}_\alpha{}^\beta \end{array}\right), \quad \left(\begin{array}{c|c} \mathbb{L}_{\dot a}{}^{\dot b} & \mathbb{Q}_{\dot\alpha}{}^{\dot b} \\ \hline \mathbb{Q}_{\dot a}^{\dagger\dot\beta} & \mathbb{R}_{\dot\alpha}{}^{\dot\beta} \end{array}\right)$$

- Fundamental representation $\boxslash = (\phi_a|\psi_\alpha) = (\phi_1, \phi_2|\psi_3, \psi_4)$
- Commutation relations

$$\begin{aligned}
\left[\mathbb{L}_a{}^b, \mathbb{J}_c\right] &= \delta_c^b \mathbb{J}_a - \frac{1}{2}\delta_a^b \mathbb{J}_c, \quad \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}_\gamma\right] = \delta_\gamma^\beta \mathbb{J}_\alpha - \frac{1}{2}\delta_\alpha^\beta \mathbb{J}_\gamma, \\
\left[\mathbb{L}_a{}^b, \mathbb{J}^c\right] &= -\delta_a^c \mathbb{J}^b + \frac{1}{2}\delta_a^b \mathbb{J}^c, \quad \left[\mathbb{R}_\alpha{}^\beta, \mathbb{J}^\gamma\right] = -\delta_\alpha^\gamma \mathbb{J}^\beta + \frac{1}{2}\delta_\alpha^\beta \mathbb{J}^\gamma, \\
\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_\beta{}^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\mathbb{C}, \quad \{\mathbb{Q}_a^{\dagger\alpha}, \mathbb{Q}_b^{\dagger\beta}\} = \epsilon^{\alpha\beta}\epsilon_{ab}\mathbb{C}^\dagger, \\
\{\mathbb{Q}_\alpha{}^a, \mathbb{Q}_b^{\dagger\beta}\} &= \delta_b^a \mathbb{R}_\alpha{}^\beta + \delta_\alpha^\beta \mathbb{L}_b{}^a + \frac{1}{2}\delta_b^a \delta_\alpha^\beta \mathbb{H}
\end{aligned}$$

- From the algebra, central charge is

$$\mathbb{H} = -ig\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right) = \sqrt{1 + 16g^2 \sin^2\frac{p}{2}}$$

- With $x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}, \quad \frac{x^+}{x^-} = e^{ip}$ $\qquad x^\pm = e^{\pm i\frac{p}{2}}\left[\frac{1+\sqrt{1+16g^2\sin^2\frac{p}{2}}}{4g\sin\frac{p}{2}}\right]$

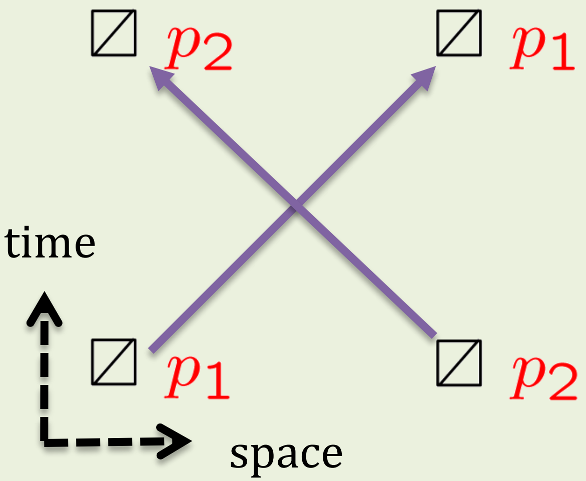
- Comparing with BMN limit

$$E = \sqrt{1 + \frac{\lambda}{J^2}n_j^2} \qquad \left(p_j = \frac{2\pi n_j}{J}\right) \qquad g \equiv \frac{\sqrt{\lambda}}{4\pi}$$

S-matrix from su(2|2) symmetry

- S-matrix $\mathbb{S} = \mathbf{S} \otimes \dot{\mathbf{S}}, \qquad \mathbf{S} = \dot{\mathbf{S}}$

Beisert (2008)



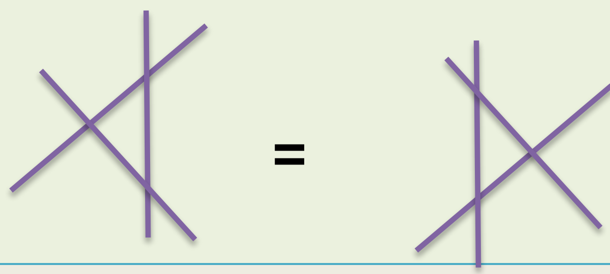
- S-matrix should commute with su(2|2)

$$\left[\mathbf{S}(p_1,p_2), \left(\begin{array}{c|c} \mathbb{L}_a{}^b & \mathbb{Q}_\alpha{}^b \\ \hline \mathbb{Q}^{\dagger\beta}_a & \mathbb{R}_\alpha{}^\beta \end{array}\right)\right] = 0$$

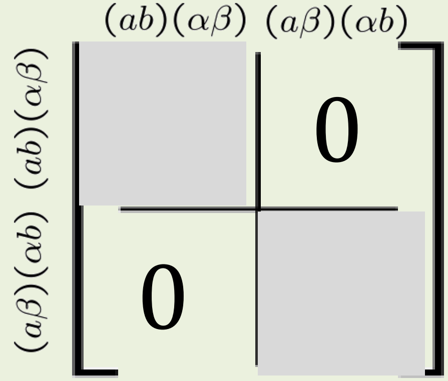
- S satisfies Yang-Baxter equation

Arutyunov, Frolov, Zamaklar (2008)

$$\mathbf{S}_{12}(p_1,p_2)\,\mathbf{S}_{13}(p_1,p_3)\,\mathbf{S}_{23}(p_2,p_3) = \mathbf{S}_{23}(p_2,p_3)\,\mathbf{S}_{13}(p_1,p_3)\,\mathbf{S}_{12}(p_1,p_2)$$



