Ch.1: Basics of Shallow Water Fluid

Sec. 1.1: Basic Equations

1. Shallow Water Equations on a Sphere

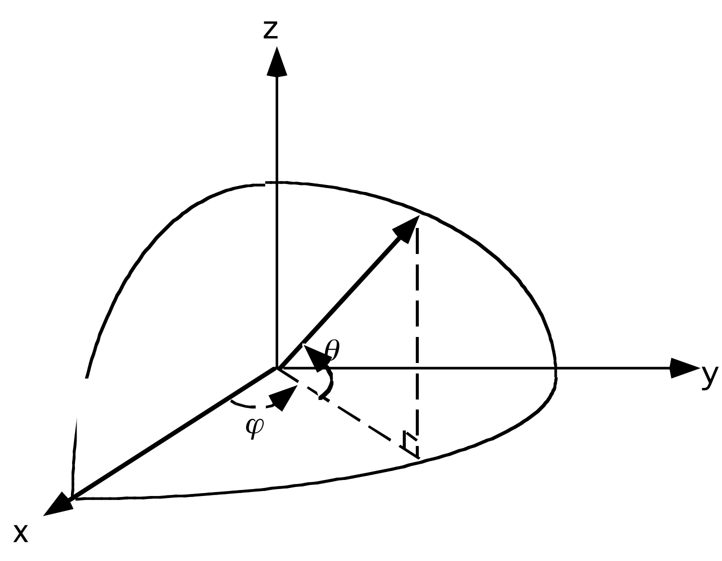
We start with the shallow water fluid of a homogeneous density and focus on the effect of rotation on the motion of the water. Rotation is, perhaps, the most important factor that distinguishes geophysical fluid dynamics from classical fluid dynamics.

There are four basic equations involved in a homogeneous fluid system. The first is the mass equation:

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla_3 \bullet \mathbf{u}_3 = 0 \qquad (1.1.1)$$

where $\nabla_3 = \mathbf{i}\partial_x + \mathbf{j}\partial_y + \mathbf{k}\partial_z, \quad \mathbf{u}_3 = (u, v, w)$. The other three equations are the momentum equations, which, in its 3-dimensional vector form can be written as:

$$\frac{d\mathbf{u}_3}{dt} + 2\grave{\mathbf{U}} \times \mathbf{u}_3 = -\frac{1}{\rho}\nabla_3 p - \mathbf{g} + \mathbf{F}$$
$$where \frac{d}{dt} = \partial_t + \mathbf{u}_3 \bullet \nabla_3 \qquad (1.1.2)$$



On the earth, it is more convenient to cast the equations on the spherical coordinate with ϕ, θ, r being the longitude, latitude and radians, respectively. That is:

$$\begin{cases} \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{r\cos\theta}\frac{\partial u}{\partial \varphi} + \frac{1}{r\cos\theta}\frac{\partial(v\cos\theta)}{\partial\theta} + \frac{\partial w}{\partial r} = 0 \\ \frac{du}{dt} - (2\Omega + \frac{u}{r\cos\theta})(v\sin\theta - w\cos\theta) = -\frac{1}{\rho r\cos\theta}\frac{\partial p}{\partial\varphi} + F_\varphi \\ \frac{dv}{dt} + (2\Omega + \frac{u}{r\cos\theta})u\sin\theta + \frac{wv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + F_\theta \\ \frac{dw}{dt} - (2\Omega + \frac{u}{r\cos\theta})u\cos\theta - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + F_r \end{cases} \qquad (1.1.3)$$

This is a complex set of equations that govern the fluid motion from ripples, turbulence to planetary waves. For the atmosphere and ocean, many approximations can be made. Don't be afraid of making approximations! Indeed, *proper approximations are the keys for the understanding of the dynamics!* You can never include everything in your equations, no matter how fast is your computer (even if you are a good programmer). Therefore, to truly understand a certain dynamic issue, you have to know what the most important is for this phenomenon and make sure you absolutely keep this term.

Here, to study large scale flows, we will make 6 approximations. (i) First of all, for a homogeneous fluid, the density is constant. So the mass equation degenerates to $\rho = const$, which, according to (1.1.1), gives the so called incompressibility condition:

$$\nabla_3 \bullet \mathbf{u}_3 = 0 \qquad (1.1.4)$$

This is a very good approximation for the ocean, because the density of the water varies by less than a few percent. This is not a good approximation for the atmosphere, because the air density decreases significantly, even within the troposphere. Essentially, (1.1.4) states that the mass conservation becomes volume conservation.

(ii) The second approximation is the thin layer approximation $r \approx a$. The thickness of the atmosphere and ocean is roughly $D \propto 10km$, which is tiny compared with the radius of the earth $a \approx 6370km$. Therefore, this is a very good approximation with an error of less than 1 percent. For convenience, we often use the new vertical coordinate $z = r - a$ that starts from the surface of the earth.

iii) The third approximation is important for large scale circulation. This is the shallow water approximation $\frac{D}{L} << 1$, where D and L are the characteristic scales of the motion in the vertical and horizontal respectively. Examples that satisfy the shallow water approximation are cyclone and ocean eddies, as well as planetary flows. A cumulus cloud has its scales of $D \sim L \sim 1-10\ km$ and therefore does not satisfy the shallow water approximation.

The shallow water approximation results in an important simplification to the vertical momentum equation, and leads to the so called hydrostatic approximation. Indeed, our scaling analysis below shows that all the terms in the w-equation is much smaller than the dynamic pressure gradient term. First, we separate the pressure into the dynamic p' and static $p_s=-\rho gz$ parts: $p=p_s+p'$. The static part satisfies the hydrostatic equation

$$\frac{1}{\rho}\frac{\partial p_s}{\partial z}=-g,$$

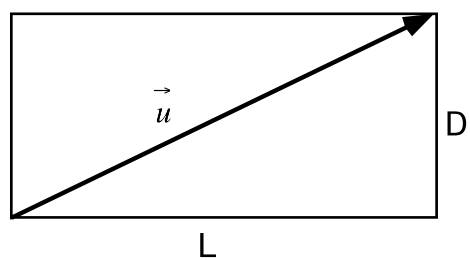
such that the vertical momentum equation reduces to

$$\frac{dw}{dt}-(2\Omega+\frac{u}{r\cos\theta})u\cos\theta-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p'}{\partial r}+F_r \qquad (1.1.5)$$

In a shallow water system, we can show that all the acceleration terms are unimportant. Take the term $u\partial_x w$ for example:

$$\frac{u\partial_x w}{\frac{1}{\rho}\frac{\partial p'}{\partial z}}\sim\frac{\frac{UW}{L}}{\frac{1}{\rho}\frac{\delta p}{D}}\leq\frac{\frac{DU^2}{L^2}}{\frac{U^2}{D}}\sim(\frac{D}{L})^2<<1$$

Here we have used the scaling relationship $\frac{W}{D}\leq\frac{U}{L}$, which can be derived from the continuity equation $w_z=-(\partial_x u+\partial_y v)$.



We have also used the scaling relationship between the dynamic pressure and the horizontal velocity as $\delta p\sim U^2$. This relation is derived from the horizontal momentum equation if one recognizes that the horizontal acceleration is driven by the dynamic pressure (in the case of weak rotation), and therefore,

$$u\partial_x u\sim\frac{1}{\rho}\frac{\partial p}{\partial x}\quad\Rightarrow\quad\frac{U^2}{L}\sim\frac{1}{\rho}\frac{\delta p}{L}\quad\Rightarrow\quad\delta p\sim\rho U^2.$$

Since all the terms in the w-equation (try other terms yourself!) are negligible relative to the pressure gradient term, (1.1.5) at the first order can be reduced to:

$$\frac{1}{\rho}\frac{\partial p'}{\partial z}=0 \qquad (1.1.6)$$

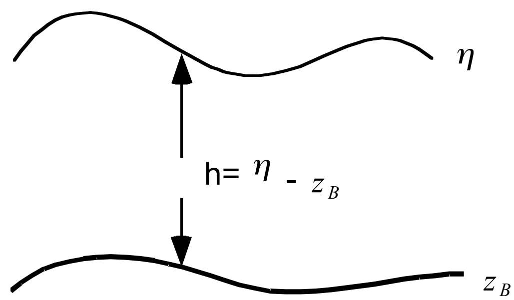
The total pressure therefore satisfies the hydrostatic approximation

$$\frac{1}{\rho}\frac{\partial p}{\partial z}=-g \qquad (1.1.7)$$

or

$$p(z)=p_{surface}+\rho g(\eta-z) \qquad (1.1.7a)$$

where η is the free surface elevation and we have neglected the pressure above the free surface. The last equation states that the pressure at a level equals the weight of the fluid above it!



Eqn.(1.1.7a) in turn simplifies the horizontal pressure gradient force as:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho}\frac{\partial p'}{\partial x} = \frac{1}{\rho}\frac{\rho g \partial\eta}{\partial x} = g\partial_x\eta .$$

Eqn. (1.1.6) (which is the result of the two approximations of shallow water and homogeneous fluid) states that the dynamic pressure gradient force is independent of the depth of the fluid. This implies the absence of vertical shear of horizontal velocities as $\partial_z u = \partial_z v = 0$. Indeed, since $\frac{1}{\rho}\frac{\partial p}{\partial x} = g\partial_x\eta$ and $\eta = \eta(x, y, t)$, we have $\frac{\partial}{\partial z}(\frac{1}{\rho}\frac{\partial p}{\partial x}) = 0$.

Thus the pressure-driven flow should not have vertical shear either. The absence of vertical shear further simplifies the horizontal momentum equations and the continuity equation as follows. For the momentum equations, the vertical advection terms are negligible now, such that the total derivative is now:

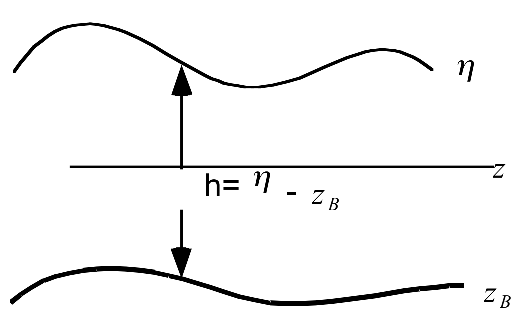
$$\frac{d}{dt} = \partial_t + \frac{u}{a\cos\theta}\partial_\varphi + \frac{v}{a}\partial_\theta , \qquad (\equiv \frac{D}{Dt}) \qquad (1.1.8)$$

For the mass equation,

$$\frac{1}{a\cos\theta}\partial_\varphi u + \frac{1}{a\cos\theta}\partial_\theta(v\cos\theta) + \partial_z w = 0$$

vertical integration ($\int_{z_B}^{\eta} dz$) leads to:

$$(\eta - z_B)[\frac{1}{a\cos\theta}\partial_\varphi u + \frac{1}{a\cos\theta}\partial_\theta(v\cos\theta)] + w(\eta) - w(z_B) = 0$$



The kinematic boundary condition on the surface is:

$$w(\eta) = \frac{d\eta}{dt} = (\partial_t + \mathbf{u} \bullet \nabla)\eta$$

where $\boldsymbol{u}=(u,v)$. The kinematic boundary condition on the bottom, for a fixed bottom topography $\partial_t z_B=0$, is

$$w(h_B) = \frac{dz_B}{dt} = (\partial_t + \mathbf{u} \bullet \nabla) z_B = \mathbf{u} \bullet \nabla z_B \,,$$

Therefore, the continuity equation becomes:

$$\frac{dh}{dt} + h\nabla \bullet \mathrm{u} = 0 \qquad \text{or} \qquad \frac{\partial h}{\partial t} + \nabla \bullet (h\mathrm{u}) = 0 \qquad (1.1.9)$$

where $h = \eta - z_B$ is the total depth of the water column.

