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STUDY S-334

GENERATION AND AIRBORNE DETECTION
OF INTERNAL WAVES FROM
AN OBJECT MOVING THROUGH
A STRATIFIED OCEAN

JASON 1968 Summer Study in Two Volumes
VOLUME II: SUPPORTING ANALYSES

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April 1969

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13. ABSTRACT <p>This study deals with fundamentals in the performance of airborne sensors for detecting the wake from the passage of a submarine through stratified water. No attempt has been made to inter the theoretical results in an operational setting.</p> <p>(1) Internal waves are generated by fluid displacement (dipole source) and wake collapse (quadrupole source). The wake source predominates for stern free shallow moving bodies, and vice versa. The associated surface strains lead to a characteristic pattern of enhancement of ambient wind waves, and this in turn effects the radiometric temperature and scattering cross sections, as measured with passive and active devices, respectively.</p> <p>(2) Under very favorable circumstances (a shallow source in a sharp thermocline) the resulting surface strain, ϵ, may exceed 10^{-3}; typically $\epsilon = 0(10^{-3})$ to $0(10^{-2})$, and diminishes rapidly with source depth. Enhancement of slope is $0(e)$, of scattering cross section is $0(10e)$, and of radiance $0(0.1e)$. Given the natural background and instrument noise levels, the detection of an anomalous pattern covering $12m \times 100m$, say, requires strains of $0(10^{-2})$ for some passive (IR and microwave) and active (radar) devices (but the limitations for the three methods differ distinctly).</p> <p>(3) There is some indication that the detectable strain might be reduced to $0(10^{-2})$ for an active device of high angular resolution and frequency (Laser) viewing the sea surface from slopes from the vertical. Active devices appear attractive also for the reason that they eliminate the last step in an already well involved series of coupled phenomena: internal wave generation - surface straining - differential roughing (reflecting scattering) - radiometric temperature (IR).</p>			

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SUPPORTING ANALYSIS A

INTERNAL WAVE WAKES OF A BODY MOVING IN A STRATIFIED FLUID

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I. INTRODUCTION

When a ship travels with constant velocity along the surface of a liquid, it creates behind it a wake which is called a "ship wave" pattern. A similar pattern is produced by a submerged object moving parallel to the surface. The usual analyses of such patterns apply to liquids of uniform density in which only one type of propagating wave, called a surface wave, is possible. We shall consider ship wave patterns in horizontally stratified liquids in which one or more propagating internal waves exist in addition to the surface wave. Keller and Levy (Ref. 1) have shown that in any such liquid the ship wave pattern is a superposition of separate patterns, one for each propagating internal or surface wave. They have also obtained formulas for the wave height and particle velocity as functions of position throughout the pattern. From these formulas one can see that for a submerged object the patterns corresponding to some of the internal waves can have larger amplitudes than that corresponding to the surface wave. Therefore we shall examine the internal wave patterns in detail for a simplified, but realistic density profile in which infinitely many propagating internal waves occur. Previously Hudimac (Ref. 2) studied the special case of a two-layer fluid in which just one propagating internal wave exists.

II. WAKE GEOMETRY

To describe the wake of a horizontally moving object, we replace the object by a point which we call the source. We introduce cartesian coordinates in the horizontal x, z plane containing the source, with the x -axis along the path of the source and the origin at the position of the source at time $t = 0$. If the speed of the source is $-v$ then the coordinates $x_0(t')$, $z_0(t')$ of the source at time t' are

$$x_0(t') = -vt', \quad z_0(t') = 0 \quad (2.1)$$

We wish to determine the wake corresponding to waves of a particular type emitted by the source, i.e., to the surface wave or to the n -th internal wave. We suppose that the source emits waves of this type with all frequencies ω and that the wave has a definite propagation constant or wave number k . It is convenient to express ω as a function of k ,

$$\omega = \omega(k) \quad (2.2)$$

The functional relation (2.2) is determined by the density profile, and will be considered later.

Let us consider the phase $-\phi(x, z, k, \tau)$ at the point x, z at time $t = 0$ of the wave of wave number k emitted by the source at time $-\tau$, $\tau \geq 0$. If the wave is emitted at phase zero then

$$-\phi(x, z, k, \tau) = kr - \omega\tau \quad (2.3)$$

Here r is defined by

$$r = \{[x-x_0(-\tau)]^2 + z^2\}^{1/2} \quad (2.4)$$

We seek those values of k and τ for which ϕ is stationary. This requirement yields the two conditions*

$$0 = -\phi_k = r - \omega_k \tau \quad (2.5)$$

$$0 = -\phi_\tau = kr_\tau - \omega \quad (2.6)$$

From these equations we find that

$$r/\tau = c_g \equiv \omega_k \quad (2.7)$$

$$v(x_0 - x)/r = c \equiv \omega/k. \quad (2.8)$$

Here we have introduced the group velocity c_g and the phase velocity c defined by the last equalities in (2.7) and (2.8). Equation (2.7) shows that the wave from the source $x_0(-\tau), 0$ travels to x, z at the group velocity c_g . Equation (2.8) shows that the trace on the x -axis, of the straight line perpendicular to the ray from $x_0(-\tau), 0$ to x, z , travels with the source velocity $-v$.

The two equations (2.7), (2.8) determine the values of k and τ which make ϕ stationary. When these values are used in (2.3), (2.3) will yield the stationary value of the phase at each point x, z . These results are just Equations (11.5) and (11.6) of Ref. 1, which we have rederived in a simpler way. We now use (2.7) to write $\tau = r/c_g$ and (2.8) to write $\omega = kc$. Then we can rewrite (2.3) as

$$-\phi = kr(1 - c/c_g) \quad (2.9)$$

*Letter subscripts denote partial differentiation.

Next (2.8) and (2.4) yield, if $c \leq v$,

$$\therefore x = rc/v, z = r(1 - c^2/v^2)^{1/2} \quad (2.10)$$

From (2.1) and (2.7),

$$x_0 = vr/c_g \quad (2.11)$$

Let us now eliminate x_0 from (2.10) by means of (2.11) and then eliminate r by means of (2.9). Thus we obtain

$$x = (\phi v/k) \left[\frac{1 - (c c_g/v^2)}{c - c_g} \right], z = (\phi/k) \left[\frac{c_g \sqrt{1 - (c/v)^2}}{c - c_g} \right] \quad (2.12)$$

Thus if $c \leq v$, (2.12) is the parametric equation of the wavefront $\phi = \text{constant}$, where ϕ is the stationary value of the phase and k is the parameter.