- S : 16 x 16 matrix



$$
\begin{aligned}
S^{a\,a}_{a\,a} &= A\,, \quad S^{\alpha\,\alpha}_{\alpha\,\alpha} = D\,,\\
S^{a\,b}_{a\,b} &= \frac{1}{2}(A-B)\,, \quad S^{b\,a}_{a\,b} = \frac{1}{2}(A+B)\,,\\
S^{\alpha\,\beta}_{\alpha\,\beta} &= \frac{1}{2}(D-E)\,, \quad S^{\beta\,\alpha}_{\alpha\,\beta} = \frac{1}{2}(D+E)\,,\\
S^{\alpha\,\beta}_{a\,b} &= -\frac{1}{2}\epsilon_{ab}\epsilon^{\alpha\beta}C\,, \quad S^{a\,b}_{\alpha\,\beta} = -\frac{1}{2}\epsilon^{ab}\epsilon_{\alpha\beta}F\,,\\
S^{a\,\alpha}_{a\,\alpha} &= G\,, \quad S^{\alpha\,a}_{a\,\alpha} = H\,, \quad S^{a\,\alpha}_{\alpha\,a} = K\,, \quad S^{\alpha\,a}_{\alpha\,a} = L
\end{aligned}
$$

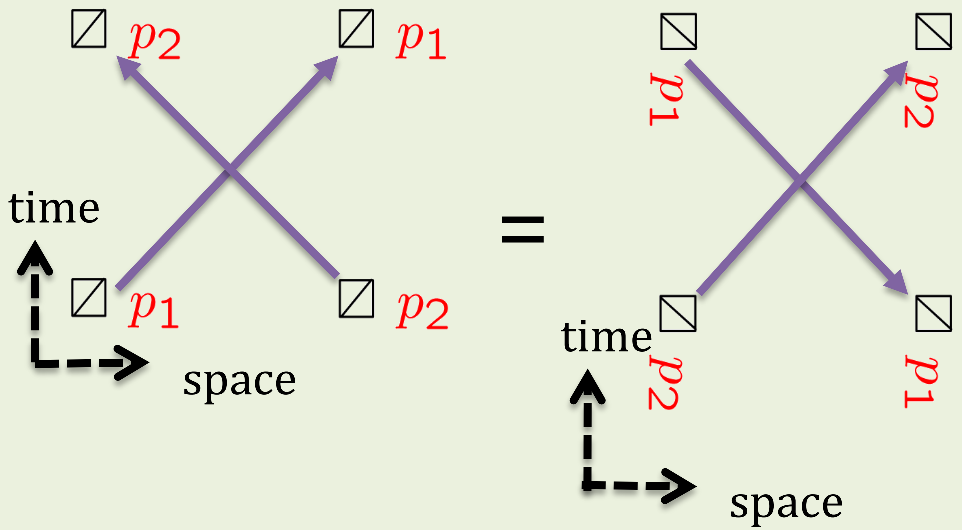
$$
\begin{aligned}
A &= S_0\frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,,\\
B &= -S_0\left[\frac{x_2^- - x_1^+}{x_2^+ - x_1^-} + 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^-x_2^- - x_1^+x_2^+)}\right]\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\,,\\
C &= S_0\frac{2ix_1^-x_2^-(x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+x_2^+(x_1^- - x_2^+)(1 - x_1^-x_2^-)}\,, \qquad D = -S_0\,,\\
E &= S_0\left[1 - 2\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^-x_2^- - x_1^+x_2^+)}\right],\\
F &= S_0\frac{2i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^-x_2^-)\tilde\eta_1\tilde\eta_2}\,,\\
G &= S_0\frac{(x_2^- - x_1^-)}{(x_2^+ - x_1^-)}\frac{\eta_1}{\tilde\eta_1}\,, \qquad H = S_0\frac{(x_2^+ - x_2^-)}{(x_1^- - x_2^+)}\frac{\eta_1}{\tilde\eta_2}\,,\\
K &= S_0\frac{(x_1^+ - x_1^-)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde\eta_1}\,, \qquad L = S_0\frac{(x_1^+ - x_2^+)}{(x_1^- - x_2^+)}\frac{\eta_2}{\tilde\eta_2}
\end{aligned}
$$

$$\eta = e^{ip/4}\sqrt{i(x^- - x^+)} \qquad \eta_1 = \eta(p_1)e^{ip_2/2}\,, \quad \eta_2 = \eta(p_2)\,, \quad \tilde\eta_1 = \eta(p_1)\,, \quad \tilde\eta_2 = \eta(p_2)e^{ip_1/2}$$

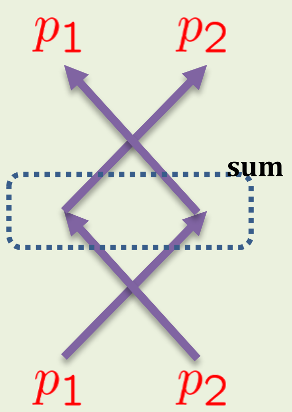
Dressing phase

- YBE, symmetry DO NOT determine the overall function

- Crossing symmetry from space←→time



- Unitarity :



$$\mathbf{S}(p_1, p_2) \cdot \mathbf{S}(p_2, p_1) = \mathbf{I}$$

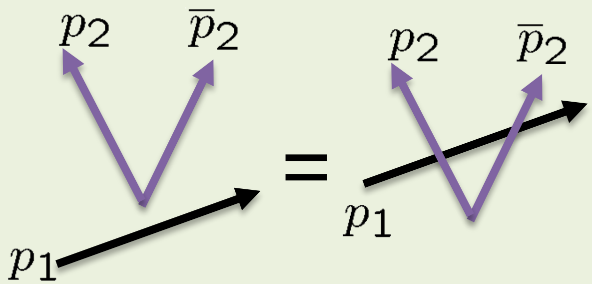
• Crossing-unitarity from a singlet operator

$$I(p) = C^{ij}(p)\, A_i^\dagger(p)\, A_j^\dagger(\bar{p}) \equiv -i\epsilon^{ab} A_a^\dagger(p)\, A_b^\dagger(\bar{p}) + \epsilon^{\alpha\beta} A_\alpha^\dagger(p)\, A_\beta^\dagger(\bar{p})$$