With all the approximations above, we have a simpler set of equations: the shallow water equations :

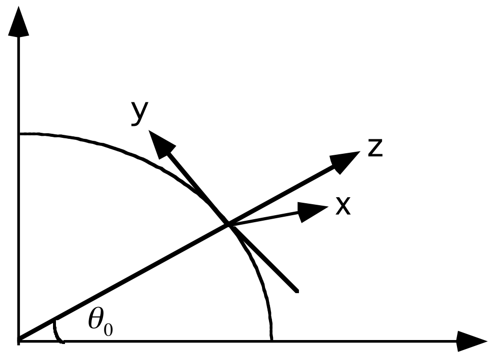
$$\begin{aligned} &\partial_t \mathbf{u} + (\mathbf{u} \bullet \nabla)\mathbf{u} + f\mathbf{k} \times \mathbf{u} = -g\nabla\eta + \mathbf{F} \\ &\partial_t h + \nabla \bullet (h\mathrm{u}) = 0 \;. \end{aligned} \qquad (1.1.10)$$

In the spherical coordinate, the shallow water equations can be written as :

$$\begin{aligned} &\frac{D}{Dt}u - (2\Omega + \frac{u}{a\cos\theta})v\sin\theta = -\frac{g}{a\cos\theta}\partial_\varphi \eta + F_\varphi \\ &\frac{D}{Dt}v + (2\Omega + \frac{u}{a\cos\theta})u\sin\theta = -\frac{g}{a}\partial_\theta \eta + F_\theta \\ &\frac{D}{Dt}h + \frac{h}{a\cos\theta}[\partial_\varphi u + \partial_\theta (v\cos\theta)] = 0 \end{aligned} \qquad (1.1.11)$$

For most purposes, the curvature term $\frac{u}{a\cos\theta}$ is much smaller than the Coriolis term, and therefore can also be neglected (except in the polar region and global scale flows, see next).

2. Local Cartesian Coordinate (β-plane)



Unless one studies global scale (in θ) circulation, most of the time, we can simplify the equations further by using the local Cartesians coordinate:

$$x = a\cos\theta_0(\varphi - \varphi_0), \quad y = a(\theta - \theta_0)$$

This gives:

$$\partial x = a\cos\theta_0 \partial\varphi$$

$$\partial y = a\partial\theta$$

The shallow water equations (1.1.11) can be written as:

$$\begin{cases} \partial_t u + \dfrac{\cos\theta_0}{\cos\theta} u\partial_x u + v\partial_y u - (2\Omega + \dfrac{u}{a\cos\theta})v\sin\theta = -g\dfrac{\cos\theta_0}{\cos\theta}\partial_x\eta + F_x \\ \partial_t v + \dfrac{\cos\theta_0}{\cos\theta} u\partial_x v + v\partial_y v + (2\Omega + \dfrac{u}{a\cos\theta})u\sin\theta = -g\partial_y\eta + F_y \\ \partial_t h + \dfrac{\cos\theta_0}{\cos\theta} u\partial_x h + v\partial_y h + h[\dfrac{\cos\theta_0}{\cos\theta}\partial_x u + \partial_y v - \dfrac{v}{a}tg\theta] = 0 \end{cases} \quad (1.1.12)$$

Furthermore, we can use the beta-plane approximation for motions of meridional scales less than the radius of the earth, $\frac{L_y}{a} \approx \frac{a(\theta - \theta_0)}{a} << 1$. Indeed, now with $\theta - \theta_0 << 1$, we can have the first order approximations as:

$$\cos\theta = \cos\theta_0 - (\theta - \theta_0)\sin\theta_0 + O[(\theta - \theta_0)^2]$$

$$\sin\theta = \sin\theta_0 + (\theta - \theta_0)\cos\theta_0 + O[(\theta - \theta_0)^2]$$

$$tg\theta = tg\theta_0 + O[\theta - \theta_0]$$

The curvature terms become negligible even compared with the advection term (since $\frac{u}{a\cos\theta} << 2\Omega$ is easily satisfied). In the u-equation, we have

$$\frac{\frac{uv}{a}tg\theta}{v\partial_y u} \propto \frac{L_y}{a}tg\theta_0 << 1$$

(except for the polar region with $\theta_0 \to \frac{\pi}{2}$!). In the mass equation, we have

$$\frac{\frac{v}{a}tg\theta}{\partial_y v} \propto \frac{L_y}{a} << 1.$$

Define

$$f_0 = 2\Omega \sin\theta_0, \beta = \frac{2\Omega}{a}\cos\theta_0$$

we can approximate the Coriolis parameter as

$$f = f_0 + \beta y.$$

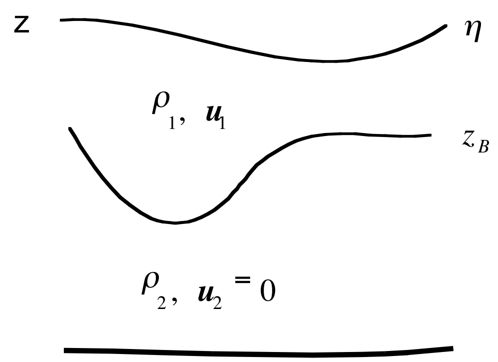
Therefore, equations (1.1.11) can be approximated as

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - fv = -g\partial_x \eta + F_x \\ \partial_t v + u\partial_x v + v\partial_y v + fu = -g\partial_y \eta + F_y \\ \partial_t h + u\partial_x h + v\partial_y h + h(\partial_x u + \partial_y v) = 0. \end{cases} \quad (1.1.13)$$

where $h=\eta-z_B$ is the layer thickness. This is the typical form of shallow water equations to be used.

<u>3. 1.5-layer model</u>

Much of the shallow water equation results can be applied to a very (seemingly) different fluid. (example, stratosphere, oceanic thermocline etc.)



Consider a general 2-layer fluid with the two layers of fluid of densities ρ_1 and ρ_2. In general, the upper layer fluid still satisfies the shallow water equations (1.1.13), except now the bottom depth of the upper layer fluid z_B also varies with time $z_B = z_B(x, y, t)$ and is an unknown variable to be determined. For a general 2-layer fluid, therefore, the problem is not closed, because we have three equations but four unknowns u, v, η, z_B. Now, consider a special type of 2-layer fluid, in which the upper layer flow is much faster than the lower layer flow. In this case, the bottom layer can be treated approximated as motionless, in which the horizontal pressure gradient vanishes. It is this vanishing pressure gradient in the lower layer provides the addition relation needed to close the upper layer problem. This is the so called 1.5-layer fluid system. In the lower layer, the total pressure at a depth z can be derived from the hydrostatic balance as

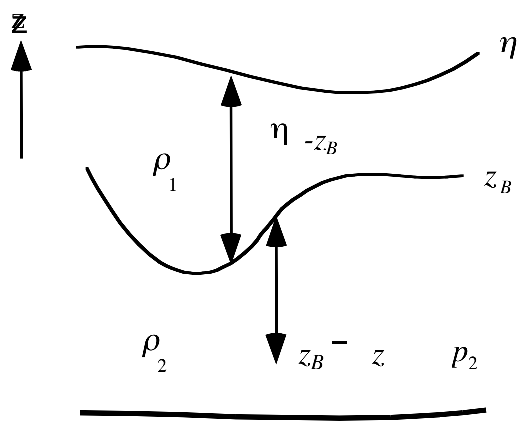
$$p_2 = \int_z^{\eta} \rho g dz = \int_{z_B}^{\eta} \rho g dz + \int_z^{z_B} \rho g dz = g(\eta - z_B)\rho_1 + g(z_B - z)\rho_2$$

The condition of zero pressure gradient in layer 2 is

$$0 = \nabla p_2 = g\rho_1 \nabla \eta - (\rho_2 - \rho_1)\nabla(-z_B)$$

Thus, the surface elevation can be represented as

$$\underline{g\rho_1 \nabla \eta = (\rho_2 - \rho_1)\nabla(-z_B)} \qquad (1.1.14)$$



The pressure gradient in the surface layer can therefore be represented in terms of the layer interface as

$$g\nabla\eta = \left(\frac{\rho_2 - \rho_1}{\rho_1}\right) g \nabla(-z_B) = g' \nabla(-z_B)$$

where

$$g' = \frac{\rho_2 - \rho_1}{\rho_1} g < g$$

($g' \ll g$ in the ocean) is called the reduced gravity. It is seen that the vanishing of the lower layer pressure gradient is now possible because the pressure gradient at the upper layer is completely compensated by that associated with the layer interface. This leads to the opposite slopes of the surface elevation and the layer interface.

Assuming the average surface is at z=0, we have the thickness of the upper layer as

$$h = \eta - z_B \approx -z_B$$

where we have used $g' << g$, so that $|\nabla\eta| << \nabla h$ from (1.1.14). Thus, the equations for the upper layer, according to (1.1.13), can be written as:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - fv = -g'\partial_x h + F_x \\ \partial_t v + u\partial_x v + v\partial_y v + fu = -g'\partial_y h + F_y \ , \\ \partial_t h + u\partial_x h + v\partial_y h + h(\partial_x u + \partial_y v) = 0. \end{cases} \qquad (1.1.15)$$

This set of equations for the 1.5-layer model therefore looks exactly the same as the one-layer system equations for layer 1 in (1.1.13), except to replace *g* and η by *g'* and *h*.

The 1.5-layer approximate is usually very good for the oceanic thermocline, because the upper ocean circulation is much faster than the abyssal flow. Indeed, the slope of the oceanic thermocline is usually opposite to that of the surface elevation (Fig.1.1) with a much smaller magnitude of the former than the latter.

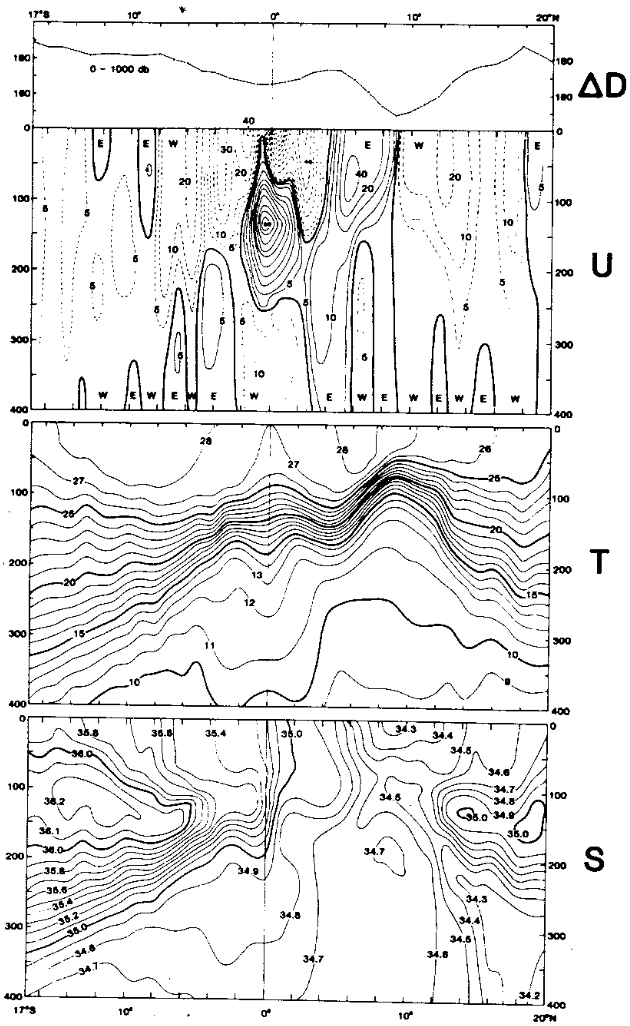


FIG. 7.33. Mean distributions of surface dynamic height (ΔD dyn. cms.) relative to 1000 db (dyn. cm.) and vertical meridional sections of zonal geostrophic flow (U cm/s), temperature (t°C) and salinity (S) between Hawaii and Tahiti, for 12 months from April 1979. (Wyrtki and Kilonsky, 1984.)

FIG.
phosp

Fig.1.1

Section 1.2: Conservation Laws

We first consider some fundamental conservation laws that should be satisfied by a general fluid system. This is also a check for the consistency of the approximations that we made to our shallow water equations.