III. LONG WAVES AND THE FAR WAKE

Suppose that for k small, the function $\omega(k)$ in (2.2) has the power series expansion

$$\omega = \omega_1 k - \omega_2 k^2 - \omega_3 k^3 + \dots \quad (3.1)$$

Then

$$c = \omega_1 - \omega_2 k + \dots \quad (3.2)$$

$$c_g = \omega_1 - 2\omega_2 k + \dots \quad (3.3)$$

Now (2.12) becomes

$$\begin{aligned} x &= \frac{\phi v}{k} \left[\frac{1 - v^{-2} (\omega_1 - \omega_2 k + \dots)(\omega_1 - 2\omega_2 k + \dots)}{\omega_2 k + 2\omega_3 k^2 - \dots} \right] \\ &= (\phi v / k^2 \omega_2) \left[\left(1 - \frac{\omega_1^2}{v^2} \right) + \left(\frac{3\omega_1 \omega_2}{v^2} - \frac{2\omega_3}{\omega_2} + \frac{2\omega_1^2 \omega_3}{v^2 \omega_2} \right) k + \dots \right] \quad (3.4) \end{aligned}$$

$$\begin{aligned} z &= (\phi v / k) \left[\frac{(\omega_1 - 2\omega_2 k + \dots)(1 - v^{-2} \omega_1^2 + 2v^{-2} \omega_1 \omega_2 k + \dots)^{1/2}}{\omega_2 k + 2\omega_3 k^2 - \dots} \right] \\ &= (\phi \omega_1 / k^2 \omega_2) \left(1 - \frac{\omega_1^2}{v^2} \right)^{1/2} \left[1 - \left(\frac{2\omega_2}{\omega_1} - \frac{\omega_1 \omega_2}{v^2 - \omega_1^2} + \frac{2\omega_3}{\omega_2} \right) k + \dots \right] \quad (3.5) \end{aligned}$$

It is clear from (3.4) and (3.5) that for long waves, for which k is small, both x and z are large.

To eliminate k we solve (3.4) for k and substitute into (3.5), obtaining

$$z = \frac{\omega_1 x}{(v^2 - \omega_1^2)^{1/2}} - \frac{2(\phi \omega_2 v^3 x)^{1/2}}{v^2 - \omega_1^2} + \dots \quad (3.6)$$

If $\omega(k)$ is not analytic around $k = 0$, (3.1) is not valid and therefore (3.6) does not apply. This is the case for ordinary surface waves in water of constant density and infinite depth, since for them $\omega(k) = (gk)^{1/2}$. Then (2.12) becomes

$$x = \frac{2\phi v}{(gk)^{1/2}} \left(1 - \frac{g}{2v^2 k}\right), \quad z = \frac{\phi}{k} \left(1 - \frac{g}{v^2 k}\right)^{1/2} \quad (3.7)$$

From (3.7) we see that k must be restricted to the range $k > g/v^2$ in order that z be real, so a small- k expansion is not applicable in this case.

IV. SHORT WAVES AND THE NEAR WAKE

For short waves or large k , we assume that $\omega(k)$ has the asymptotic expansion

$$\omega(k) = N - N_2 k^{-2} - N_4 k^{-4} - \dots \quad (4.1)$$

Then
$$c = Nk^{-1} - N_2 k^{-3} - N_4 k^{-5} + \dots \quad (4.2)$$

$$c_g = 2N_2 k^{-3} + 4N_4 k^{-5} + \dots \quad (4.3)$$

Upon using (4.2) and (4.3) in (2.12) and (2.13) we obtain

$$\begin{aligned} x &= (\sigma v/k) \left[\frac{1 - v^{-2}(Nk^{-1} - N_2 k^{-3} - \dots)(2N_2 k^{-3} + \dots)}{Nk^{-1} - 3N_2 k^{-3} - \dots} \right] \\ &= (\sigma v/N) \left[1 + \frac{3N_2}{N} k^{-2} + \dots \right] \end{aligned} \quad (4.4)$$

$$\begin{aligned} z &= (\sigma/k) \left[\frac{(2N_2 k^{-3} + 4N_4 k^{-5} + \dots) (1 - v^{-2}N^2 k^{-2} + 2v^{-2}NN_2 k^{-4} + \dots)^{1/2}}{Nk^{-1} - 3N_2 k^{-3} - \dots} \right] \\ &= (2\sigma N_2/Nk^3) \left[1 + \left(\frac{2N_4}{N_2} - \frac{N^2}{2v^2} + \frac{3N_2}{N} \right) k^{-2} + \dots \right] \end{aligned} \quad (4.5)$$

Solving (4.4) for k and substituting the result into (4.5) yields

$$z = \frac{2N^2}{(27v^3 N_2 \sigma)^{1/2}} \left(x - \frac{\sigma v}{N} \right)^{3/2} + \dots \quad (4.6)$$

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$$\begin{aligned} z &= (\sigma/k) \left[\frac{(2N_2 k^{-3} + 4N_4 k^{-5} + \dots) (1 - v^{-2}N^2 k^{-2} + 2v^{-2}NN_2 k^{-4} + \dots)^{1/2}}{Nk^{-1} - 3N_2 k^{-3} - \dots} \right] \\ &= (2\sigma N_2/Nk^3) \left[1 + \left(\frac{2N_4}{N_2} - \frac{N^2}{2v^2} + \frac{3N_2}{N} \right) k^{-2} + \dots \right] \end{aligned} \quad (4.5)$$

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$$z = \frac{2N^2}{(27v^3 N_2 \sigma)^{1/2}} \left(x - \frac{\sigma v}{N} \right)^{3/2} + \dots \quad (4.6)$$

From (4.6) we see that each wavefront $\phi = \text{constant}$ has a cusp at $x = \phi v/N$ on the path $z = 0$.

In the case of ordinary surface waves in water of constant density and finite or infinite depth, (4.1) does not hold so neither does (4.6). For infinite depth (3.7) yields for k large,

$$z = \pm gx^2/4v^2\phi + \dots \quad (4.7)$$

Then all wavefronts enter the origin $x = 0$ on the path $z = 0$. The result (4.7) also holds for the finite depth case when the density is constant.

V. EXAMPLE

A. THE DISPERSION EQUATION

Let $v(y)$ be the y dependent factor of the vertical component of fluid velocity in a time harmonic wave of angular frequency ω and wavenumber k in a fluid of density $\rho_0(y)$ and depth h . Then $v(y)$ satisfies the equations

$$v_{yy} - g^{-1}N^2 v_y + k^2(\omega^{-2}N^2 - 1)v = 0, \quad 0 \geq y \geq -h \quad (5.1)$$

$$v_y(0) = k^2 \omega^{-2} g v(0) \quad (5.2)$$

$$v(-h) = 0. \quad (5.3)$$

(ref. 1, Eqs. 5.14 - 5.18). here $N^2(y)$ is the "Brunt-Väisälä" frequency defined by

$$N^2 = (-g)(\rho_0)_y / \rho_0 \quad (5.4)$$

This problem has nontrivial solutions only if k^2 is an eigenvalue.