↑ Charge conjugation

$$x^\pm(\bar{p}) = \frac{1}{x^\pm(p)}$$

$$\mathbb{H},\ p \ \rightarrow\ -\mathbb{H},\ -p$$



$$\begin{aligned} A_i^\dagger(p_1)\, I(p_2) &= C^{jk}(p_2)\, A_i^\dagger(p_1)\, A_j^\dagger(p_2)\, A_k^\dagger(\bar{p}_2) \\ &= C^{jk}(p_2)\, S_{ij}^{i'j'}(p_1,p_2)\, A_{j'}^\dagger(p_2)\, A_{i'}^\dagger(p_1)\, A_k^\dagger(\bar{p}_2) \\ &= \boxed{C^{jk}(p_2)\, S_{ij}^{i'j'}(p_1,p_2)\, S_{i'k}^{i''k'}(p_1,\bar{p}_2)}\, A_{j'}^\dagger(p_2)\, A_{k'}^\dagger(\bar{p}_2)\, A_{i''}^\dagger(p_1) \\ &\equiv I(p_2)\, A_i^\dagger(p_1) \qquad \propto C^{j'k'}\delta_i^{j''} \end{aligned}$$

$$S_0(p_1,p_2)\, S_0(p_1,\bar{p}_2) = \frac{\left(\frac{1}{x_1^-} - x_2^-\right)(x_1^- - x_2^+)}{\left(\frac{1}{x_1^+} - x_2^-\right)(x_1^+ - x_2^+)}$$

$$S_0(p_1\,,p_2)^2 \equiv \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\,\frac{1 - \frac{1}{x_1^+ x_2^-}}{1 - \frac{1}{x_1^- x_2^+}}\,\sigma(p_1\,,p_2)^2$$

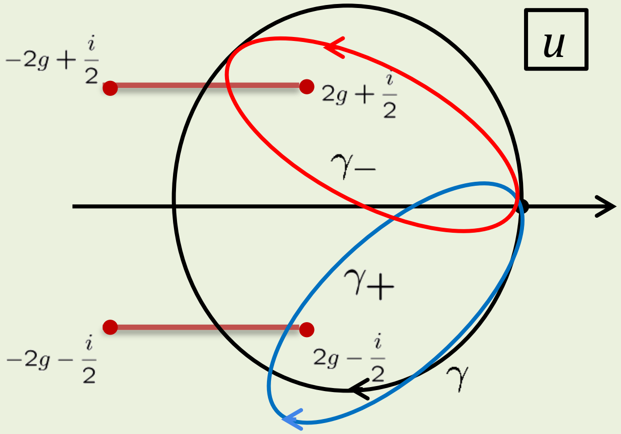
$$\sigma(p_1,p_2)\,\sigma(\bar{p}_1,p_2) = \frac{1 - \frac{1}{x_1^+ x_2^+}}{1 - \frac{x_1^-}{x_2^-}}\;\frac{1 - \frac{x_1^-}{x_2^+}}{1 - \frac{1}{x_1^+ x_2^-}}$$

Janik (2006)

- Zhukovsky map $\quad x + \frac{1}{x} = \frac{u}{g}, \qquad x^{\pm} + \frac{1}{x^{\pm}} = \frac{1}{g}\left(u \pm \frac{i}{2}\right)$

- For real u : $|x^{\pm}| > 1$

- cuts occur when



- Crossing cuts :

$$\begin{aligned} \gamma \quad &: \quad x^{\pm} \to \frac{1}{x^{\pm}} \\ \gamma_- \quad &: \quad x^{-} \to \frac{1}{x^{-}} \\ \gamma_+ \quad &: \quad x^{+} \to \frac{1}{x^{+}} \end{aligned}$$

- Janik relation can be written as

$$\sigma(x,y)\,\sigma^{\gamma}(x,y) = \frac{1-\frac{1}{x^+y^+}}{1-\frac{x^-}{y^-}}\,\frac{1-\frac{x^-}{y^+}}{1-\frac{1}{x^+y^-}} \qquad x^{\pm}=x\left(u\pm\frac{i}{2}\right),\ y^{\pm}=x\left(v\pm\frac{i}{2}\right)$$

- Apply γ_ contour

$$\sigma^{\gamma_-}(x,y)\,\sigma^{\gamma_+}(x,y) = \frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^-y^-}}\,\frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^+y^-}} \qquad (*)$$

- Define a translation

$$D=e^{\frac{i}{2}\partial_u}\ :\ Df(u)=f(u+i/2)=e^{D\ln f}=f^D$$

- RHS of (*)

$$\frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^-y^-}}\,\frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^+y^-}} = \frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^+y^-}}\,\frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^-y^-}} = \left(\frac{1-\frac{1}{xy^+}}{1-\frac{1}{xy^-}}\right)^{D}\left(\frac{1-\frac{1}{xy^+}}{1-\frac{1}{xy^-}}\right)^{D^{-1}} = \left(\frac{1-\frac{1}{xy^+}}{1-\frac{1}{xy^-}}\right)^{D+D^{-1}} = \left(\frac{x-\frac{1}{y^+}}{x-\frac{1}{y^-}}\right)^{D+D^{-1}}$$

- Define

$$\sigma(x,y) = \exp\left\{i\left[\chi(x^+,y^-)+\chi(x^-,y^+)-\chi(x^+,y^+)-\chi(x^-,y^-)\right]\right\}$$
$$\sigma_1(x,y) = \exp\left\{i\left[\chi(x,y^-)-\chi(x,y^+)\right]\right\}$$

$$\sigma^{\gamma_-}(x,y) = \exp\left\{i\left[\chi(x^+,y^-)+\chi(1/x^-,y^+)-\chi(x^+,y^+)-\chi(1/x^-,y^-)\right]\right\} = \frac{\sigma_1(x^+,y)}{\sigma_1(1/x^-,y)}$$
$$\sigma^{\gamma_+}(x,y) = \exp\left\{i\left[\chi(1/x^+,y^-)+\chi(x^-,y^+)-\chi(1/x^+,y^+)-\chi(x^-,y^-)\right]\right\} = \frac{\sigma_1(1/x^+,y)}{\sigma_1(x^-,y)}$$