1. Energy Conservation.

To derive the energy equation, we first put the total derivative of any variable A into its mass conservation form.

$$\begin{aligned} hD_tA &= h\partial_t A+hu\partial_x A+hv\partial_y A+A[\partial_t h+\partial_x(uh)+\partial_y(hv)] \\ &= \partial_t(hA)+\partial_x(huA)+\partial_y(hvA) \end{aligned} \tag{1.2.1}$$

where we have used the mass equation $\partial_t h+\partial_x(uh)+\partial_y(vh)=0$. The kinetic energy equation can be derived by first multiplying u and v onto the u- and v- equations, respectively and then sum them up as

$$D_t[(u^2+v^2)/2] = -g[u\partial_x\eta+ v\partial_y\eta]+uF_x+vF_y$$

work by pressure grad. Work by source/sink

Multiplied by h, and with (1), we have the mass conservation form of the kinetic energy of the water column per unit area $K= h(u^2+v^2)/2$ as:

$$\partial_t K+\partial_x(uK)+\partial_y(vK)= -g[hu\partial_x\eta+ hv\partial_y\eta]+huF_x+hvF_y \tag{1.2.2}$$

The potential energy in each column of water per unit area is $P=g(\eta+z_B)h/2 \equiv g(\eta^2-z_B^2)/2$ (prove it, notice z_B is independent of time now). The potential energy equation can be derived by multiplying the mass equation by $g\eta$ as:

$$\partial_t P+g\eta[\partial_x(uh)+\partial_y(vh)]=0 \tag{1.2.3}$$

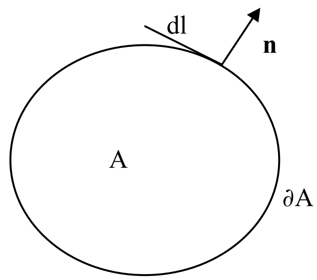
Therefore, the equation for total energy E=K+P is derived by adding (1.2.2) and (1.2.3) as:

$$\partial_t E+\partial_x(uK)+\partial_y(vK)= -\partial_x (hug\eta)+ \partial_y (hvg\eta)+huF_x+hvF_y$$

Integrating within a domain A with a solid or periodic boundary, we have the conservation of the total energy (in the absence of external source/sink) as:

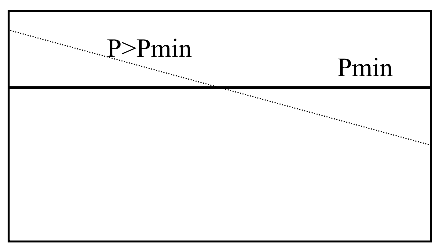
$$\partial_t \int_A E dA=\partial_t\int_A(K+P)dA = \int_A h\boldsymbol{u}\bullet\boldsymbol{F}dA \tag{1.2.4}$$

where we have used the divergence theorem: $\int_A \nabla\bullet(uS)dA=\int_{\partial A} S\boldsymbol{u}\bullet\boldsymbol{n}dl$ with S being any variable, ∂A the boundary of the domain A, $\boldsymbol{n}$ the unit vector outwards, and dl the line element around the boundary.



2. Available Potential Energy (APE)

How much of the potential energy can be changed to the kinetic energy ? Not all of them for sure. There is a basic part of the potential energy that can't be converted to the kinetic energy. This is the minimum potential energy. (see figure, the leveled surface state is the minimum potential energy state). (Indeed, the absolute value of potential energy has no meaning, because one can choose the reference height arbitrarily.)



Therefore, we define the part of P that can be converted to K as the APE, or

$$APE=P-P_{min}$$

In the shallow water system, define $\eta_m=\int\eta dA/A$, we have

$$d\eta_m/dt=0 \qquad (1.2.5)$$

(total mass conservation). Thus, the APE is $APE=g(\eta-\eta_m)^2/2$. Since $dAPE/dt=\partial_t\int_A g[\eta^2-2\eta\eta_m+\eta_m^2]dA=\partial_t\int_A g\eta^2 - 2g\eta_m\,\partial_t\int_A\eta=\partial_t\int_A g\eta^2=dP/dt$, (see (1.2.5)), the total energy conservation equation (1.2.4) can be written as:

$$\partial_t\int_A(K+APE)dA=\int_A h\mathbf{u}\bullet\mathbf{F}dA$$

3. Bernoulli Equation

In the absence of source and sink, the kinetic energy equation is:

$$D_t[(u^2+v^2)/2] = -g[u\partial_x\eta+ v\partial_y\eta].$$

The mass equation $\partial_t H+\partial_x(uH)+\partial_y(Hv)=0$ can be rewritten as

$$D_t\,(g\eta)+gh(\partial_x u+\partial_y v) - g[u\partial_x z_B+ v\partial_y z_B]=0,$$

we have

$$\begin{aligned}D_t[(u^2+v^2)/2+g\eta] &= -g[u\partial_x\eta+ v\partial_y\eta]+ g[u\partial_x z_B+ v\partial_y z_B] -gh(\partial_x u+\partial_y v)\\ &= -g[u\partial_x\,(\eta- z_B)+ v\partial_y(\eta- z_B)]-gh(\partial_x u+\partial_y v)= -g[\partial_x\,(uh)+ \partial_y(vh)]=g\partial_t\,\eta\end{aligned}$$

Thus, for steady flow, $\partial_t\,\eta=0$, and therefore we have the Bernoulli equation in the shallow water as:

$$D_t B= D_t(k_e+p_e)=D_t[(u^2+v^2)/2+g\eta]=0.$$

This states that the Bernoulli function B, which is the sum of the kinetic energy of a water parcel $k_e=(u^2+v^2)/2$ and the potential energy $p_e=g\eta,$ is conserved. following the motion of a steady circulation. (Note: this is the conservation of a particle, while total energy conservation is in a fixed domain).

4. Angular Momentum Conservation:

In the spherical coordinate, we can also show that the angular momentum of a particle $M=\Omega a^2 cos\theta^2+u\ a\ cos\theta$ is conserved. Multiply the u-eq. in (1.1.3) by $cos\theta$, we can show that:

$$D_t M= -g\partial_\varphi\eta+F_\varphi a\ cos\theta$$

Thus, if there is no source/sink in zonal momentum, the zonally integrated angular momentum is conserved.

$$D_t(\int M\ d\varphi)=0.$$

Here, we have used the condition that the pressure is continuous around the globle and therefore the integrated derivative along the latitudinal circle is zero. One should notice that if one neglects the curvature term $u/cos\theta$ in the term of $(2\Omega +u/ cos\theta)vsin\theta$ in the u-equation, the angular momentum is no longer conserved. For example, in the beta-plane model, $\int M\ d\varphi$ is not conserved.

Section 1.3: Circulation, Vorticity and Kelvin's Theorem.

(ref. Pedloksy, section 2.2, and Holton section 4.1).

1. Vorticity and Circulation:

How to measure the rotation rate of a fluid parcel? Unlike a solid body, different parts of the fluid usually does not rotate at the same rate, because of the velocity shears. One way is to calculate the integrated circulation of the velocity around the boundary of the surface domain A of a fluid parcel:

$$\Gamma = \int_{\partial A} \boldsymbol{u} \bullet \mathbf{dr} = \int_A \nabla \times \boldsymbol{u} \bullet \boldsymbol{n}\, dA = \int_A \boldsymbol{\omega} \bullet \boldsymbol{n}\, dA, \qquad (1.3.1)$$

where **u** is the 3-D velocity field in a non-rotating frame, and

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \qquad (1.3.2)$$

is the vorticity of this velocity field.

In comparison, for a solid body, we have $\mathbf{u} = \mathbf{\Omega} \times \mathbf{r}$ valid at any point. Notice the vector multiplication

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

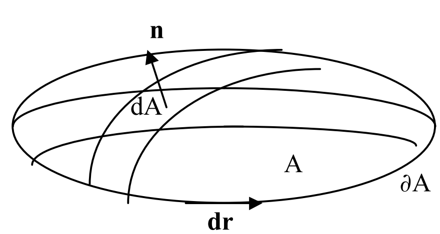
one can prove that vorticity is twice the rotation rate

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = 2\boldsymbol{\Omega}. \qquad (1.3.3)$$

For a fluid parcel, each point rotates at different rate. Therefore, vorticity is approximately the averaged circulation (or averaged rotation). This can also be seen clearly if one allows the domain of circulation to shrink to a point. Then, from (1.3.1), the area averaged circulation becomes

$$\boldsymbol{\omega} \bullet \boldsymbol{n} = \int_{\partial A} \boldsymbol{u} \bullet \boldsymbol{dr}/A \qquad \text{for } A \to 0$$

Thus, vorticity can be thought as the area-averaged circulation.



2. Kelvin Theorem.

Kelvin theorem predicts the change of the circulation and is a fundamental theory in fluid dynamics. Let's study how the circulation on a material surface varies with time. (denote $d_t \equiv d/dt$)

$$d_t\Gamma = \int_{\partial A} d_t\, \mathbf{u}\bullet \mathbf{dr} + \int_{\partial A} \mathbf{u}\bullet\, d_t\, (\mathbf{dr}).$$

Notice that $\int_{\partial A} \mathbf{u} \bullet d_t\, (\mathbf{dr}) = \int_{\partial A} \mathbf{u}\bullet\, d\mathrm{u} = \int_{\partial A} d|\mathbf{u}|^2/2 = 0$, where we have used

$d_t\, (d\mathbf{r}) = d\, (d_t\, \mathbf{r}) = d\mathbf{u}$, we have:

$$d_t\Gamma = \int_{\partial A} d_t\, \mathbf{u} \bullet \mathbf{dr} = - \underbrace{\int_{\partial A} \nabla p/\rho \bullet \mathbf{dr}}_{\text{pressure grad}} + \int_{\partial A} \mathbf{F} \bullet \mathbf{dr} \qquad (1.3.4)$$

where we have used the momentum equation

$$d_t\, \mathbf{u} = - \nabla p/\rho + \mathbf{F}.$$

For a homogeneous fluid, the averaged pressure gradient vanishes,

$$\int_{\partial A} \nabla p/\rho \bullet \mathrm{d}\mathbf{r} = 1/\rho \int_{\partial A} \nabla p \bullet \mathbf{dr} = 1/\rho \int_{\partial A} dp = 0.$$

If, furthermore, there is no source and sink ($\boldsymbol{F}=0$), circulation is conserved following the water parcel:

$$d_t\Gamma = 0, \quad or \quad \Gamma = const \qquad (1.3.5)$$

This is the Kelvin's theorem.

3. Kelvin's theorem in a rotating frame

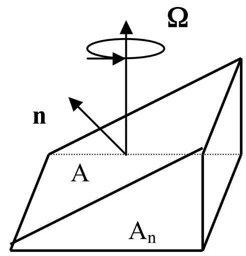
In a rotating frame, the absolute velocity in the non-rotating frame is $\mathbf{u}_a = \mathbf{u} + \mathbf{\Omega}\times\mathbf{r}$ (prove it !), where **u** is the **relative** velocity in the **rotating frame** now. The absolute circulation is then (see Pedlosky, also Question ***Q1.3***):

$$\Gamma_a = \int_{\partial A} \mathbf{u}_a \bullet \mathbf{dr} = \int_{\partial A} \mathbf{u}\bullet\mathbf{dr} + \int_{\partial A} (\mathbf{\Omega}\times\mathbf{r})\bullet\mathbf{dr} = \Gamma + 2\Omega A_n\,. \qquad (1.3.6)$$

where A_n is the area projected by the surface A onto a plane normal to the rotation vector **Ω..** Here, we have used (1.3.3) and the Stokes' theorem such that:

$$\int_{\partial A} (\mathbf{\Omega}\times\mathbf{r})\bullet\mathrm{d}\mathbf{r} = \iint_A \nabla\times(\mathbf{\Omega}\times\mathbf{r})\bullet\mathbf{n}\, dA = \iint_A 2\mathbf{\Omega}\bullet\mathbf{n}\, dA = 2\mathbf{\Omega} A_n$$

Thus, the absolute circulation consists of a relative circulation and a planetary circulation.



The Kelvin's theorem becomes:

$$d_t\Gamma_a = d_t\,(\Gamma+2\Omega A_n) \;= 0 \qquad (1.3.7)$$

Or

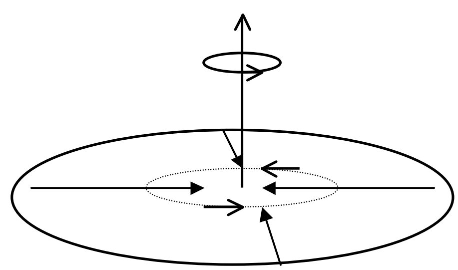
$$\Gamma+2\Omega A_n = const. \qquad (1.3.8)$$

Eqn. (1.3.7) can also be proven directly in the rotating frame, if the Coriolis force is included in the momentum equation.