We shall take for $\rho_0(y)$ the following function

$$\begin{aligned} \rho_0(y) &= \rho_1, \quad 0 \geq y \geq y_1 \\ &= \rho_1 \exp[(N^2/g)(y_1 - y)], \quad y_1 \geq y \geq y_2 \\ &= \rho_2 = \rho_1 \exp[(N^2/g)(y_1 - y_2)], \quad y_2 \geq y \geq -h \end{aligned} \quad (5.5)$$

The layer between y_1 and y_2 is the thermocline, within which N^2 is constant, and outside it $N^2 = 0$. Now the coefficients in (5.1) are piece-wise constant so (5.1) can be solved explicitly. If we ignore the term $g^{-1}N^2v_y$ in (5.1), the solution is simply

$$v(y) = \sinh(ky + \alpha) \quad 0 \geq y \geq y_1 \quad (5.6)$$

$$v(y) \approx C \cos kvy + D \sin kvy \quad y_1 \geq y \geq y_2 \quad (5.7)$$

$$v(y) = B \sinh k(y + h) \quad y_2 \geq y \geq -h \quad (5.8)$$

Here v is defined by

$$v^2 = (N^2/w^2) - 1 \quad (5.9)$$

Condition (5.3) is satisfied by (5.8), while (5.2) yields

$$\tanh \alpha = w^2/kg \quad (5.10)$$

At y_1 and y_2 both v and v_y must be continuous. This requirement yields the four conditions

$$C \cos kvy_1 + D \sin kvy_1 = \sinh(ky_1 + \alpha) \quad (5.11)$$

$$C \cos kvy_2 + D \sin kvy_2 = B \sinh k(y_2 + h) \quad (5.12)$$

$$-v C \sin kvy_1 + vD \cos kvy_1 = \cosh(ky_1 + \alpha) \quad (5.13)$$

$$-v C \sin kvy_2 + vD \cos kvy_2 = B \cosh k(y_2 + h) \quad (5.14)$$

We now combine (5.11) and (5.13) to obtain

$$\begin{aligned} & [C \cos kvy_1 + D \sin kvy_1] \cosh(ky_1 + \alpha) \\ & + v [C \sin kvy_1 - D \cos kvy_1] \sinh(ky_1 + \alpha) = 0. \end{aligned} \quad (5.15)$$

Similarly we get from (5.12) and (5.14)

$$\begin{aligned}
& [C \cos kvy_2 + D \sin kvy_2] \cosh k(y_2+h) \\
& + v[C \sin kvy_2 - D \cos kvy_2] \sinh k(y_2+h) = 0 \quad (5.16)
\end{aligned}$$

In order for (5.15) and (5.16) to have nontrivial solutions for C and D, the determinant of the coefficient matrix must vanish. This yields the dispersion equation

$$\begin{aligned}
& \tan kv(y_1-y_2)[1+v^2 \tanh(ky_1+\alpha) \tanh k(y_2+h)] \\
& = v[\tanh(ky_1+\alpha) - \tanh k(y_2+h)] \quad (5.17)
\end{aligned}$$

B. FAR WAKE

Let us examine (5.17) for k small, tentatively assuming that $\omega \sim kc(0)$ as k tends to zero. Then (5.9) yields $v \sim N/kc(0)$, (5.10) yields $\alpha \sim kc^2/g$ and (5.17) becomes at $k = 0$

$$\tan [(N/c)(y_1-y_2)] = \frac{Nc^{-1}[y_1-y_2-h+(c^2/g)]}{1 + N^2c^{-2}[y_1+(c^2/g)](y_2+h)} \quad (5.18)$$

This is an equation for $c(v)$ which has infinitely many solutions which we shall call $c_n(0)$, $n = 0, 1, 2, \dots$. To describe them we write

$$c_n(0) = \frac{Ns}{n\pi + a_n}, \quad -\frac{\pi}{2} < a_n < \frac{\pi}{2} \quad (5.19)$$

Here we have introduced the thermocline thickness s defined by

$$s = y_1 - y_2 \quad (5.20)$$

Then (5.18) becomes the following transcendental equation for a_n :

$$\tan a_n = \frac{s[(s-h)(n\pi+a_n) + \frac{Ns^2}{g(n\pi+a_n)}]}{s^2 + [y_1(n\pi+a_n)^2 + N^2s^2/g](y_2+h)}, \quad -\pi/2 < a_n < \pi/2 \quad (5.21)$$

For n large (5.21) yields

$$a_n \sim -\frac{s(h-s)}{n\mu y_1(h+y_2)} \quad (5.22)$$

If $h \gg s$ and $h \gg |y_2|$, (5.22) becomes

$$a_n \sim -\frac{s}{n\mu y_1}, \quad n \gg 1 \quad (5.23)$$

For $n = 0$, (5.21) becomes

$$\tan a_0 = \frac{s[(s-h)a_0 + \frac{N^2 s^2}{g a_0}]}{s^2 + (y_1 a_0^2 + \frac{N^2 s^2}{g})(y_2+h)} \quad (5.24)$$

If $|y_2| \ll h$ and $N^2 s/g \ll 1$, we can replace $\tan a_0$ by a_0 in (5.24). The resulting biquadratic equation has as its two positive solutions

$$a_0^- \sim \frac{Ns}{(gh)^{\frac{1}{2}}} \quad (5.25)$$

$$a_0^+ \sim \left(\frac{s}{-y_1}\right)^{\frac{1}{2}} \quad (5.26)$$

Let us now use the results (5.23), (5.25), and (5.26) in (5.19) and introduce the effective gravity g' defined by

$$g' = N^2 s \approx \frac{\rho_2 - \rho_1}{\rho_1} g \quad (5.27)$$

Then (5.19) yields

$$c_0^-(0) \sim (gh)^{\frac{1}{2}} \quad (5.28)$$

$$c_0^+(0) \sim (g'|y_1|)^{\frac{1}{2}} \quad (5.29)$$

$$c_n(0) \sim \frac{(g's)^{1/2}}{n\pi} \left(1 - \frac{s}{n^2 \pi^2 |y_1|} \right), \quad n \gg 1 \quad (5.30)$$

Since $\omega \sim kc$, it follows that $c_g(0) = c(0)$ for each mode.