- LHS of (*) = RHS of (*)

$$\frac{\sigma_1(x^+,y)}{\sigma_1(x^-,y)}\,\frac{\sigma_1(1/x^+,y)}{\sigma_1(1/x^-,y)} = \frac{[\sigma_1(x,y)\sigma_1(1/x,y)]^D}{[\sigma_1(x,y)\sigma_1(1/x,y)]^{D^{-1}}} = [\sigma_1(x,y)\sigma_1(1/x,y)]^{D-D^{-1}} = \left(\frac{x-\frac{1}{y^+}}{x-\frac{1}{y^-}}\right)^{D+D^{-1}}$$

$$\sigma_1(x,y)\sigma_1(1/x,y) = \left(\frac{x-\frac{1}{y^+}}{x-\frac{1}{y^-}}\right)^{\frac{D+D^{-1}}{D-D^{-1}}}$$

$$\sigma_1(x,y)\sigma_1(1/x,y) = \exp\left\{i\left[\chi(x,y^-)-\chi(x,y^+)+\chi(1/x,y^-)-\chi(1/x,y^+)\right]\right\} = \frac{\exp\left\{i\left[\chi(x,y^-)+\chi(1/x,y^-)\right]\right\}}{\exp\left\{i\left[\chi(x,y^+)+\chi(1/x,y^+)\right]\right\}}$$

$$e^{i[\chi(x,y)+\chi(1/x,y)]} = \left(\frac{x-\frac{1}{y}}{\sqrt{x}}\right)^{-f(D)}, \quad f(D) = \frac{D+D^{-1}}{D-D^{-1}}$$

$$e^{i[\chi(x,y)+\chi(1/x,y)+\chi(x,1/y)+\chi(1/x,1/y)]} = \left(\frac{x-\frac{1}{y}}{\sqrt{x}}\cdot\frac{x-y}{\sqrt{x}}\right)^{-f(D)} = \left(x+\frac{1}{x}-y-\frac{1}{y}\right)^{-f(D)} = (u-v)^{-f(D)}$$

- Using $f(D)=\frac{D+D^{-1}}{D-D^{-1}}=\frac{D^{-2}}{1-D^{-2}}-\frac{D^{2}}{1-D^{2}}=\sum_{n=1}^{\infty}D^{-2n}-\sum_{n=1}^{\infty}D^{2n}$

$$(u-v)^{\sum_{n=1}^{\infty}D^{-2n}-\sum_{n=1}^{\infty}D^{2n}}=\prod_{n=1}^{\infty}\frac{u-v-in}{u-v+in}=\frac{\Gamma(1+iu-iv)}{\Gamma(1-iu+iv)}$$

$$e^{i[\chi(x,y)+\chi(1/x,y)+\chi(x,1/y)+\chi(1/x,1/y)]}=\frac{\Gamma(1+iu-iv)}{\Gamma(1-iu+iv)}$$

- In terms of *u, v near the cuts*

$$\chi(u+i0,v+i0)+\chi(u-i0,v+i0)+\chi(u+i0,v-i0)+\chi(u-i0,v-i0)=\frac{1}{i}\ln\frac{\Gamma(1+iu-iv)}{\Gamma(1-iu+iv)}$$

- Riemann-Hilbert problem $\xi(u+i0)-\xi(u-i0)=f(u)\to\xi(u)=\int_{\Gamma}\frac{dw}{2\pi i}\frac{f(w)}{w-u}$ 

$$\chi(u)\equiv\left(x(u)-\frac{1}{x(u)}\right)\xi(u),\;\; F(u)\equiv\left(x(u)-\frac{1}{x(u)}\right)f(u)\;\to\;\chi(u+i0)+\chi(u-i0)=F(u)\;\to\;\chi(u)=K_u\star F\;\equiv\;\int_{-2g+i0}^{2g+i0}\frac{dw}{2\pi i}\frac{x(u)-\frac{1}{x(u)}}{x(w)-\frac{1}{x(w)}}\frac{1}{w-u}F(w)$$

$$\chi(u,v)=\frac{1}{i}K_v\star K_u\star\frac{\Gamma(1+iu-iv)}{\Gamma(1-iu+iv)}$$

Zhukovsky map

$$z=x(w),\quad z+\frac{1}{z}=\frac{w}{g},\quad x=x(u),\quad x+\frac{1}{x}=\frac{u}{g}$$

$$K_u\star F\;\equiv\;\int_{-2g+i0}^{2g+i0}\frac{dw}{2\pi i}\frac{x(u)-\frac{1}{x(u)}}{x(w)-\frac{1}{x(w)}}\frac{1}{w-u}F(w)$$

$$=\oint_{|z|=1}\frac{dz}{2\pi i}\frac{1}{x-z}F\left(g(z+1/z)\right)-\frac{1}{g}\int_{-2g+i0}^{2g+i0}\frac{dw}{2\pi i}\frac{1}{x(w)-\frac{1}{x(u)}}F(w)$$

antisymmetrize

$$K_v\star K_u\star F=\oint_{|z'|=1}\frac{dz'}{2\pi i}\frac{1}{y-z'}\oint_{|z|=1}\frac{dz}{2\pi i}\frac{1}{x-z}F\left(g(z+1/z),g(z'+1/z')\right)+(\text{symmetric in } u\leftrightarrow v,\; x\leftrightarrow y)$$

BES dressing phase

Beisert-Hernandez-Lopez, Beisert-Eden-Staudacher

- Integral Representation: **Dorey, Hofman, Maldacena (2006)**

$$\sigma(x_1,x_2) = \exp\left\{i\left[\chi(x_1^+,x_2^-)+\chi(x_1^-,x_2^+)-\chi(x_1^+,x_2^+)-\chi(x_1^-,x_2^-)\right]\right\}$$

$$\chi(x,y) = -i\oint_{|z|=1}\frac{dz}{2\pi i}\oint_{|z'|=1}\frac{dz'}{2\pi i}\frac{1}{x-z}\frac{1}{y-z'}\frac{\ln\Gamma\left[1+ig\left(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}\right)\right]}{\ln\Gamma\left[1-ig\left(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}\right)\right]}$$

- Originally conjectured based on weak and strong coupling results

- Direct derivation from Janik relation **Volin**

Checks

- Weak coupling limit : S-matrix from coordinate Bethe ansatz
  - Lecture by R. Nepomechie
- Strong coupling limit: worldsheet string theory
  - Dressing phase
  - Matrix structure