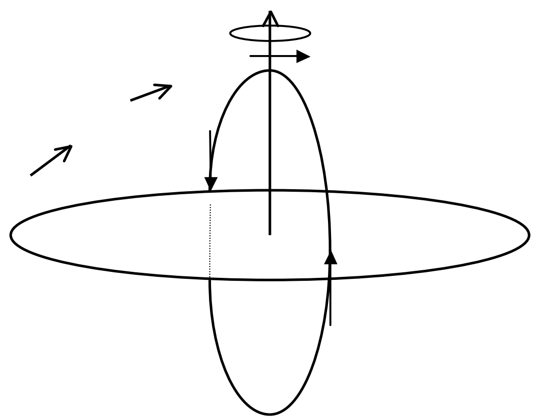
4. Examples:

The Kelvin theorem is very powerful. It states that if there is a circulation, it will be there by itself forever (in the absence of dissipation). Below are some examples of application.

Example 1: Contraction spin-up: When the fluid converges towards the sink, the circulation will accelerate because of the decrease of the area A_n.

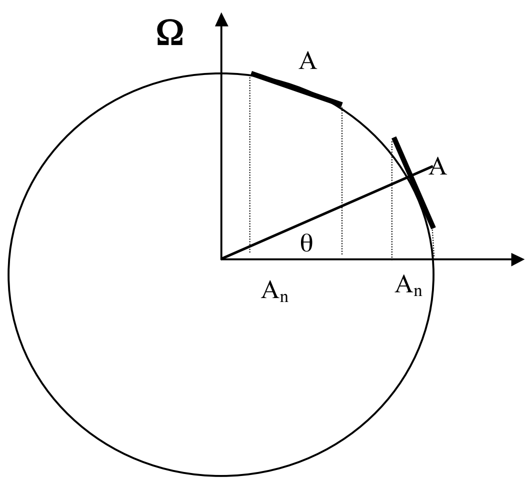


Example 2: Tilting: When the surface of circulation tilts away from the plane of rotation, circulation increases because of the decrease of A_n.



Example 3: Rossby wave

On a sphere, we have the area of a latitudinal belt of air as $A_n = A\sin\theta$. Now, Kelvin's theorem gives the so called Rossby wave (see Section 2.2).



Since $d_t(\Gamma + 2\Omega A_n) = 0$, we have $d_t\Gamma = -2\Omega\, d_t A_n = -2\Omega A\cos\theta\, d_t\theta = -A\beta v$, where $\beta = 2\Omega\cos\theta/a$ and $v = a\, d_t\theta$. Since relative vorticity is the averaged circulation $\zeta = \Gamma/A$, we have:

$$d_t\zeta + \beta v = 0.$$

This will be seen later as the equation for the Rossby wave.

Sec. 1.4: Potential Vorticity Conservation

1. Vorticity Equation

Vorticity equation governs the local change of fluid rotation from an Eularea view and is therefore practical. From the momentum equations

$$\begin{cases}\partial_t u + u\partial_x u + v\partial_y u - fv = -g\partial_x \eta + F_x \\ \partial_t v + u\partial_x v + v\partial_y v + fu = -g\partial_y \eta + F_y\end{cases}$$

we apply the vorticity operation $\partial_x(\partial_t v \quad eq) - \partial_y(\partial_t u \quad eq)$ to eliminate the pressure gradient term. This gives the equation for the relative vorticity $\zeta = \partial_x v - \partial_y u$. First, notice that

$$u\partial_x u + v\partial_y u = -v(\partial_x v - \partial_y u) + \partial_x(\frac{u^2}{2} + \frac{v^2}{2}) = -v\zeta + \partial_x(\frac{u^2 + v^2}{2})$$

$$u\partial_x v + v\partial_y v = +u(\partial_x v - \partial_y u) + \partial_y(\frac{u^2}{2} + \frac{v^2}{2}) = u\zeta + \partial_y(\frac{u^2 + v^2}{2})$$

we can rewrite the u, v equations as

$$\partial_t u - (f + \zeta)v = -\partial_x B + F_x$$
$$\partial_t v + (f + \zeta)u = -\partial_y B + F_y$$

where

$$B = \frac{u^2 + v^2}{2} + g\eta$$ is the Bernoulli function.

Now, $\partial_x(v \quad eq) - \partial_y(u \quad eq)$ $\Rightarrow$ eliminate B, we have the vorticity equation

$$(\partial_t + u\partial_x + v\partial_y)(f + \zeta) = -(f + \zeta)(\partial_x u + \partial_y v) + curl\,\vec{F} \qquad (1.4.1)$$

where $curl\,\vec{F} = \partial_x F_y - \partial_y F_x$.

Or

$$\frac{D}{Dt}\zeta_a = -\zeta_a \nabla \bullet \vec{u} + curl\,\vec{F} \qquad (1.4.2)$$

where $\zeta_a = \zeta + f$ is the absolute vorticity. Again, we see that a divergent flow, in the presence of background vorticity field, can generate vorticity. This is similar to the Kelvin's theorem.

2. Potential Vorticity Equation

From the continuity equation, we have

$$\frac{D_t h}{h} = -\nabla \bullet \mathrm{u}$$

Notice the vorticity equation (1.4.1) or (1.4.2), we have

$$\frac{\frac{D}{Dt}\zeta_a}{\zeta_a} = \frac{D_t h}{h} + \frac{curl\mathrm{F}}{\zeta_a}$$

$$\frac{D}{Dt}(\ln \zeta_a) = \frac{D}{Dt}(\ln h) + \frac{curl\mathrm{F}}{\zeta_a}$$

$$\frac{D}{Dt}\ln(\frac{\zeta_a}{h}) = \frac{curl\mathrm{F}}{\zeta_a}$$

Or, finally

$$\frac{D}{Dt}q = \frac{D}{Dt}(\frac{\zeta + f}{h}) = \frac{curl\mathrm{F}}{h} \tag{1.4.3}$$

where

$$q = \frac{\zeta + f}{h}$$

is the potential vorticity. Thus, in the absence of source and sink terms, we have

$$\boxed{\frac{D}{Dt}q = 0}$$

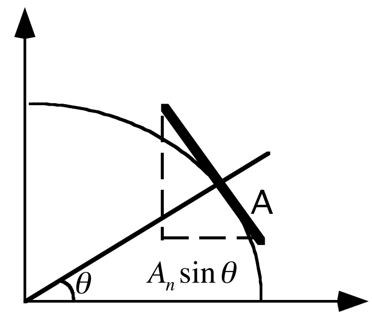
The PV is conserved along a particle trajectory. This is a very strong constrain on fluid motion. (Rossby, 2-D, 1940; later, Ertel, 3-D, 1942).

3. P.V. Conservation and Kelvin's Theorem

The conservation of PV can be derived directly from the Kelvin's theorem.

$$\frac{D}{Dt}(\Gamma + 2\Omega A_n) = 0$$

Now $A_n = A\sin\theta$, and $\Gamma \approx \zeta A$.



For a small area element A, we have

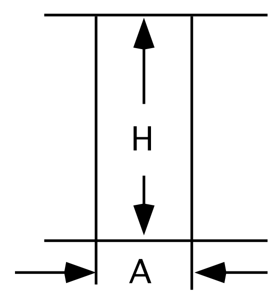
$$\frac{d}{dt}(\zeta A + 2\Omega \sin\theta A) = \frac{d}{dt}[(\zeta + f)A] = 0\,.$$

The total mass conservation can be written as

$$hA = const\,.$$

Therefore, we have the PV conservation

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0$$



Thus, PV conservation, in principle, is the same as the Kelvin's theorem. They represent two different views of the fluid rotation: the PV provides a microscopic view, while the circulation (and Kelvin theorem) the macroscopic view. This law on PV is of fundamental importance to GFD.

<u>4. Angular momentum conservation and PV conservation</u>:

The PV conservation can also be understood intuitively from the angular momentum conservation

$$\omega r^2 = const.$$

For a water column of a fixed volume (or mass),

$$\pi \rho r^2 H = M = mass = const.$$

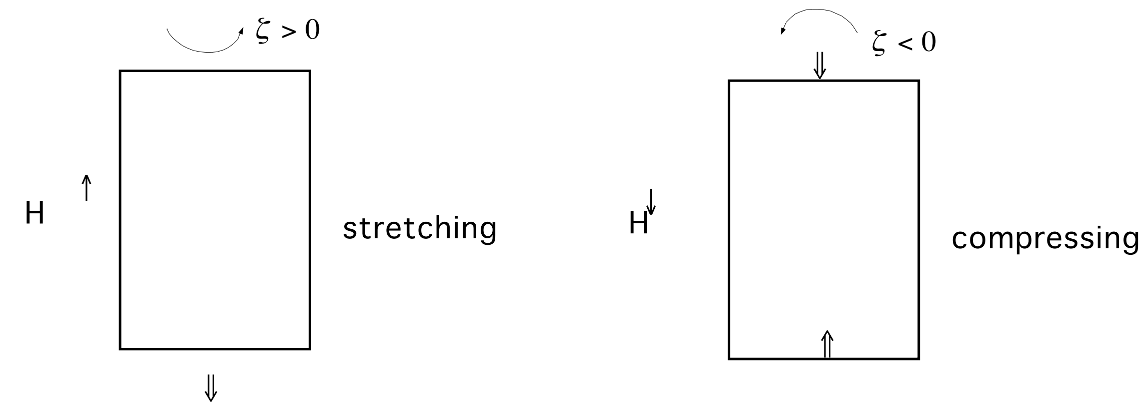
Thus,

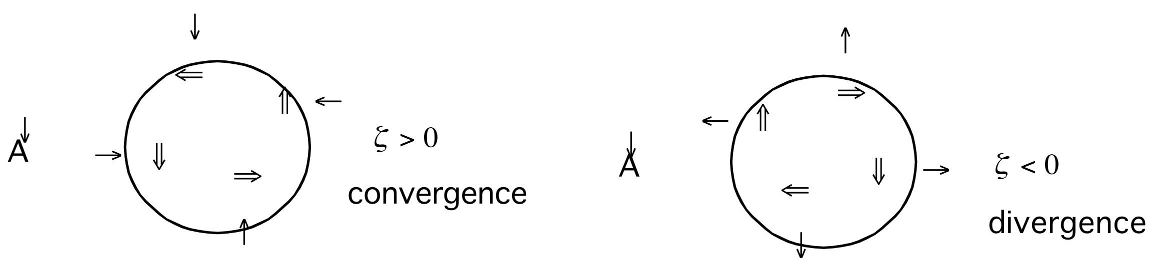
$$\frac{\omega}{H} = const$$

This recovers PV conservation. Thus, PV conservation is simply angular momentum conservation in the case of a solid body.

<u>5. Applications</u>

(i) Stretching—Contraction:
Stretching of a water column generates positive vorticity, while compressing of a water column generates negative vorticity, according to PV conservation.





On, say, a f-plane, $f = f_0 = const$, we therefore have

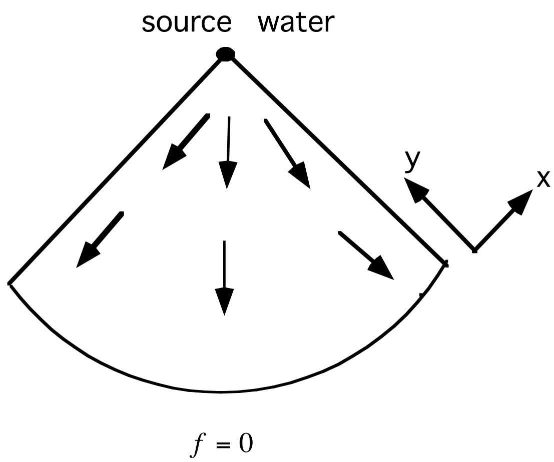
$$H\uparrow \quad \Rightarrow \quad \zeta\uparrow$$

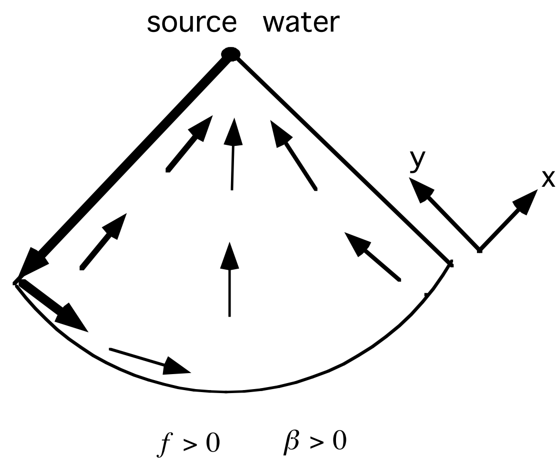
$$H\downarrow \quad \Rightarrow \quad \zeta\downarrow$$

Application: intensification of the center of a storm, figure skating.