C. NEAR WAKE

Now we shall examine (5.17) for k large, assuming that $\omega \sim N$ as k tends to infinity. Then (5.17) becomes

$$\tan kvs \sim -2v \quad (5.31)$$

Since v is small, the solutions of (5.31) are

$$kvs \sim n\pi - 2v \quad (5.32)$$

By using the definition (5.9) of v we obtain from (5.32)

$$\omega = N \left[1 + \left(\frac{n\pi}{ks+2} \right)^2 \right]^{-1/2} \sim N \left[1 - \frac{1}{2} \left(\frac{n\pi}{ks+2} \right)^2 \right] \sim N - N_2 k^{-2} - \dots \quad (5.33)$$

Here $N_2 = (n\pi/s)^2 N$. Therefore from (5.33) we obtain

$$c = \frac{\omega}{k} \sim \frac{N}{k} - \frac{N}{2k} \left(\frac{n\pi}{ks+2} \right)^2 \quad (5.34)$$

$$c_g = \frac{d\omega}{dk} \sim \frac{Ns(n\pi)^2}{(ks+2)^3} \quad (5.35)$$

These results hold only for $n \neq 0$ as we see from (5.32).

D. SURFACE WAVES

If ω tends to infinity as k does, we must proceed differently. Then $v^2 \sim -1$ and (5.17) becomes

$$\tanh ks [1 - \tanh(ky_1 + \alpha) \tanh k(y_2 + h)] \sim \tanh(ky_1 + \alpha) - \tanh k(y_2 + h) \quad (5.36)$$

Thus

$$1 - \tanh(ky_1 + \alpha) \sim \tanh(ky_1 + \alpha) - 1 \quad (5.37)$$

It follows that α must tend to $+\infty$ as k does. Then (5.10) yields

$$\omega \sim (kg)^{\frac{1}{2}} \quad (5.38)$$

This is the result for the surface wave.

E. DEEP OCEANS

In the oceans the depth is so large that $kh \gg 1$ even for the smallest practical value of k . In this case it is possible to simplify some of the preceding results. For example, in (5.17), we can set $\tanh k(y_2+h) = 1 + \dots$. Then the solution of (5.17) for small k can be carried beyond the leading term given in (5.29) and (5.30) with the result

$$\omega = (g'|y_1|)^{\frac{1}{2}} k \left[1 - \frac{|y_1|}{2} k + \dots \right], \quad n = 0 \quad (5.39)$$

$$\omega = \frac{(g's)^{\frac{1}{2}}}{n\pi} k \left[1 - \frac{s}{n^2 n^2} k + \dots \right], \quad n \gg 1 \quad (5.40)$$

In writing these results we have assumed that $\omega^2/g|y_1| \ll 1$, since in the oceans this number is typically of the order 10^{-3} .

VI. WAVE HEIGHT DUE TO A MOVING SOURCE OR DIPOLE

The wave height $\eta_{\text{source}}(x, z, t)$ has been determined for a unit point source of fluid moving with the constant velocity $-v$ at the depth y_0 . The asymptotic form of η_{source} far from the source is

$$\eta_{\text{source}}(x+vt, z) \sim \frac{1}{r^{\frac{1}{2}}} \sum \frac{B}{(v^2 - c^2)^{\frac{1}{2}} v^{\frac{1}{2}}} \cos(kr - \omega\tau + \pi/4) \quad (6.1)$$

(Ref. 1, Eq. 11.14). The sum in (1) is over the modes of wave propagation, and for each mode over the roots k and τ of (2.7) and (2.8). The functions $\omega(k)$ and $c(k) = \omega/k$ are determined for each mode as in Section 5 and B is given by Ref. 1, (10.9).

$$B = - \left| \frac{2}{\pi k_{(v)}} \right|^{\frac{1}{2}} \omega^{-1} \frac{[\omega^2 - N^2(0)] w(y_0)}{[\omega^2 - N^2(0)] w_k(0) - g w_{k,y}(0)} \quad (6.2)$$

Here $w(y) = \rho_0(y) v(y)$ where $v(y)$ is a nontrivial solution of (5.1) and (5.3).

If the source is a dipole of unit strength oriented along its direction of motion, the wave height η_{dipole} can be obtained by differentiating (6.1) with respect to $-x$. Only the phase $-\phi$ of the cosine need be differentiated and in view of (2.5) its derivative is $-kr_x + \omega\tau_x$. Alternatively we can obtain $-\phi_x$ from (2.12). In either way we obtain

$$\eta_{\text{dipole}}(x+vt, z) \sim - \sum \frac{k(c-c_g) B}{v(1-cc_g v^{-2})(v^2 - c^2)^{\frac{1}{2}}} \sin(kr - \omega\tau + \frac{\pi}{4}) \quad (6.3)$$

With the help of (2.3), (2.10) and (2.13), the equations (6.1) and (6.3) can be written in the shorter forms:

$$\eta_{\text{source}} = \sum \frac{B}{z^{\frac{1}{2}} v} \cos(\phi - \pi/4) \quad (6.4)$$

$$\eta_{\text{dipole}} = \sum \frac{B \phi r}{x z v} \sin(\phi - \pi/4) \quad (6.5)$$

In the example of Sec. V, we have $N(0) = 0$, $v_k = y \cosh(ky + \alpha)$, and $v_{ky} = \cosh(ky + \alpha) + ky \sinh(ky + \alpha)$. Then (6.2) becomes

$$B = \frac{w}{g \cosh \alpha} \left| \frac{2}{\pi k_{ww}} \right|^{\frac{1}{2}} \frac{\rho_0(y_0) v(y_0)}{\rho_0(0)} \quad (6.6)$$

VII. ILLUSTRATION

A. NORMALIZATION

Figures (1) and (2) are drawn for a density distribution consisting of a "thermocline" of thickness $s = y_1 - y_2$ within which N^2 is constant, and outside which $N^2 = 0$ (i.e., the case treated in Section V). We collect the dimensionless formulae used in the construction. All distances are normalized with respect to the thermocline thickness s :

$$X = x/s, Y = y/s, Z = z/s; K = ks \quad (7.1)$$

and all frequencies to the "Vaisälä" frequency N , so that

$$\Omega = \omega/N, C = \Omega/K = c/(Ns), V = v/(Ns) \quad (7.2)$$

$F = v/c = VK/\Omega$ is a "Froude Number," measuring source velocity relative to the speed of internal waves. We consider only $F > 1$.