- Weak coupling expansion

$$\chi(x,y) = -i\oint_{|z|=1}\frac{dz}{2\pi i}\oint_{|z'|=1}\frac{dz'}{2\pi i}\frac{1}{x-z}\frac{1}{y-z'}\frac{\ln\Gamma\left[1+ig\left(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}\right)\right]}{\ln\Gamma\left[1-ig\left(z_1+\frac{1}{z_1}-z_2-\frac{1}{z_2}\right)\right]}$$

$$\chi(x,y) = -\sum_{r,s=1}^{\infty}\frac{c_{r,s}(g)}{x^r y^s} \qquad z=e^{i\phi},\ z'=e^{i\phi'}$$

$$\begin{aligned} c_{r,s}(g) &= i\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^{2\pi}\frac{d\phi'}{2\pi}e^{ir\phi+is\phi'}\frac{\ln\Gamma\left[1+2ig(\cos\phi-\cos\phi')\right]}{\ln\Gamma\left[1-2ig(\cos\phi-\cos\phi')\right]} \\ &= 2\sin\left(\frac{\pi}{2}(r-s)\right)\int_0^{\infty}dt\frac{J_r(2gt)J_s(2gt)}{t(e^t-1)} \end{aligned}$$

$$c_{r,s}(g) = \sum_{n=1}^{\infty} g^{r+s+2n}\cdot 2(-1)^n\sin\left(\frac{\pi}{2}(r-s)\right)\frac{(2n+r+s-1)!(2n+r+s)!}{n!(n+r)!(n+s)!(n+r+s)!}\zeta(2n+r+s)$$

$$\begin{aligned} \sigma^2(u,v) &= \exp\left\{2i\left[\chi(x^+,y^-)+\chi(x^-,y^+)-\chi(x^+,y^+)-\chi(x^-,y^-)\right]\right\} \\ &= 1+256\zeta(3)g^6\frac{(u-v)(4uv-1)}{(1+4u^2)^2(1+4v^2)^2}+\mathcal{O}(g^8) \end{aligned}$$

- Strong coupling expansion

$$c_{r,s}(g) = \sum_{n=1}^{\infty} g^{1-n} \cdot \frac{\zeta(n)((-1)^{r+s}-1)\Gamma(\frac{1}{2}(n-r+s-1))\Gamma(\frac{1}{2}(n+r+s-3))}{2(-2\pi)^n\Gamma(n-1)\Gamma(\frac{1}{2}(-n-r+s+3))\Gamma(\frac{1}{2}(-n+r+s+1))}$$
$$= g\frac{\delta_{s,r-1}-\delta_{s,r+1}}{rs} + \frac{(-1)^{r+s}-1}{\pi}\,\frac{1}{r^2-s^2} + \mathcal{O}(g^{-1})$$

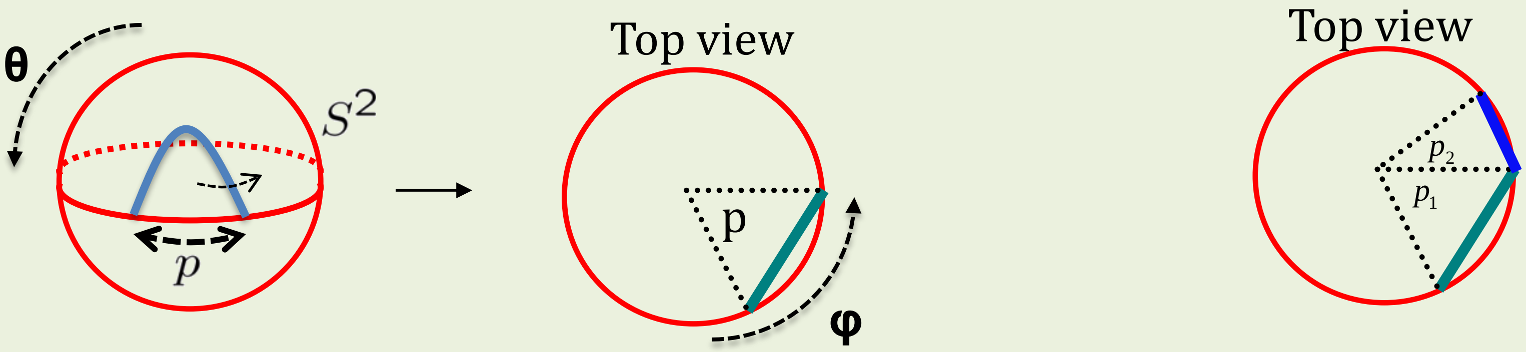
$$\chi^{(0)}(x,y) = \left(x+\frac{1}{x}-y-\frac{1}{y}\right)\ln\left(1-\frac{1}{xy}\right)-\frac{1}{x}+\frac{1}{y}$$

$$\sigma(u,v) \approx \frac{1-\frac{1}{x^-y^+}}{1-\frac{1}{x^+y^-}}\left(\frac{1-\frac{1}{x^-y^-}}{1-\frac{1}{x^-y^+}}\;\frac{1-\frac{1}{x^+y^+}}{1-\frac{1}{x^+y^-}}\right)^{i(v-u)}$$

Arutyunov, Frolov, Staudacher (2004)

$$S_0(p_1,p_2)^2 \equiv \frac{x_1^- - x_2^+}{x_1^+ - x_2^-}\frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\sigma(p_1,p_2)^2$$

- Scattering of two giant magnons



- Scattering amplitude of two sine-Gordon solitons:

$$S(p_1,p_2) = \left(\frac{\sin^2\frac{p_1+p_2}{4}}{\sin^2\frac{p_1-p_2}{4}}\right)^{4ig\left(\cos\frac{p_1}{2}-\cos\frac{p_2}{2}\right)}$$

– Exact S-matrix for su(2) sector

$$S(p_1,p_2) = A(p_1,p_2)^2 = S_0^2\left(\frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\right)^2 \approx \left(\frac{1-\frac{1}{x_1^- x_2^-}}{1-\frac{1}{x_1^- x_2^+}}\,\frac{1-\frac{1}{x_1^+ x_2^+}}{1-\frac{1}{x_1^+ x_2^-}}\right)^{2i(u_1-u_2)}$$

$$x_j^\pm \approx e^{\pm i\frac{p_j}{2}}, \qquad u_j \approx 2g\cos\frac{p_j}{2}$$

Worldsheet S-matrix

Klose,McLoughlin,Roiban,Zarembo (2007)