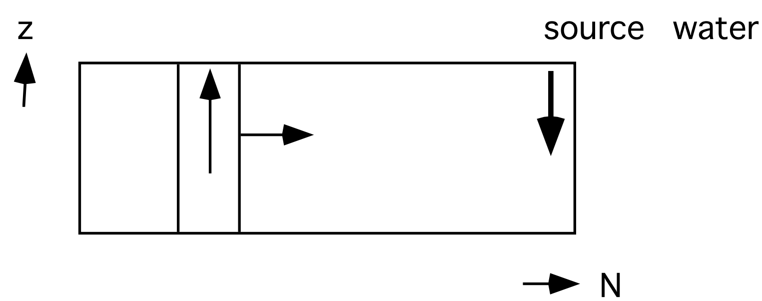
(ii) Abyssal circulation (Stommal-Arons model).

The thermohaline circulation at the abyss is forced by sinking water at polar region. For a deep water column, this means that the water column will be stretched H↑ because of the accumulation of waters. Thus, $f\uparrow$ ($\zeta << f$) , or and the water moves northward. This is rather surprising because the interior flow goes towards the source water, opposite to the non-rotating case.

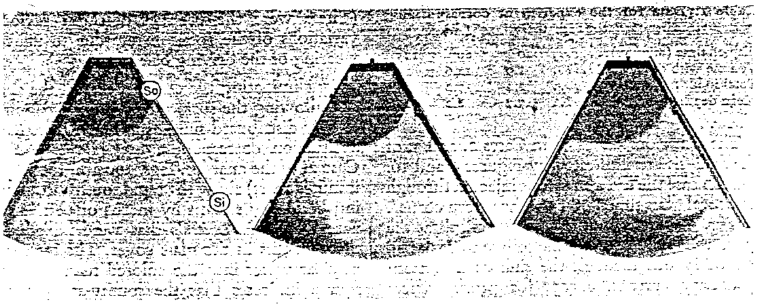




$f = 0$ $f \neq 0, \beta \neq 0$

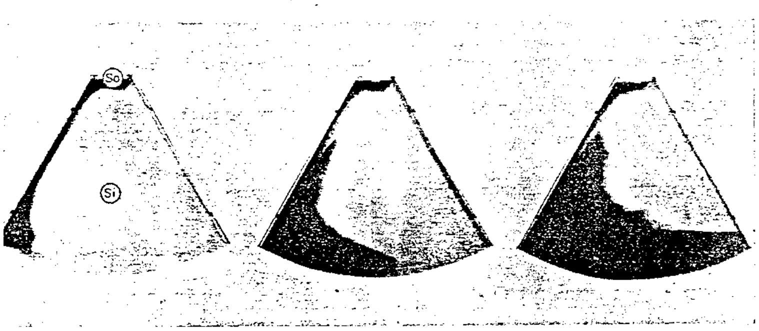


However, the source water at the pole has to get to the lower latitude to satisfy the mass conservation. The equatorward transport is carried in a narrow western boundary current. This flow pattern is confirmed in lab rotating tank experiment (Fig.1.2).



ig. 7. Photographs at 20, 80, and 220 minutes after the introduction of dye, showing the path followed by the urce water ($S_0 = 50$ cc./min.) in flowing from a slot in the eastern wall near the apex to the sink ($S_1 = 5/6\ S_0$) in the same wall near the rim (corresponding to Fig. 2).

Fig. 8 illustrates the circulation with the urce at the apex of the sector and with no ternal sink, i.e., with a uniform rise of the ater level in the tank (a uniformly distributed nk). (Compare with Fig. 3 and Case 1. of the athematical analysis.) In accord with the eory, the interior of the basin filled from the m and continuity was maintained by an in- stern current S_0 from the source, which as augmented by a recirculation (theoretically f strength S_0 for $l = 1$) from the interior water. vidence for this recirculation is the clear streak Figs. 8b, c which penetrates toward the rim to the dyed fluid near the western wall. By aluating the volume of fluid-containing dye, e transport of the total western boundary current has been determined for comparison with the source strength. The data of Table I show that: a) at 20 minutes the dyed fluid was composed almost entirely of source water, b) at 60 minutes the dyed fluid occupied twice the total volume of the input, which implies mixing with an equal volume of clear water (presumably that from the recirculated component of the boundary current), and c) at 125 minutes the dyed volume was less than twice that of the source input, a fact which is consistent with the decrease in the factor l as the mean depth of the fluid ($\bar{h}$) gradually increased. These results are in rough agreement with the quantitative predictions of the theory.



ig. 8. Photographs at 20, 60, and 80 minutes with $S_0 = 120$ cc./min. The source was at the apex and there was no external sink (corresponding to Fig. 3).

Fig.1.2

Sec.1.5 Shallow Water Waves on a f-Plane

Consider small amplitude motions on an f-plane with a flat bottom z_B=0 and in turn a constant depth H. The basic state is motionless $U = V = 0, \quad H = const$. The perturbed variables are therefore:

$$\eta = H + \eta', \quad u = u', \quad v = v', \quad \eta' << H$$

Linearizing the shallow water equations (1.1.13), we have:

$$\begin{cases} \partial_t u' - f_0 v' = -g\partial_x \eta' \\ \partial_t v' + f_0 u' = -g\partial_y \eta' \\ \partial_t \eta' + H(\partial_x u' + \partial_y v') = 0 \end{cases}$$

The vorticity and divergence equations can be derived from $\partial_x(v \quad eq) - \partial_y(u \quad eq)$ and $\partial_x(u \quad eq) + \partial_y(v \quad eq)$ as

$$\partial_t(\partial_x v' - \partial_y u') + f_0(\partial_x u' + \partial_y v') = 0,$$

and

$$\partial_t(\partial_x u' + \partial_y v') - f_0(\partial_x v' - \partial_y u') = -g\nabla^2\eta'$$

respectively. Note that vorticity can generate divergence, and vise versa.

Using the divergence and vorticity equations $\partial_t(div \quad eq) + f(vort \quad eq)$, we have

$$(\partial_{tt} + f_0^2)(\partial_x u' + \partial_y v') = -g\nabla^2\partial_t\eta'$$

Substitute in the continuity equation to eliminate divergence, we have finally,

$$\boxed{\partial_t[(\partial_{tt} + f_0^2) - c_0^2\nabla^2]\eta' = 0}$$

where $c_0 = \sqrt{gH}$ is the gravity wave speed.

Assuming free waves of the form $\eta \propto \mathrm{Re}\{\eta_0 e^{i(kx+ly-\omega t)}\}$, we have

$$\omega[f_0^2 - \omega^2 + c_0^2 K^2]\eta_0 = 0$$

where the total wave number is $K^2 = k^2 + l^2$. For nontrivial solution $\eta_0 \neq 0$, so we have the dispersion relationship

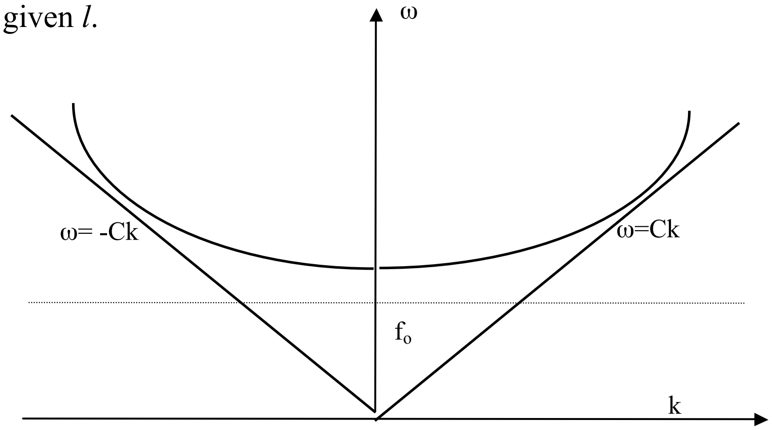
$$\omega[f_0^2 - \omega^2 + c_0^2 K^2] = 0 \qquad (1.5.1)$$

There are three roots:

$$\omega_1 = 0$$
$$\omega_{2,3}^2 = f_0^2 + c_0^2 K^2 \qquad (1.5.2a,b)$$

They represent two very different types of waves.

Dispersion diagram
for a given *l*.



1) The geostrophic mode.

The first group is the low frequency mode: $\omega_1 = 0$. The eigenfunction for this mode can be derived from (1.5.1) by setting $\partial_t = 0$ as

$$\begin{cases} -fv' = -g\partial_x \eta' \\ fu' = -g\partial_y \eta' \\ \partial_x u' + \partial_y v' = 0 \end{cases}$$

This mode is in geostrophic balance and is therefore a low frequency geostrophic mode. If the Coriolis parameter varies with latitude (the so called beta-effect $df/dy = \beta \neq 0$), or a varying bottom topography $z_{By} \neq 0$, this mode will be modified and will be called the Rossby wave. (see later).

2). The Inertial-Gravity wave

The second group are $\omega_{2,3}$. For these waves, $\omega^2 > f^2$. So these are high frequency modes (faster than about the rotation period). These are the gravity waves modified by the rotation. They are called the Inertial-Gravity waves (also Poincare wave in oceanography).

For short waves with $K^2 >> (\frac{f_0}{c_0})^2 = \frac{1}{L_D^2}$ (or $L>>L_D$), we have approximately $\omega^2 = c_0^2 K^2$. The inertial gravity wave reduces to the shallow water gravity wave and does not feel much of the rotation. On the other limit, for very long inertial-gravity waves, $K^2 << (\frac{f_0}{c_0})^2 = \frac{1}{L_D^2}$, we have approximately $\omega^2 = f^2$. This is simply the inertial oscillation.

Here we see an important scale

$$L_D = \frac{c_0}{f_0} = \frac{\sqrt{gH}}{f_0}, \qquad (1.5.3)$$

the so called Rossby deformation radius. This scale separates the motions that are affected by the rotation. Large scale motions with $L >> L_D$ feel rotation, while small scale waves with $L << L_D$ do not feel rotation.

Typical values for the deformation radius can be calculated as follows. For shallow water system (they resemble the so called barotropic mode), we have $g=10m\cdot s^{-2}$, $f_0 = 10^{-4}s^{-1}$ So,

$$H_{atmosphere} = 10km \qquad \to \qquad L_D = 3000km$$

$$H_{ocean} = 4km \qquad \to \qquad L_D = 2000km$$

For the 1.5 layer system (they resemble the so called 1st baroclinic mode), we have the so called internal deformation radius

$$L_{ID} = \frac{c_0}{f_0} = \frac{\sqrt{g'H}}{f_0} \qquad (1.5.4)$$

$$g'_{atmosphere} \approx \frac{\Delta\rho g}{\rho} \approx \frac{g}{10}, \quad H \propto 10km, \quad \to \quad L_D \approx 1000km$$

$$g'_{ocean} \approx \frac{\Delta\rho g}{\rho} \approx \frac{g}{1000} \approx 10^{-2}m\cdot s^{-1}, \quad H_{\substack{thermocline\\ ocean}} \approx 1km, \quad \to \quad L_D \approx 50km$$

Thus, rotation is important for baroclinic oceanic processes even at very small scales (*L ~10 km*), while it is important only for those atmospheric processes of *L≥ 1000 km*.

Sec.1.6: Geostrophic Adjustment

One general question is why the observed large scale atmosphere and oceanic flows are always close to geostrophic balance? From the last section, we have seen that, on an f-plane, there is a low frequency free mode that is in geostrophic balance. If there is an initial imbalance of geostrophy (or ageostrophic disturbance) forced by external forcing or some other processes, how can the system always recovers back to the observed geostrophic balance? This is the geostrophic adjustment problem that was first studied by Rossby on the Gulf Stream problem in the late 1930s.