The dimensionless amplitude along any line of constant phase $\phi = 0, 2\pi, \dots$ is given by

$$\frac{\eta_{\text{source}}}{s} = \frac{2^{-\frac{1}{2}}}{Ns} \frac{B}{S^{1/2} Z^{\frac{1}{2}} V} \quad (6.4')$$

$$\frac{\eta_{\text{dipole}}}{s} = - \frac{2^{-\frac{1}{2}}}{Ns^3} \frac{B\phi(1-F^{-2})^{-\frac{1}{2}}}{XV} \quad (6.5')$$

where

$$B = \left(\frac{2s}{\pi}\right)^{\frac{1}{2}} \frac{N^2 \Omega}{g \cosh \alpha} X_{\Omega}^{-\frac{1}{2}} \left\{ \frac{\rho_0(y_0) v(y_0)}{\rho_0(0)} \right\} \quad (6.6')$$

The expression $\rho_0(y_0)v(y_0)/\rho_0(0)$ in (6.6') depends on the depth of the source, $y_0 = sY_0$, in accordance with (5.5) - (5.8). The simplest case is that of the source above the thermocline, $Y_0 \geq Y_1$ in which case we find

$$\frac{\rho_0(y_0)v(y_0)}{\rho_0(0)} = \sinh(KY_0 + \alpha), \quad \tanh \alpha = \frac{N^2 s}{g} \frac{\Omega^2}{K} \quad (7.3)$$

B. NEAR WAKE

Equation (4.6) with N_2 determined by (5.33) can be written in the dimensionless form

$$z \sim \frac{\phi}{n\pi} \left| \frac{2(X - \phi V)}{3\phi V} \right|^{\frac{3}{2}} \quad (7.4)$$

Here $\phi = 2\pi, 4\pi, \dots$, and $n = 1, 2, \dots$ designate successive crests for various modes. The cases $\phi = 0$ and $n = 0$ are beyond the scope of the present approximation. The wave crest can now be constructed for any specified n and ϕ .

For given n, ϕ, X , and Z , the amplitudes can be obtained as follows. First we eliminate N_2 between (4.3) and (4.5) to obtain $z = c_g \phi / N + \dots$, with c_g determined by (5.35). This leads to the dimensionless formula

$$Z \sim \phi (n\pi)^2 (K+2)^3 \quad (7.5)$$

from which K can be calculated. Furthermore from (5.33) and from the definition of F we have

$$\Omega \sim 1 - \frac{1}{2} \left(\frac{n\pi}{K+2} \right)^2 \quad (7.6)$$

$$F = VK/\Omega \quad (7.7)$$

By differentiating (7.6) we find

$$X_{\Omega} \sim 3(n\pi)^{-1} (K+2)^5 \quad (7.8)$$

C. FAR WAKE

Equation (3.6) can now be written

$$z = (F_0^2 - 1)^{-\frac{1}{2}} [X - 2X_0^{\frac{1}{2}} X^{\frac{3}{2}} + \dots] \quad (7.9)$$

where F_0 and X_0 are given by

$$F_0 = \frac{v}{w_1} \quad (7.10)$$

$$X_0 = \frac{gw_2 v^3}{w_1^2 s (v^2 - w_1^2)} \quad (7.11)$$

The mode $n = 0$ corresponds essentially to a thermocline displacement, and the modes $n = 1, 2, \dots$ to thermocline distortions. We need to treat these cases separately.

From (3.1), (5.39) and (5.40) we find

$$w_1 = (g'|y_1|)^{\frac{1}{2}}, \quad w_2 = (g'|y_1|)^{\frac{1}{2}} |y_1|/2 \text{ for } n = 0 \quad (7.12)$$

$$w_1 = (g's)^{\frac{1}{2}}/n\pi, \quad w_2 = (g's)^{\frac{1}{2}} s (n\pi)^{-3} \text{ for } n \gg 1 \quad (7.13)$$

By using (7.12) and (7.13) in (7.10) we obtain

$$F_0 = |y_1|^{-\frac{1}{2}} v \text{ for } n = 0; \quad F_0 = n\pi v \text{ for } n \gg 1 \quad (7.14)$$

We now define $\delta(n)$ by

$$\delta(n) = 2 \text{ for } n = 0; \quad \delta(n) = 1 \text{ for } n \gg 1 \quad (7.15)$$

Then from (7.11) - (7.15) we obtain

$$X_0 = \frac{\phi V^2 F_0}{\delta(n)(F_0^2 - 1)} \quad (7.16)$$

We now use (7.12) and (7.13) in (3.5) to get

$$Z \sim \frac{\delta \phi F_0 (F_0^2 - 1)^{1/2}}{V^2 K^2} \quad (7.17)$$

From (7.17) we can find K in terms of Z and ϕ . Then from (5.39) and (5.40) we find

$$\Omega = (V/F_0)K - \delta^{-1}(V/F_0)^3 K^2 + \dots \quad (7.18)$$

Differentiation of (7.18) yields

$$K_{\Omega\Omega} \sim 2\delta^{-1}[1 - 2\delta^{-1}(V/F_0)^2 K]^{-3} \quad (7.19)$$

D. RESULTS

We have computed wakes for the following cases:

Source depth	$y_0 = 30$ m
Thermocline depth	$y_1 = 50$ m
Thermocline thickness	$s = 10$ m
"Vaisala" frequency	$N = 10^{-2}$ sec ⁻¹

Figure (1) and (2) portray the near wake for the cases $V = \sqrt{10}$ and 10, corresponding to source velocities $v = VN_s = 0.316$ m/s and 1 m/s, respectively. The X-axis extends from $X = 0$ to 200, corresponding to 2 km full scale; the horizontal Z-axis is drawn to the same scale. With increasing v , the wake field is rapidly concentrated along the source axis, particularly for large n and ϕ . We have (improperly) used the $n \gg 1$ approximation for the cases $n = 1, 2$. The case $n = 0$ is beyond the scope of the present treatment for the near wake, and

the far wake is off-scale in the example shown. The computed source functions diminish rapidly with distance from the source axis. Unlike the case of a surface (Kelvin) wake, internal sources moving at quite moderate velocities through typically stratified fluids produce internal wakes that are sharply concentrated along the source axis.