- So far we considered S-matrix from gauge theory spin chains
- String perturbative computation (large λ) of S-matrix is also possible
  - Fluctuation around BMN in light-cone gauge
  - Keep the terms with R^{-2}
  - Effective Lagrangian contains
    - Quadratic terms in terms of oscillator algebra [BMN limit]
    - Quartic interaction terms

$$L = \frac{1}{2}(\partial_a \vec{x})^2 - \frac{1}{2}\vec{x}^2 + \frac{1}{4}\frac{\alpha'}{R^2}\left[\vec{z}^2(\partial_a \vec{z})^2 - \vec{y}^2(\partial_a \vec{y})^2 + (\vec{y}^2 - \vec{z}^2)(\dot{\vec{x}}^2 + \vec{x}'^2)\right], \qquad \vec{x} = \vec{y},\, \vec{z}$$

$$\frac{\alpha'}{R^2} \sim \frac{1}{\sqrt{\lambda}}$$

- Can compute scattering amplitudes on the worldsheet

- (ex) $y_i(p_1)y_j(p_2) \to y_k(p_2')y_l(p_1')$
- Redefinition:

$$X = y_1 + iy_2, \quad \overline{X} = y_1 - iy_2, \quad Y = y_3 + iy_4, \quad \overline{Y} = y_3 - iy_4$$

- Interactions: $L_{\text{int}} \supset \frac{1}{2\sqrt{\lambda}} \sum_{i=1}^{4} y_i^2 \cdot \sum_{j=1}^{4} {y_j'}^2 = \frac{1}{2\sqrt{\lambda}} (X\overline{X} + Y\overline{Y})(X'\overline{X}' + Y'\overline{Y}')$

- Mode expansions:

$$Z(\sigma,\tau) = \int \frac{dp}{\sqrt{2\epsilon(p)}} \left[A_Z^\dagger(p) e^{i(p\sigma - \epsilon\tau)} + A_{\overline{Z}}(p) e^{-i(p\sigma-\epsilon\tau)} \right], \quad Z = X,\, Y, \quad \epsilon(p) = \sqrt{1+p^2}$$

- Scattering processes $Z_1(p_1)Z_2(p_2) \to Z_2'(p_2')Z_1'(p_1')$

$$\frac{1}{2\sqrt{\lambda}} \langle 0 | A_{Z_2'}(p_2') A_{Z_1'}(p_1') : \int d\sigma d\tau (X\overline{X} + Y\overline{Y})(X'\overline{X}' + Y'\overline{Y}') : A_{Z_1}^\dagger(p_1) A_{Z_2}^\dagger(p_2) | 0 \rangle$$

- Kinematical factors

$$\frac{1}{\sqrt{\epsilon_1\epsilon_2\epsilon_1'\epsilon_2'}}\int d\sigma d\tau e^{i(p_1+p_2-p_1'-p_2')\sigma}e^{i(\epsilon_1+\epsilon_2-\epsilon_1'-\epsilon_2')\tau}$$

$$=\frac{1}{\sqrt{\epsilon_1\epsilon_2\epsilon_1'\epsilon_2'}}\delta(p_1+p_2-p_1'-p_2')\delta(\epsilon_1+\epsilon_2-\epsilon_1'-\epsilon_2')$$

$$=\frac{1}{\epsilon_1\epsilon_2}\frac{1}{\frac{d\epsilon_1}{dp_1}-\frac{d\epsilon_2}{dp_2}}\left[\delta(p_1-p_1')\delta(p_2-p_2')+\delta(p_1-p_2')\delta(p_2-p_1')\right]$$

$$=\frac{1}{\epsilon_2 p_1-\epsilon_1 p_2}\left[\delta(p_1-p_1')\delta(p_2-p_2')+\delta(p_1-p_2')\delta(p_2-p_1')\right]$$