1. Equilibrium State

Let's consider the simplest case, small amplitude flow on an f-plane, with a constant depth:

$$\begin{cases} \partial_t u - fv = -g\partial_x \eta \\ \partial_t v + fu = -g\partial_y \eta \\ \partial_t \eta + H(\partial_x u + \partial_y v) = 0 \end{cases}$$

If the system eventually reaches steady state, we have a flat surface equilibrium for $f = 0$:

$$\begin{cases} \eta_{\infty x} = 0 \\ \eta_{\infty y} = 0 \\ u_{\infty x} + v_{\infty y} = 0 \end{cases} \qquad (1.6.1)$$

and the geostrophic balance for $f \neq 0$,

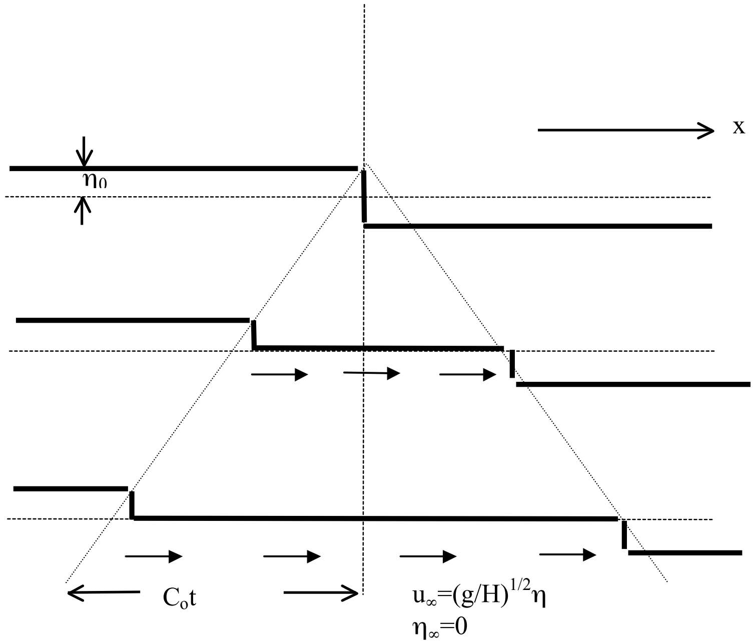
$$\begin{cases} -fv_\infty = -g\eta_{\infty x} \\ fu_\infty = -g\eta_{\infty y} \\ \partial_x u_\infty + \partial_y v_\infty = 0 \end{cases} \qquad (1.6.2)$$

In the latter case of $f \neq 0$, the first two equations satisfy the last equation automatidcally. Therefore, the final equilibrium is degenerated (geostrophic degeneracy) in the sense that there are 3 unknowns but only 2 equations. In addition, it is possible now to have $\eta_{\infty x} \neq 0, \eta_{\infty y} \neq 0$, so the final state has a finite available potential energy (APE). But how much? Some information is missing! This will be seen is the potential vorticity (PV).

2. *f=0* adjustment

First, we study the cases without rotation. We will study the simpler case with $\partial_y = 0$.

Case 1: An "jump" initial condition: $\eta_i = \eta_0\, sign(x)$, $u_i = v_i = 0$.



$$\begin{cases} \partial_t u = -g\eta_x \\ \partial_t v = 0 \\ \partial_t \eta = -H\partial_x u \end{cases}$$

Since $\eta_{\infty x} = 0 \neq \eta_i$, the initial state is not in equilibrium. The final state can be obtained as $\eta_\infty = const = 0$, $v_\infty = 0$, $u_\infty = const$, where the final velocity will be derived from the energy conservation below.

The KE and APE equations can be derived as:

$$\partial_t H\frac{u^2}{2} = -gHu\eta_x, \text{ and } g\partial_t\frac{\eta^2}{2} = -gH\eta u_x.$$

The total energy equation is therefore

$$\partial_t(KE + APE) = \partial_t(\frac{Hu^2}{2} + \frac{g\eta^2}{2}) = -\partial_x(gHu\eta)$$

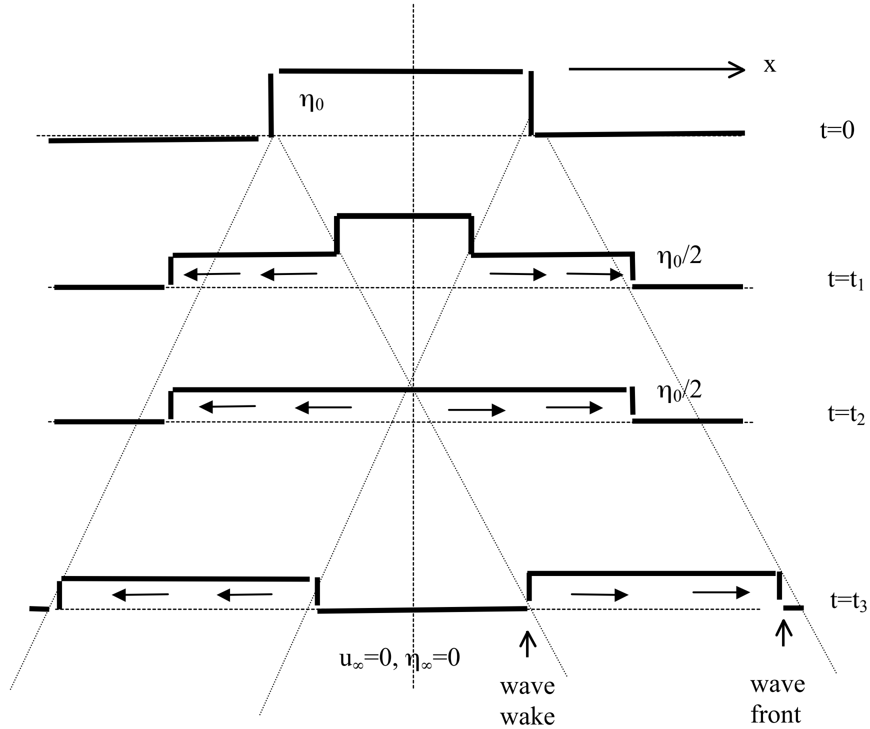
Before wave front, $u=0,\ \eta\neq 0$, after wave front, $u\neq 0,\ \eta=0$. Thus, the energy flux $u\eta = 0$ anywhere any time. Within each section, the total energy is conserved locally

$$\partial_t\int_{x_1}^{x_2}(KE + APE)dx = -gHu\eta\Big|_{x_1}^{x_2} = 0$$

Thus at final equilibrium which has no APE ($\eta_\infty = 0$), all the initial APE is converted to KE. This gives the final velocity as $u_\infty = \eta_0\sqrt{g/H}$. (One should notice that this local total energy conservation is usually not true, as will be seen in the next case. It occurs here because of the initial condition of an anti-symmetric elevation of infinite length.)

Case 2: An initial condition with a finite "bump".

One can speculate that the finite initial disturbance will eventually radiates away through gravity waves, leaving neither KE nor APE.



We can estimate the adjustment time as follows. From the equation $\partial_{tt}\eta - c^2\eta_{xx} = 0$, where $c = \sqrt{gH}$ is the gravity wave speed, the final equilibrium is reached when $\partial_{tt} << c^2\partial_{xx}$. Thus, the adjustment time T satisfies $\frac{1}{T^2} << \frac{c^2}{L^2}$ and in turn $T >> \frac{L}{c} = T_g$, where T_g is time for gravity wave to arrive.

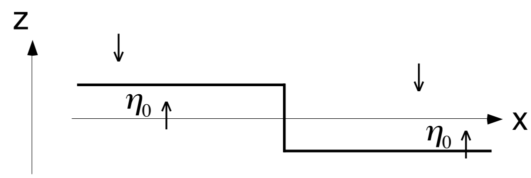
At a fixed position of, say, x > 0, the velocity experiences 5 stages and reaches the equilibrium after the passing of four wave fronts (fronts and wakes).

3. f≠0 adjustment

With $\partial_y=0$, the final equilibrium is

$$\begin{cases} -fv_\infty = -g\partial\eta_{\infty x} \\ u_\infty = 0 \\ u_{\infty x} = 0 \end{cases}$$

This state is undetermined. Rossby (1940) found the solution from the piece of missing information – the PV conservatoin.



The linearized vorticity equation can be derived from $\partial_x(v \quad eq) - \partial_y(u \quad eq)$ as

$$\partial_t \zeta = -f(\partial_x u + \partial_y v).$$

Substitute in the continuity equation, we have the linearized version of PV conservation.

$$\partial_t(\zeta - f\frac{\eta}{H}) = 0$$

Thus

$$\partial_x v_\infty - f\frac{\eta_\infty}{H} = \zeta_\infty - f\frac{\eta_\infty}{H} = initial\ PV = 0 - f\frac{\eta_i(x)}{H}$$

Since

$$v_\infty = \frac{g}{f}\eta_{\infty x}$$

we have

$$\partial_{xx}\eta_\infty - \frac{\eta_\infty}{L_D^2} = -\frac{\eta_i(x)}{L_D^2}$$

where $L_D^2 = \frac{gH}{f^2}$ is the deformation radius. The general solution is therefore:

$$\eta_\infty = \begin{cases} -\eta_0 + A_+ e^{\frac{x}{L_D}} + B_+ e^{-\frac{x}{L_D}}, & for \quad x > 0 \\ -\eta_0 + A_- e^{\frac{x}{L_D}} + B_- e^{-\frac{x}{L_D}}, & for \quad x < 0 \end{cases}.$$

The coefficients will be determined by boundary conditions as follows.

• Radiation condition means that perturbation energy should propagate away from the source region (*x=0* here), or the response has to be finite at infinity,

$\eta_\infty < \infty \quad at \quad x \to \pm\infty$. This requires $A_+ = B_- = 0$. The solution is therefore

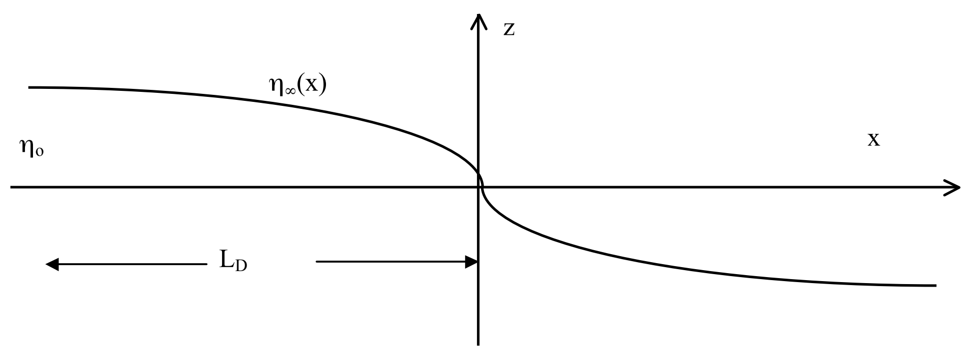
$$\eta_\infty = \begin{cases} -\eta_0 + B_+ e^{-\frac{x}{L_D}}, & for \quad x > 0 \\ -\eta_0 + A_- e^{\frac{x}{L_D}}, & for \quad x < 0 \end{cases}$$

• The continuity condition requires the continuity of η and v across $x = 0$ (therefore η_x and v_x are finite).

Thus, we have the final solution as

$$\eta_\infty = \begin{cases} \eta_0 (1 - e^{\frac{x}{L_D}}), & x < 0 \\ -\eta_0 (1 - e^{-\frac{x}{L_D}}), & x > 0 \end{cases}$$

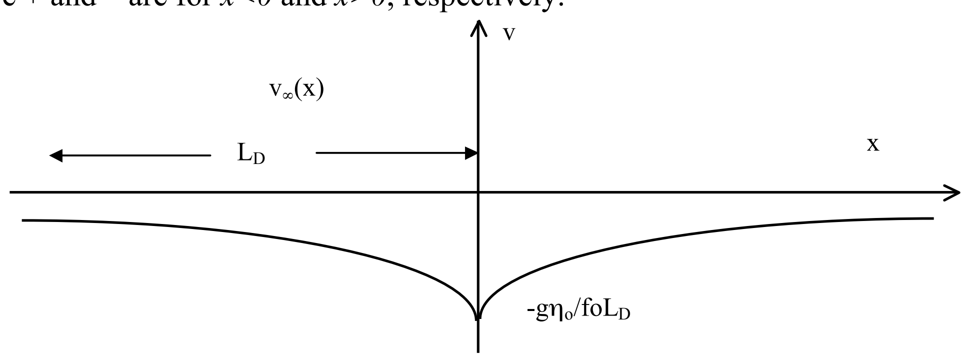
Final solution after adjustment



The velocity field is in geostrophic balance as:

$$v_\infty = \frac{g}{f}\partial_x h_\infty = -\frac{g\eta_0}{fL_D} e^{\pm\frac{x}{L_D}},$$

where + and – are for $x<0$ and $x>0$, respectively.