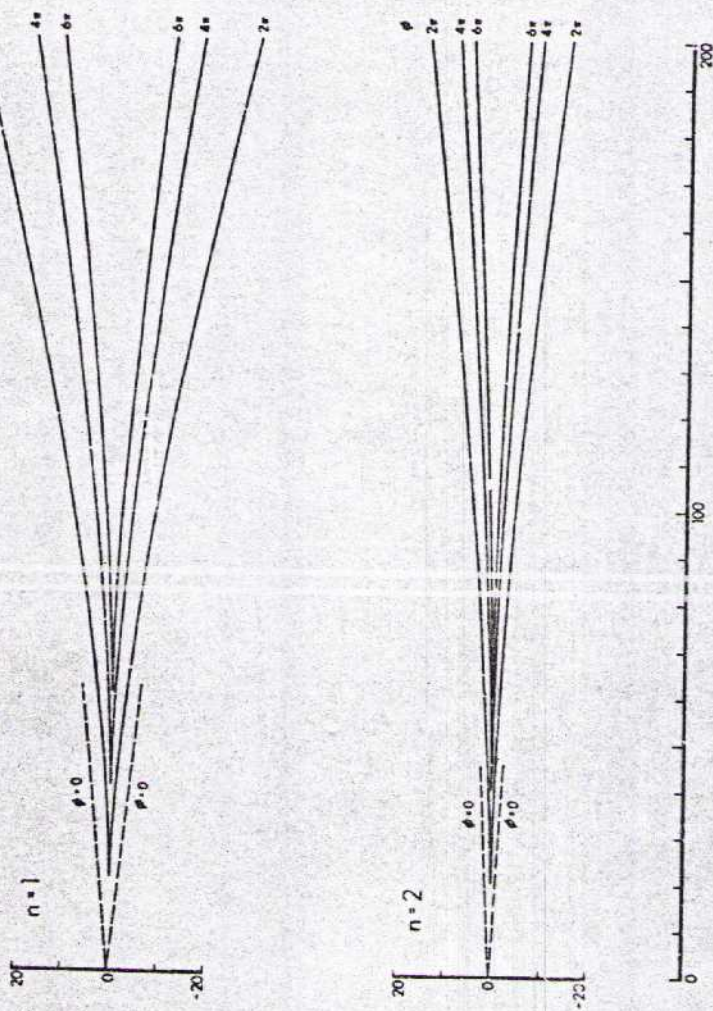


FIGURE 1. The near field for a source velocity $V = 3.16$. The dashed curves marked $\phi = 0$ correspond to the far field solution and are sketched for orientation only.

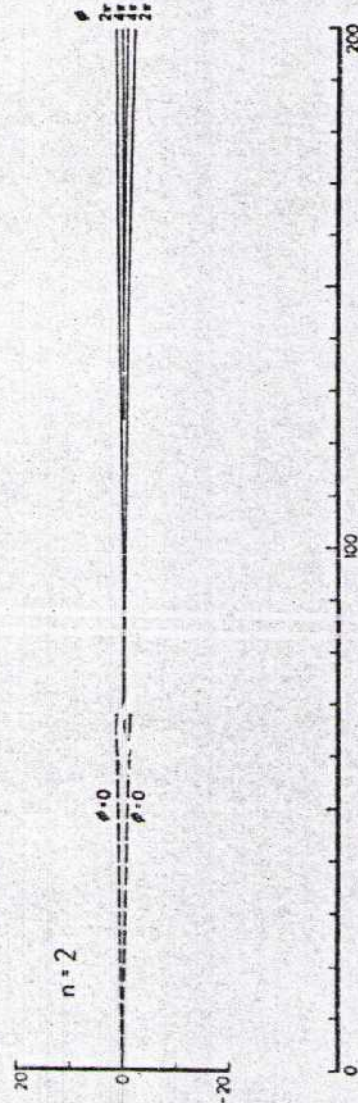
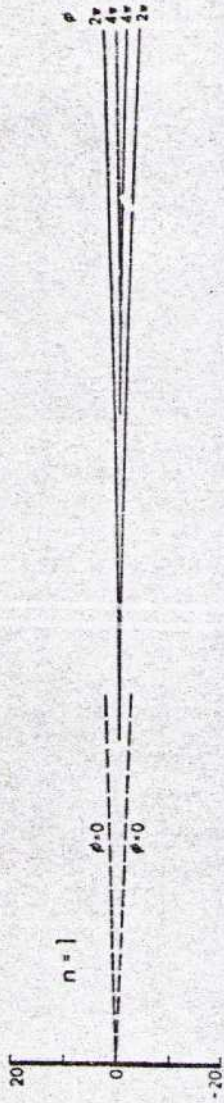


FIGURE 2. The near field for γ source velocity $V = 10$.

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SUPPORTING ANALYSIS B

INTERNAL WAVES GENERATED BY A MOVING SOURCE

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ABSTRACT

The internal waves produced by either a moving body or the collapsing wake behind a moving body in a stratified fluid are calculated asymptotically (at large distances behind the source) on the hypotheses of small disturbances, the Boussinesq approximation, and the slender-body approximation (the transverse dimensions of the body and wake are small compared with the wavelengths of the significant internal waves).

Explicit results are given for two, complementary models: (a) a constant- N model, in which the density gradient is constant and (b) a thin thermocline model, in which the density gradient peaks sharply in a thin layer and is elsewhere negligible. The internal-wave spectrum is continuous in (a) and discrete in (b); however, only the dominant mode is included in the explicit results given for (b).

A WKB solution also is given for a thermocline model. This approximation does not give an adequate representation of the dominant mode but does provide estimates of the contributions of the higher modes that are neglected in the thin-thermocline model. These contributions of the higher modes that are neglected in the thin-thermocline model. These contributions are typically negligible relative to that of the dominant mode in the neighbourhood of the maximum, free-surface disturbance.

I. INTRODUCTION

We consider the disturbance generated by a horizontally moving source in an incompressible, inviscid, vertically stratified fluid. This disturbance comprises the near field, which dies out more or less rapidly with distance from the source, and the radiated field which consists of internal gravity waves. We focus primarily on the radiated field, but emphasize that there may be situations of interest in which the amplitude of the near field is not small compared with that of the radiated field. In particular, the radiated field in a steady flow (uniform translation of the source) appears only in the lee of the source, so that the near field must be taken into account in calculating the disturbance forward of, or directly over, the source.

The appropriate similarity parameter for the generation of internal waves by a moving source is the reduced frequency (or inverse Froude number)

$$\Omega = Nl/U \quad (1.1)$$

where N is a characteristic value of the intrinsic (or "Väisälä") frequency of internal waves (see Eq. 2.9 below), l is a characteristic length of the source, and U is its speed. The frequency spectrum of the internal waves is $(0, N_{\max})$. The intensity is typically a rapidly increasing function of Ω (and, therefore, a decreasing function of U) for $\Omega < \Omega_c$, say, where Ω_c is a characteristic value of Ω , of order unity, at which nonlinear phenomena intervene. Internal-wave generation is weak for $\Omega \gg \Omega_c$, and as $\Omega \rightarrow \infty$ ($U \rightarrow 0$) the flow tends to a plug type, in which a horizontal column of fluid is pushed in front of the body.

We develop the equations of motion in Sec. II on the hypotheses of small disturbances and the Boussinesq approximation (in which only the buoyancy effects of density stratification are included, the inertial effects being neglected). We obtain formal solutions of these equations in Sec. III with the aid of integral transforms and specialize these to a moving dipole (by which a body may be approximated if $\Omega \ll 1$) in Sec. IV and to a slender, collapsing wake (a region of stirred fluid) aft of a moving body in Sec. V. We give explicit calculations of the internal-wave field for a constant-N model in Sec. VI and for a thermocline model, in which N peaks sharply in a region of limited vertical extent, in Sec. VII and Sec. VIII.