- Scattering processes $\quad Y(p_1)Y(p_2)\rightarrow Y(p_2)Y(p_1)$

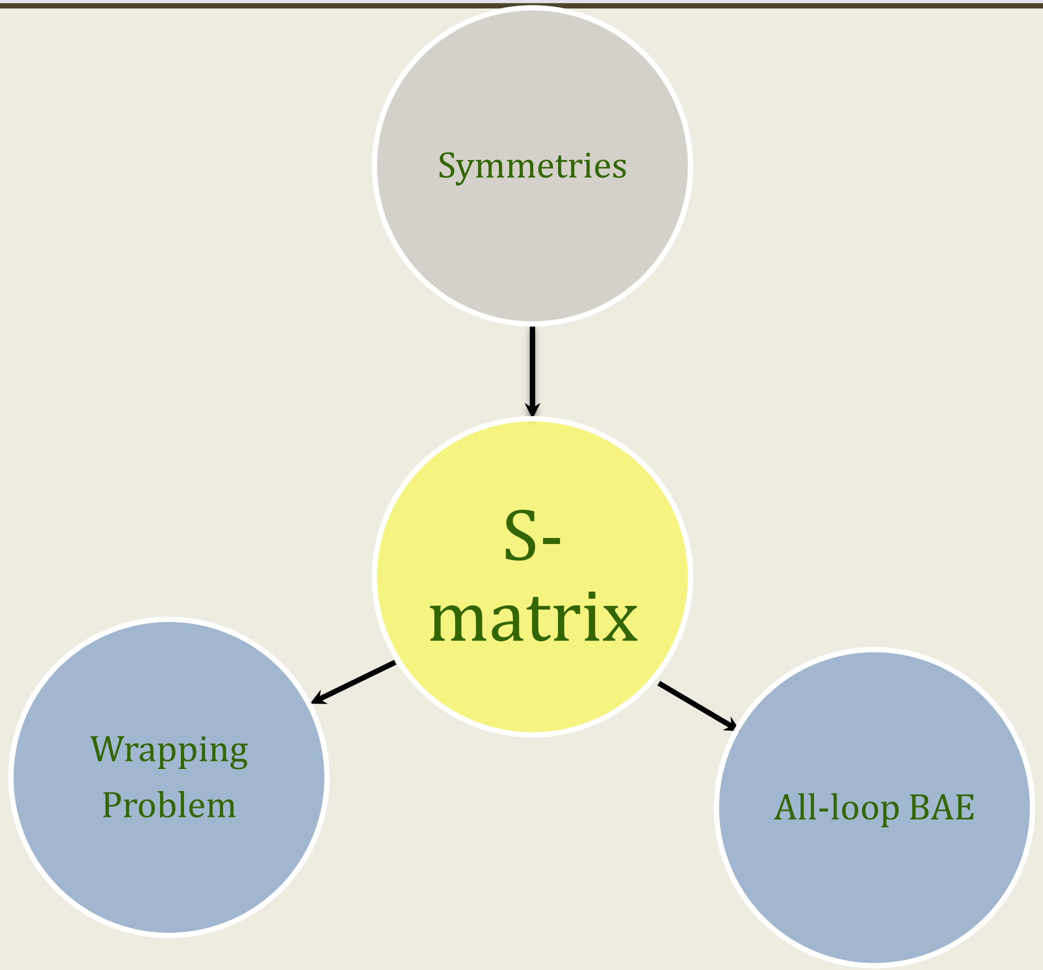
p_2 $\quad p_2$

$$\frac{1}{2\sqrt{\lambda}}\langle 0|A_Y(p_2)A_Y(p_1):Y\,\overline{Y}\,Y'\,\overline{Y}':A_Y^\dagger(p_1)A_Y^\dagger(p_2)|0\rangle$$

$$=\frac{1}{2\sqrt{\lambda}}\frac{p_1^2+p_2^2+p_1p_2+p_2p_1}{\epsilon_2 p_1-\epsilon_1 p_2}\left[\delta(p_1-p_1')\delta(p_2-p_2')+\delta(p_1-p_2')\delta(p_2-p_1')\right]$$

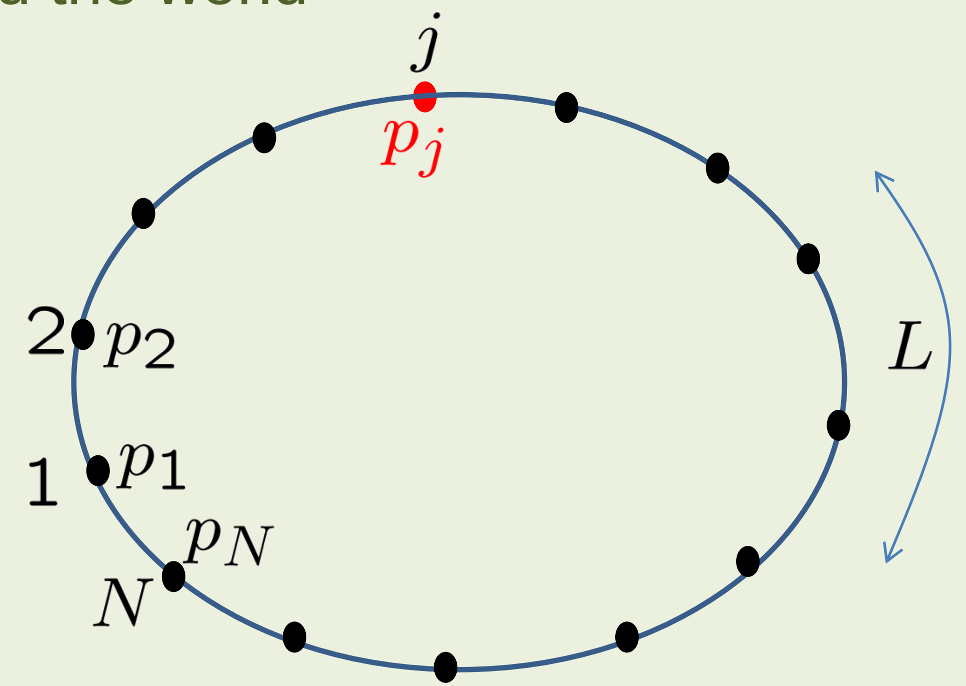
- Match with all the S-matrix elements

Applications of S-matrix



Periodic BC

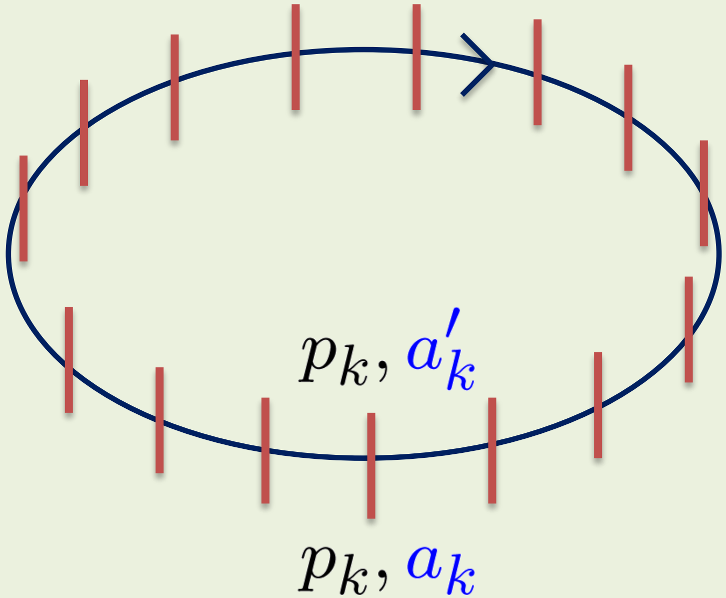
- Going around the world

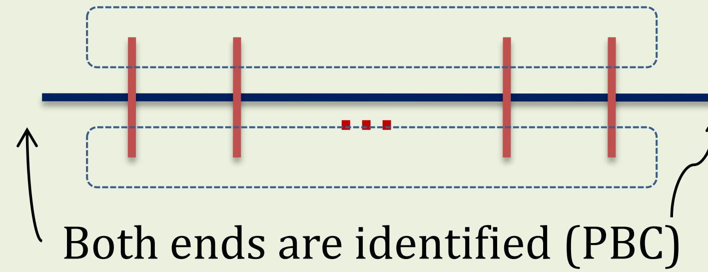


- At each crossing, S-matrix

$$e^{ip_jL}\prod_{k\neq j,1}^{N} S(p_j,p_k)=1$$

- When the scattering is non-diagonal



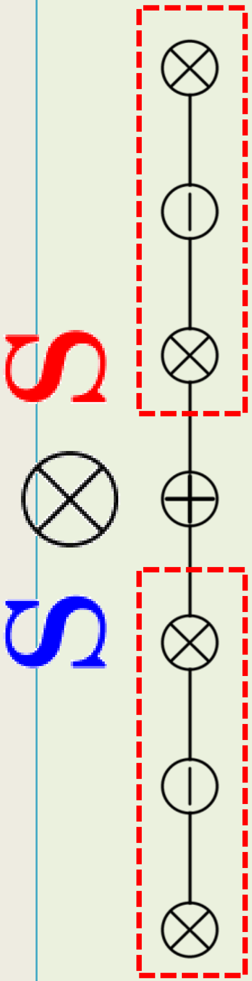


$$e^{ip_j L} S^{a'_1 b_2}_{a_1 b_1}(p_1, p_j) S^{a'_2 b_3}_{a_2 b_2}(p_2, p_j) \cdots S^{a'_N b_1}_{a_N b_N}(p_N, p_j) = 1$$