Several points are noteworthy here.

- Deformation radius L_D: we have seen in sec1.5 that large scale motions with $L > L_D$ is affected by rotation. Here, we further see that L_D also determines the influence distance of an ageostrophic anomaly.

• Adjustment time: From $(\partial_{tt} + f^2)\eta - c_0^2 \partial_{xx}\eta = 0$, we see that $\partial_{tt} << f^2$ gives the final steady state. Thus, when $t > \frac{1}{f} \approx$ 1 day, the adjustment is completed and the final state is in geostrophic balance, independent of the spatial scale. Therefore, independent of spatial scales, geostrophic adjustment is very fast (within about a day). This is the fundamental reason why the observed flow are always in geostrophic balance. Simply put it, any imbalance from geostrophy will be adjusted quickly to a new balance within about a day or so.

• Energetics: Since $\partial_x \eta_\infty \neq 0$, APE exists in the final state ! (different from the non-rotating case). But, not all the lost APE are converted to KE. Indeed, the initial energy is:

$$\begin{cases} APE_i = \frac{1}{2} g \int_{-L}^{L} \eta_i^2 dx = \frac{1}{2} g \int_{-L}^{L} \eta_0^2 dx = g\eta_0^2 L \\ KE_i = 0 \end{cases}$$

In the final state:

$$APE_\infty = \frac{1}{2} g \int_{-L}^{L} \eta_\infty^2 dx = \frac{1}{2} g \eta_0^2 [\int_{-L}^{0} (1 - e^{\frac{x}{L_D}})^2 dx + \int_{0}^{L} (1 - e^{-\frac{x}{L_D}})^2 dx$$

$$= g\eta_0^2 \int_0^L (1 - e^{-\frac{x}{L_D}})^2 dx$$

$$= g\eta_0^2 \left\{ L - 2L_D(1 - e^{-\frac{L}{L_D}}) + \frac{L_D}{2}(1 - e^{-\frac{2L}{L_D}}) \right\}$$

Let $L \to \infty$ (or $L >> L_D$), we have

$$APE_\infty = g\eta_0^2 (L - \frac{3}{2} L_D) > 0$$

Thus the change of APE is:

$$\Delta(APE) = APE_\infty - APE_i = \frac{3}{2} g \eta_0^2 L_D, \qquad (L \to \infty)$$

Total loss of APE is finite even through the initial APE is infinite. In addition, the final state also has KE.

$$KE_\infty = \frac{H}{2} \int_{-\infty}^{+\infty} V^2 dx = H \int_0^{+\infty} V^2 dx = H(\frac{g\eta_0}{fL_D})^2 \int_0^{+\infty} e^{-\frac{2x}{L_D}} dx$$

$$= g\eta_0^2 \frac{L_D}{2} \int_0^{+\infty} e^{-\frac{2x}{L_D}} d(\frac{2x}{L_D}) = \frac{g\eta_0^2}{2} L_D = \frac{1}{3} \Delta(APE)$$

Thus, only 1/3 of the lost APE is converted to KE. Where does the rest of APE go? They are radiated away to $x \to \pm\infty$ (or dissipated elsewhere in a finite domain) by transient inertial-gravity waves. This loss of initial energy by wave radiation is similar to the *f=0* case that has a finite initial disturbance (case 2).

• Transients: The transients are the I-G waves, which have the dispersion relation:

$$\omega^2 = f^2 + c^2K^2$$

The group velocity is:

$$2\omega\frac{\partial\omega}{\partial k} = 2c^2k$$

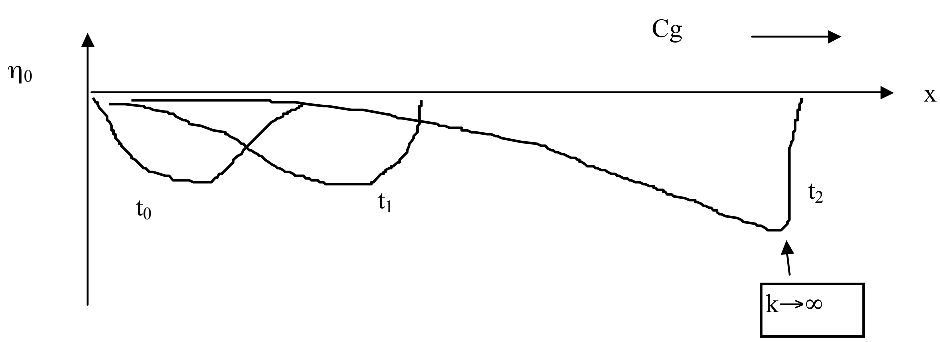
$$c_g = \frac{\partial\omega}{\partial k} = \frac{c^2k}{\omega} = \frac{\pm c^2k}{\sqrt{f^2 + c^2K^2}}$$

We see that

i) I-G waves radiate in (all the) both directions !

ii) $|c_g| \xrightarrow{\min} 0$ for $k \to 0$; $|c_g| \xrightarrow{\max} c$ for $k \to \infty$

Thus, shortwaves disperse fast, long wave slow.



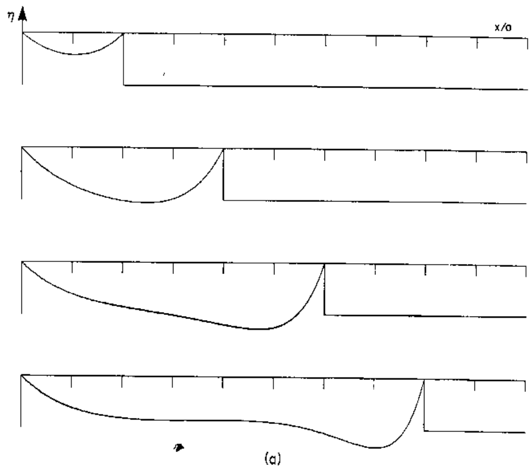


Fig. 7.3. Transient profiles for (a) η, (b) u, and (c) v for adjustment under gravity of a fluid with an initial infinitesimal discontinuity in level of $2\eta_0$ at $x = 0$. The solution is shown in the region $x > 0$, where the surface was initially depressed, at time intervals of $2f^{-1}$, where f is twice the rate of rotation of the system about a vertical axis. The marks on the x axis are at intervals of a Rossby radius, i.e., $(gH)^{1/2}/f$, where g is the acceleration due to gravity and H is the depth of fluid. The solutions retain their initial values until the arrival of a wave front that travels out from the position of the initial discontinuity at speed $(gH)^{1/2}$. When the front arrives, the surface elevation rises by η_0 and the u component of velocity rises by $(g/H)^{1/2}\eta_0$ just as in the nonrotating case depicted in Fig. 5.9a. This is because the first waves to arrive are the very short waves, which are unaffected by rotation. Behind the front, however, is a "wake" of waves produced by dispersion, which in the case of u, have the slope given by the Bessel function (7.3.14). This is the point impulse solution to the Klein–Gordon equation. The "width" of the front narrows in inverse proportion with time. Well behind the front, the solution adjusts to the geostrophic equilibrium solution depicted in Fig. 7.1.

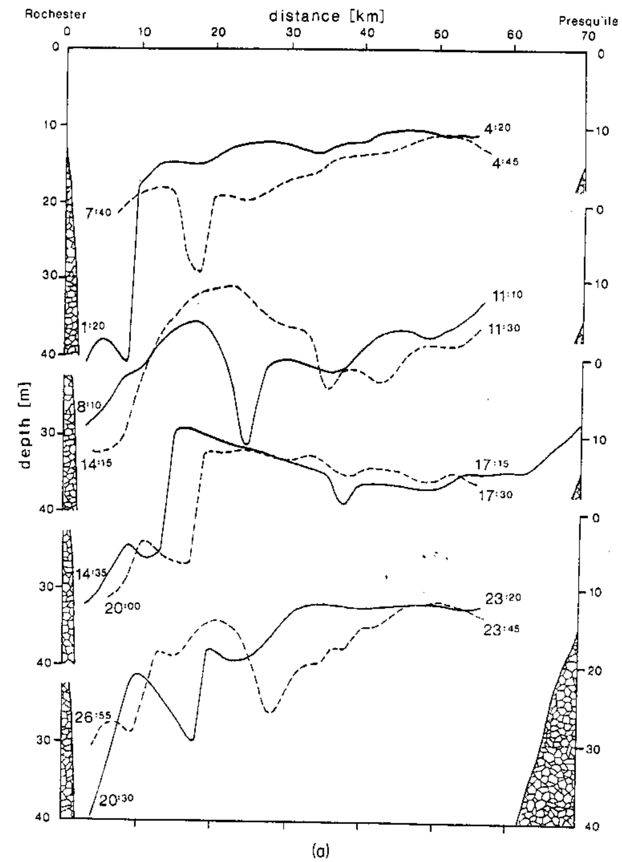


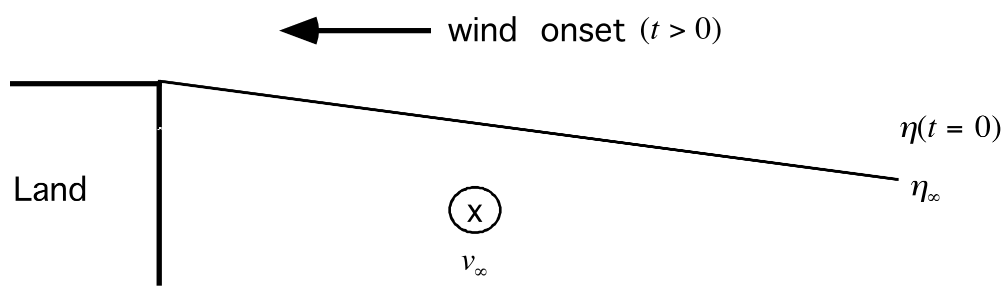
Fig. 7.5. (a) An (internal) Poincaré wave front observed in Lake Ontario following a storm on 9 August 1972. Lines show the thermocline depth as measured by the 10° isotherm. Times of the beginning and end of each transect are shown. The first transect shows the large downwelling produced by the passage of the storm, and subsequent sections show the geostrophic adjustment process involving radiation of Poincaré waves. (b) Results of a (nonlinear two-layer) model simulation of this event by Simons (1978). The diagrams are from Simons (1978, 1980) and may be compared with the solution shown in Fig. 7.3 for a very simple initial condition.

In summary, rotation produces the following effects:

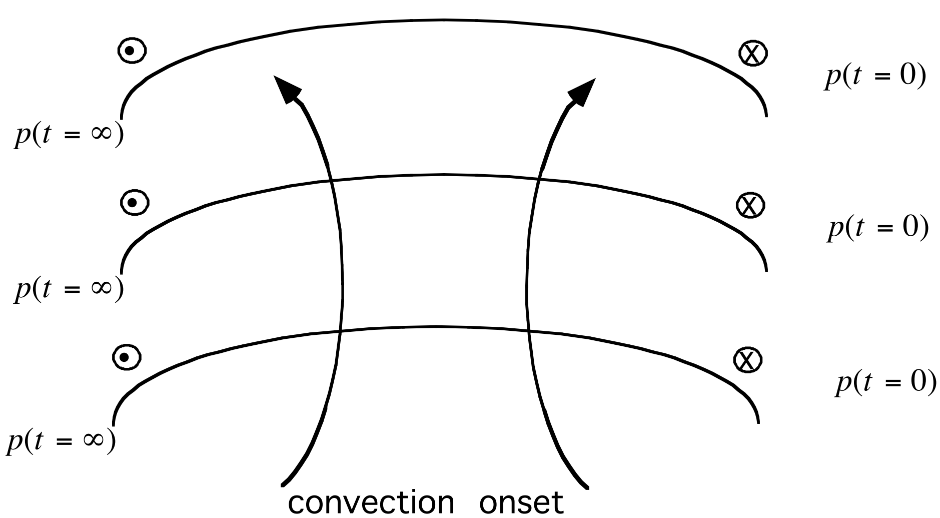
- releases less APE (hold it to build geostrophic balance)
- final state has (geostrophic) motion
- fast adjustment ($t \le 1$day)
- spatial scale about L_D (can be very small for the ocean)
- final state depends on initial condition (usually varies for different I.C.)