The constant-N model is characterized by a continuous spectrum (since we assume the fluid to be either infinite or semiinfinite) and may be representative for laboratory configurations, although finite-depth effects could be important in such configurations. The thermocline model is characterized by a discrete spectrum and affords a more realistic model for the ocean; we give explicit results only for the dominant mode on the hypothesis that the thickness of the thermocline is small compared with both its depth and the wavelength. We give a WKB solution for the thermocline model in Sec. IX. This solution does not give an adequate description of the dominant mode for a thin thermocline, but it does provide adequate estimates for the higher modes.

The disturbance produced by a moving body has been calculated previously by Hudimac (Ref. 1) for a two-layer model of the ocean and by Keller and Levy (Ref. 2), Lighthill (unpublished papers), and Mei (Ref. 6) for various models. There is a close analogy between two-dimensional, time-dependent disturbances and three-dimensional disturbances produced by a uniformly translating source. Keller has obtained results similar to (but more general and less explicit than) those reported here. Many reports from Hydronautics, Inc. also deal with the problem, both experimentally and theoretically. Nevertheless, it appears that some of the results given here are new. Perhaps the most interesting are the asymptotic approximations to the respective,

lateral strains produced at the free surface by the displacement (dipole) and wake (quadrupole) effects of a submarine that is small compared with the length of the internal waves, i.e., $\Omega \ll 1$. Thus we have

$$|\eta_d| \sim 0.4a^2 \ell b^k (h+|d-h|)^{-\frac{9}{4}} (N_h/U)^{\frac{1}{2}} x_d^{-\frac{1}{2}} \quad (1.2)$$

$$|\eta_q| \sim 0.8a^4 b^{-\frac{5}{4}} (h+|d-h|)^{-\frac{11}{4}} (N_d/N_h)^2 (U/N_h)^{\frac{1}{2}} x_q^{-\frac{1}{2}}, \quad (1.3)$$

where a , ℓ , d and U are the radius, length, depth and speed of the submarine, b and h are the thickness and depth of the thermocline, N_d and N_h are the intrinsic frequencies at depths d and h , and x_d and x_q are the respective distances behind the submarine and the plane in which its wake begins to collapse.

II. EQUATIONS OF MOTION

We consider small disturbances in an inviscid, incompressible, Boussinesq fluid in which the (hydrostatic) equilibrium distributions of density and pressure are $\rho_0(z)$ and $p_0(z)$ and z is measured positive upwards. Invoking the requirement that particle density be conserved and linearizing the equations of motion, we obtain

$$\rho = \rho_0(z - \psi) + \rho_0'(z)\psi \quad (2.1)$$

$$\nabla \cdot \underline{v} = m, \quad (2.2)$$

and

$$\rho \frac{d\underline{v}}{dt} = -\nabla p - g[0, 0, \rho], \quad (2.3)$$

where ρ denotes the density, ψ the vertical displacement of a particle, \underline{v} the velocity, m the source strength per unit volume, and p the pressure, each as a function of the Cartesian coordinates (x, y, z) and the time t ; letter subscripts denote partial differentiation, and the triplet $[-, -, -]$ denotes the Cartesian components of a vector. We seek a solution of (2.1)-(2.3) for a prescribed source density that is introduced at $t = 0$, an initial displacement $\psi_0(x, y, z)$, and an initial velocity $\underline{v}_0(x, y, z)$.

Let ϕ be a potential such that

$$p = p_0 - \rho_0 \phi_t \quad (2.4a)$$

and

$$\underline{v} = \underline{v}_0 + [\phi_x, \phi_y, \phi_t] \quad (2.4b)$$

Substituting (2.4b) into (2.2) and invoking the continuity equation for \underline{v}_0 ,

$$\nabla \cdot \underline{v}_0 = 0, \quad (2.5)$$

we obtain

$$\Delta \phi + \psi_{zt} = m, \quad (2.6)$$

where

$$\Delta \equiv \partial_x^2 + \partial_y^2 \quad (2.7)$$

is the two-dimensional Laplacian, and the operators ∂_x and ∂_y imply partial differentiation with respect to x and y . We have assumed that $m = 0$ at $t = 0$; if $m = m_0$ at $t = 0$, we need only replace the right-hand side of (2.5) by m_0 and m by $m - m_0$ in (2.6). Substituting (2.4a,b) into (2.1) and the z -component of (2.3) [the x - and y -components of (2.3) are satisfied identically by (2.4a,b)], eliminating ϕ through (2.6), and invoking the Boussinesq approximation (thereby neglecting ρ_0' except where it is multiplied by g), we obtain

$$\psi_{zztt} + (\partial_t^2 + N^2)\Delta\psi = m_{zt}, \quad (2.8)$$

where

$$N^2 = N^2(z) \equiv -g\rho_0'(z)/\rho_0(z) \quad (2.9)$$

is the square of the intrinsic ("Väisälä") frequency.

We seek the solution of (2.8) for the initial conditions (which follow from our definitions)

$$\psi = \psi_0, \quad \phi = \psi_t = m = 0 \quad (2.10)$$

and the boundary conditions

$$|\phi|, |\psi| < \infty \quad (|x|, |y| \rightarrow \infty) \quad (2.11a)$$

and

$$\psi = 0 \quad (z = 0, -D), \quad (2.11b)$$

corresponding to a free surface at $z = 0$ (which acts approximately as a rigid boundary for internal waves) and a rigid bottom at $z = -D$.

A convenient measure of the disturbance at the free surface is the lateral strain,

$$\eta = \int_{-\infty}^t \phi_{yy}(x, y, 0, \tau) d\tau, \quad (2.12)$$

which plays a significant role in calculating the interaction between the internal waves and pre-existing surface waves.

III. FORMAL SOLUTION

We define the transforms

$$\psi = \mathfrak{L}\mathfrak{F}_x\mathfrak{F}_y\psi, \quad M = \mathfrak{L}\mathfrak{F}_x\mathfrak{F}_y m, \quad \psi_0 = \mathfrak{F}_x\mathfrak{F}_y\psi_0, \quad (3.1a,b,c)$$

where

$$\mathfrak{L}(\) = \int_0^{\infty} e^{-\sigma t}(\) dt, \quad \mathfrak{L}^{-1}(\) = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} e^{\sigma t}(\) d\sigma \quad (R\sigma > 0), \quad (3.2a,b)$$

$$\mathfrak{F}_x(\) = \int_{-\infty}^{\infty} e^{-i\alpha x}(\) dx, \quad \mathfrak{F}_x^{-1}(\) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\alpha x}(\) d\alpha, \quad (3.3a,b)$$

and similarly for \mathfrak{F}_y , with x and α replaced by y and β , respectively. Transforming (2.8) and invoking (2.10) and (2.11a), we place the result in the form

$$(\partial_z^2 - \lambda^2)(\psi - \sigma^{-1}\psi_0) = \sigma^{-1}M_z + \kappa^2\sigma^{-3}N^2\psi_0, \quad (3.4)$$

where

$$\lambda = \kappa\sqrt{1+(N/\sigma)^2} \quad (R\lambda > 0) \quad (3.5)$$

$$\kappa = \sqrt{\alpha^2 + \beta^2} \quad (\kappa \geq 0), \quad (3.6)$$

and ∂_z implies partial differentiation with respect to z . The boundary conditions for ψ are given by (2.11b).