- Transfer matrix $e^{ip_j L} \cdot T(p_j | p_1, \cdots, p_N)^{\vec{a}'}_{\vec{a}} = 1$
- Need to diagonalize transfer matrix [Lecture by Nepomechie]
  - Main difference is "inhomogeneity"

Asymptotic Bethe-Yang equation

Beisert-Staudacher



$$
\begin{aligned}
1 &= \prod_{k=1}^{K_2} \frac{u_{1j}-u_{2k}+\frac{i}{2}}{u_{1j}-u_{2k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{1-1/x_{1j}x_{4k}^{+}}{1-1/x_{1j}x_{4k}^{-}} \\
1 &= \prod_{k=1}^{K_2} \frac{u_{2j}-u_{2k}-i}{u_{2j}-u_{2k}+i} \prod_{k=1}^{K_3} \frac{u_{2j}-u_{3k}+\frac{i}{2}}{u_{2j}-u_{3k}-\frac{i}{2}} \prod_{k=1}^{K_1} \frac{u_{2j}-u_{1k}+\frac{i}{2}}{u_{2j}-u_{1k}-\frac{i}{2}} \\
1 &= \prod_{k=1}^{K_2} \frac{u_{3j}-u_{2k}+\frac{i}{2}}{u_{3j}-u_{2k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{3j}-x_{4k}^{+}}{x_{3j}-x_{4k}^{-}} \\
\left(\frac{x_{4j}^{+}}{x_{4j}^{-}}\right)^{L} &= \prod_{k=1}^{K_4} \sigma^2(x_{4j},x_{4k}) \frac{u_{4j}-u_{4k}+i}{u_{4j}-u_{4k}-i} \\
&\times \prod_{k=1}^{K_1} \frac{1-1/x_{4j}^{-}x_{1k}}{1-1/x_{4j}^{+}x_{1k}} \prod_{k=1}^{K_3} \frac{x_{4j}^{-}-x_{3k}}{x_{4j}^{+}-x_{3k}} \prod_{k=1}^{K_5} \frac{x_{4j}^{-}-x_{5k}}{x_{4j}^{+}-x_{5k}} \prod_{k=1}^{K_7} \frac{1-1/x_{4j}^{-}x_{7k}}{1-1/x_{4j}^{+}x_{7k}} \\
1 &= \prod_{k=1}^{K_6} \frac{u_{5j}-u_{6k}+\frac{i}{2}}{u_{5j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{5j}-x_{4k}^{+}}{x_{5j}-x_{4k}^{-}} \\
1 &= \prod_{k=1}^{K_6} \frac{u_{6j}-u_{6k}-i}{u_{6j}-u_{6k}+i} \prod_{k=1}^{K_5} \frac{u_{6j}-u_{5k}+\frac{i}{2}}{u_{6j}-u_{5k}-\frac{i}{2}} \prod_{k=1}^{K_7} \frac{u_{6j}-u_{7k}+\frac{i}{2}}{u_{6j}-u_{7k}-\frac{i}{2}} \\
1 &= \prod_{k=1}^{K_6} \frac{u_{7j}-u_{6k}+\frac{i}{2}}{u_{7j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{1-1/x_{7j}x_{4k}^{+}}{1-1/x_{7j}x_{4k}^{-}}
\end{aligned}
$$

Simpler form of BAE

- Dynamic transformation **Beisert, Roiban; Hentschel, Plefka, Sundin**

$$K_{1,7} \to K_{1,7}-1, \quad K_{3,5} \to K_{3,5}+1, \quad L \to L-1$$

$$L' = L - K_1 - K_7$$

$$1 = \prod_{k=1}^{K_2} \frac{u_{2j}-u_{2k}-i}{u_{2j}-u_{2k}+i} \prod_{k=1}^{K_3+K_1} \frac{u_{2j}-u_{3k}+\frac{i}{2}}{u_{2j}-u_{3k}-\frac{i}{2}}$$

$$1 = \prod_{k=1}^{K_2} \frac{u_{3j}-u_{2k}+\frac{i}{2}}{u_{3j}-u_{2k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{3j}-x_{4k}^+}{x_{3j}-x_{4k}^-}$$

$$\left(\frac{x_{4j}^+}{x_{4j}^-}\right)^{L'} = \prod_{k=1}^{K_4} \sigma^2(x_{4j},x_{4k}) \frac{u_{4j}-u_{4k}+i}{u_{4j}-u_{4k}-i} \prod_{k=1}^{K_3+K_1} \frac{x_{4j}^- - x_{3k}}{x_{4j}^+ - x_{3k}} \prod_{k=1}^{K_5+K_7} \frac{x_{4j}^- - x_{5k}}{x_{4j}^+ - x_{5k}}$$

$$1 = \prod_{k=1}^{K_6} \frac{u_{5j}-u_{6k}+\frac{i}{2}}{u_{5j}-u_{6k}-\frac{i}{2}} \prod_{k=1}^{K_4} \frac{x_{5j}-x_{4k}^+}{x_{5j}-x_{4k}^-}$$

$$1 = \prod_{k=1}^{K_6} \frac{u_{6j}-u_{6k}-i}{u_{6j}-u_{6k}+i} \prod_{k=1}^{K_5+K_7} \frac{u_{6j}-u_{5k}+\frac{i}{2}}{u_{6j}-u_{5k}-\frac{i}{2}}$$

But asymptotic BAE is wrong!

- Derived from PBC based on S-matrix
- So, valid only when infinite L
- What if L is finite?