4. Applications

- Coastal jet: Initial wind piles up water against the coast with downstream surface currents. Later (after a day or so), the geostrophic adjustment leaves an along shelf geostrophic current.



- Atmospheric convection will eventually (after $t \ge \frac{1}{f}$) produces a cyclonic circulation.



Questions for Chapter 1

Q1.1*:* If we only retain the curvature term in the u-equation of the shallow water system (1.1.15), is the total energy still conserved as shown in (1.2.4)? Suppose this term is small compared with other terms, should we still keep this term? Why?

Q1.2**:** A more general way of solving the eigenvalue (dispersion relationship) is to solve the free modes directly from the linear shallow water equations. Plug *(u',v',_')=(u0,v0,_0)e^{i(kx+ly-_t)}* directly into the f-plane shallow water equations

$$\begin{cases} \partial_t u' - f_0 v' = -g\partial_x \eta' \\ \partial_t v' + f_0 u' = -g\partial_y \eta' \\ \partial_t \eta' + H(\partial_x u' + \partial_y v') = 0 \end{cases}$$

Show that the eigenvalues are the same as in (1.5.1).

Q1.3. In a homogeneous fluid,

a) For any two vectors **A** and **B**, prove the identity:

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\nabla\bullet\mathbf{B} + (\mathbf{B}\bullet\nabla)\mathbf{A} - \mathbf{B}\nabla\bullet\mathbf{A} - (\mathbf{A}\bullet\nabla)\mathbf{B} \,.$$

b) In a rotating frame, the momentum equation is:

$$d_t\,\mathbf{U} = -2\boldsymbol{\Omega}\times\mathbf{U} - \nabla p/\rho + \mathbf{F} \,.$$

Using the identity derived from a) and the Stokes' theorem to prove directly the Kelvin's theorem: $d_t\,(\Gamma+2\Omega A_n) = 0$. (All vectors are three-dimensional vectors).

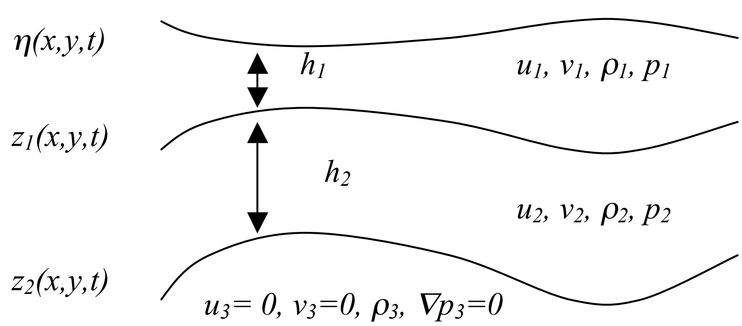
Exercises for Chapter 1

E1.1. (Hydrostatic approximation in a rotating fluid) On a f-plane (in eqn. (1.1.13)), consider a strong rotation fluid in which the pressure gradient force is balanced mainly by the Coriolis force, $fv \sim -\partial_x p/\rho$ (as opposed to by the inertial acceleration $u\partial_x u \sim -\partial_x p/\rho$ as in the handout),
a). Using scaling analysis, find the condition under which the hydrostatic approximation is valid in the primitive equation.
b). Compare with the weak rotation system that was discussed in Sec. 1.1 (the equation after (1.5)), which system is easier to reach hydrostatic balance?

E1.2. (Local plane equation) (a) What are the major approximations under which the shallow water equations on a sphere (1.1.11) (the Laplacian tidal equation) can be reduced to the Cartesian coordinate equations (1.1.13)? (b) What is the latitude region where you expect the Cartesian coordinate equation (1.1.13) to perform the poorest? (c) For a typical wind speed of 10 m/s and an ocean current of 1 cm/s, use scaling analysis to estimate the latitude region where the Cartesian coordinate equation may have serious problem.

E1.3. (Surface pressure effect) In the presence of an atmospheric sea level pressure gradient, derive the oceanic equations as in (a) a shallow water model, (b) a 1.5-layer model.

E1.4. (2.5-layer model) A 2.5-layer fluid is a special 3-layer fluid in which there is no motion (no pressure gradient) in layer 3 (see the figure). In a shallow water system where the hydrostatic approximation is valid, show that



a) the pressure in each layer can be represented as

$p_1(x,y,z,t) = p_a+g\rho_1(\eta-z)$, $p_2(x,y,z,t) = p_1(z_1)+g\rho_2(z_1-z)$, $p_3(x,y,z,t) = p_2(z_2)+g\rho_2(z_2-z)$

b) the condition of no-motion in layer 3 leads to the pressure gradients in layer 1 and 2 as

$$\nabla p_1= -g(\rho_2-\rho_1)\ \nabla z_1 -g(\rho_3-\rho_2)\ \nabla z_2$$

$$\nabla p_2= -g(\rho_3-\rho_2)\ \nabla z_2$$

c) the continuity equations for layer 1 and 2 can be derived from the incompressible equations $\partial_x u_1+\partial_y v_1 +\partial_z w_1 = 0$ and $\partial_x u_2+\partial_y v_2 +\partial_z w_2 = 0$ as $\partial_t h_1+\partial_x (u_1 h_1)+\partial_y (v_1 h_1) = 0$ and $\partial_t h_2+\partial_x (u_2 h_2)+\partial_y (v_2 h_2) = 0$, respectively.

d) finally, the 2.5-layer system is governed by

$$\begin{cases} D_t u_1 - fv_1 = -g_1'\partial_x h_1 - g_2'\partial_x h \\ D_t v_1 + fu_1 = -g_1'\partial_y h_1 - g_2'\partial_y h \\ D_t h_1 + h_1(\partial_x u_1 + \partial_y v_1) = 0 \end{cases} \quad \text{and} \quad \begin{cases} D_t u_2 - fv_2 = -g_2'\partial_x h \\ D_t v_2 + fu_2 = -g_2'\partial_y h \\ D_t h_2 + h_2(\partial_x u_2 + \partial_y v_2) = 0 \end{cases}$$

where $h=h_1+h_2$, $h_1=\eta-z_1\approx -z_1$ and $h_2= z_1-z_2$ are layer thickness, and $g'_1=(\rho_2-\rho_1)/\rho_0\approx (\rho_2-\rho_1)/\rho_1$, $g'_2=(\rho_3-\rho_2)/\rho_0\approx (\rho_3-\rho_2)/\rho_1\approx (\rho_3-\rho_2)/\rho_2$ are interface reduced gravities.

E1.5. (Vorticity of a solid body) For a solid rotating body, we have $\mathbf{U} = \mathbf{\Omega}\times\mathbf{r}$, where $\mathbf{\Omega}=(\Omega_x, \Omega_y, \Omega_z)$ is the angular velocity and $\mathbf{r}=(x, y, z)$ is the position vector. Prove that the vorticty is twice its angular veolocity, i.e. $\boldsymbol{\omega} = \nabla\times\boldsymbol{U} = 2\boldsymbol{\Omega}$.

E1.6. (Divergence equation):

a) Derive the divergence equation from the shallow water system (1.1.13a,b) as

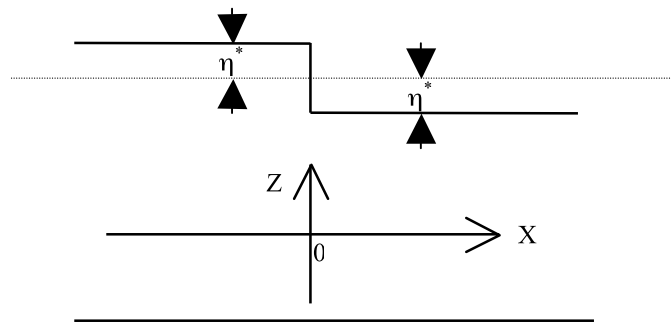
$$\partial_t (\partial_x u+\partial_y v) = (f+\zeta)\zeta - [\boldsymbol{u}\times\nabla(f+\zeta)]\bullet\boldsymbol{k} -\nabla^2[g\eta+(u^2+v^2)/2]+ \nabla\bullet\boldsymbol{F}$$

b) Discuss the differences between the divergence equation and the vorticity equation (1.4.1).

E1.7. (Adjustment process of non-rotating fluid) A linear non-rotating fluid satisfies the equations: $\partial_t u = -g\partial_x\eta$, $\partial_t\eta + H\,\partial_x u = 0$. For an initial disturbance of the form u = 0, $\eta=\eta_o(x)$, study the solutions for 2 initial conditions below.

Case 1: The initial condition is

$$\eta_o(x) = \begin{cases} -\eta^* & x > 0 \\ \eta^* & x < 0 \end{cases}$$



a) Prove that the evolution solution is $\eta(x,t) = \begin{cases} -\eta^* & Ct < x \\ 0 & -Ct < x < Ct. \\ \eta^* & x < -Ct \end{cases}$

and draw the schematic figures of the evolution at different stages.

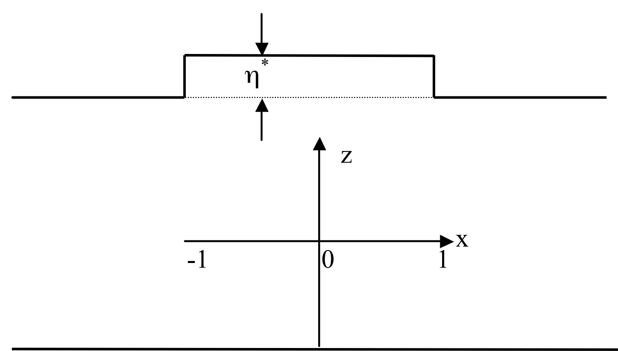
b) What is the final equilibrium state?

c) Discuss the physics.

d) What is the ratio of the kinetic energy to available potential energy at each location at different time? (optional)

Case 2: The initial condition is

$$\eta_o(x) = \begin{cases} 0 & |x| > 1 \\ \eta^* & |x| \le 1 \end{cases}$$



a) Prove the solution is

(i) for *0 < Ct < 1*

$$\eta(x,t) = \begin{cases} 0 & 1 + Ct < x \\ -\eta^*/2 & 1 - Ct < x < 1 + Ct \\ \eta^* & -1 + Ct < x < 1 - Ct \\ \eta^*/2 & -1 - Ct < x < -1 + Ct \\ 0 & x < -1 - Ct \end{cases}$$

ii) for $Ct >= 1$

$$\eta(x,t) = \begin{cases} 0 & 1 + Ct < x \\ -\eta^* / 2 & -1 + Ct < x < 1 + Ct \\ 0 & 1 - Ct < x < -1 + Ct \\ \eta^* / 2 & -1 - Ct < x < 1 - Ct \\ 0 & x < -1 - Ct \end{cases}$$

Repeat b) c) d) the same as in case 1

{Hint: for a general initial condition $\eta(x, t=0)= \eta_0(x)$, and $\partial_t \eta (x, t=0)= \eta_1(x)$, the general solution of a wave equation $\partial_{tt}\eta - c^2 \partial_{xx}\eta = 0$ is:

$$\eta(x,t) = \left\{ [\eta_0(x-ct) + \eta_0(x+ct)] + \int_{x-ct}^{x+ct} \eta_1(s)ds \right\} / 2 \qquad \}.$$

***E1.8*:** (Energy partitioning of waves). The linear perturbation equation on a f-plane is:

$$\partial_t u - fv = -g\partial_x \eta, \ \partial_t v + fu = -g\partial_y \eta, \ \partial_t \eta + H(\partial_x u + \partial_y v) = 0.$$

This set of equations contain two sets of modes: the inertial-gravity wave and the geostrophic mode (see Sec.1.5).

a) Find the ratio of the kinetic energy and available potential energy (averaged over one wave length) for an inertial-gravity wave. What are the energy ratios at the long and short wave limits? What happens for non-rotating fluid?

b) Find the energy ratio for the geostrophic mode. What are the energy ratios at the long and short wave limits?

c) Compare and discuss the energy ratios of the two modes.