The Green's function for (3.4) and (2.11b) is determined by

$$(\partial_z^2 - \lambda^2)G(z, \zeta) = \delta(z - \zeta) \quad (3.7)$$

and

$$G(0, \zeta) = G(-D, \zeta) = 0 \quad (3.8)$$

and yields the formal solution (after integrating the term in M by parts):

$$\psi(z) - \sigma^{-1}\psi_0(z) = -\sigma^{-1} \int G_\zeta(z, \zeta) M(\zeta) d\zeta + \kappa^2 \sigma^{-3} \int G(z, \zeta) N^2(\zeta) \psi_0(\zeta) d\zeta \quad (3.9)$$

We have suppressed the explicit dependence of the transforms on α , β and σ ; the integrals are over the domains of M and ψ_0 , which we assume to be of finite extent, and δ is Dirac's delta function.

Transforming (2.6), we obtain

$$\psi(z) = \kappa^{-2} (\sigma \psi_z - \psi_{0z} - M), \quad (3.10)$$

which completes the reduction of the formal solution to the determination of the Green's function and the evaluation of inverse transforms.

IV. MOVING DIPOLE

We now consider the disturbance produced by the dipole

$$m = UD \partial_x \delta(x+Ut) \delta(y) \delta(z+d) \quad (t > 0), \quad (4.1)$$

which is introduced at $x = y = t = 0$ and $z = -d$ and moves along the negative- x axis with the uniform speed U . The parameter D is the dipole moment and has the dimensions of volume (see below). Transforming (4.1) in accordance with (3.1b), we obtain

$$M = UDi\alpha(\sigma - i\alpha U)^{-1} \delta(z+d) \quad (4.2)$$

Substituting (4.2) into (3.9) and assuming the fluid to be initially undisturbed ($\psi_0 = 0$), we obtain

$$\Psi = -UDi\alpha\sigma^{-1}(\sigma - i\alpha U)^{-1} G_\zeta(z, \zeta) \Big|_{\zeta = -d} \quad (4.3)$$

The asymptotic limit of ψ as $t \rightarrow \infty$ is determined by the pole of the Laplace transform at $\sigma = i\alpha U$ (corresponding to $\partial_t \sim U\partial_x$ in the equations of motion), which yields

$$\mathcal{L}^{-1}\Psi \sim DG_\perp(z) e^{i\alpha Ut} \quad (t \rightarrow \infty), \quad (4.4)$$

where

$$G_\perp(z) = -G_\zeta(z, \zeta) \Big|_{\zeta = -d, \sigma = i\alpha U} \quad (4.5)$$

We determine the behaviour of λ , qua function of α , in (4.5) from the antecedent requirement that $\Re\lambda > 0$ as σ approaches the imaginary axis from the right:

$$\lambda_1 \equiv \lambda|_{\sigma = i\alpha U} = \kappa \sqrt{1 - (k/\alpha)^2} \quad (|\alpha| > k) \quad (4.6a)$$

$$= i\kappa\alpha^{-1} \sqrt{k^2 - \alpha^2} \quad (|\alpha| < k), \quad (4.6b)$$

where

$$k = k(z) = N(z)/U \quad (4.7)$$

(Ψ also has an essential singularity at $\sigma = 0$, which makes no contribution to the wave field, and branch points associated with the branch points of λ , qua function of σ , which contribute transients that die out at least as rapidly as $1/t$.) Taking the inverse Fourier transform of (4.4), we obtain

$$\psi \sim D\mathfrak{F}_{x+Ut}^{-1} \mathfrak{F}_y^{-1} G_1(z) \equiv \psi_d(x+Ut, y, z), \quad (4.8)$$

where the subscript d implies dipole.

Substituting (4.2) and (4.3) into (3.10) and proceeding as above, we obtain the corresponding result

$$\begin{aligned} \phi &\sim UD\mathfrak{F}_{x+Ut}^{-1} \mathfrak{F}_y^{-1} \{i\alpha/\kappa^2 [G_{1z}(z) - \delta(z+d)]\} \\ &\equiv \phi_d(x+Ut, y, z) \end{aligned} \quad (4.9)$$

Substituting (4.9) into (2.12), we obtain

$$\eta \sim -D\mathfrak{F}_{x+Ut}^{-1} \mathfrak{F}_y^{-1} \{(\beta/\kappa)^2 C_{1z}\}|_{z=0} \equiv \eta_d(x+Ut, y) \quad (4.10)$$

We apply these results to: (i) small bodies of characteristic length a and arbitrary shape and (ii) slender bodies of characteristic transverse and axial lengths a and l , where, by hypothesis,

$$ka \equiv N(-d)a/U \ll 1 \quad (4.11)$$

and

$$\epsilon = a/l \ll 1 \quad (4.12)$$

We add that a slender body for which $kl \ll 1$ is also small.

The solution to the problem of a small body moving with uniform speed U follows from the fact that the flow in the neighbourhood of the body is locally potential ($ka \ll 1$ implies that the effects of stratification are negligible over a region of scale a). Invoking the well-known result that the potential flow past a body is equivalent to that induced by a dipole at distances R that are large compared with a , we may match the potential-flow solution to the solution (4.8) in an intermediate region $a \ll R \ll 1/k$ and then use (4.8) and (4.9) to determine the far field (Rayleigh-scattering approximation). The dipole moment is given by Lamb (Sec. 121a, Ref. 5).

$$D = V + V_* \quad (ka \ll 1), \quad (4.13)$$

where V is the volume of the body and $\rho_0 V_*$ its virtual mass with respect to axial translation in a homogeneous fluid of density $\rho_0(-d)$.

The solution to the slender-body problem follows by analogy with the corresponding problem in aerodynamics, cf. Ward (Ref. 7). Omitting the details, we obtain

$$\psi(x, y, a, t) \sim \int S(\xi) [\psi_d(x + Ut - \xi, y, z) / D] d\xi$$

$$(ka \ll 1, a \ll l) \quad (4.14)$$

and an analogous result for ϕ , where $S(x)$ is the cross-sectional area of the body, and the integral extends over the body. We remark that (4.14) reduces to ψ_d if $kl \ll 1$, corresponding to the fact that $V_* \ll V$ for a slender body.

V. COLLAPSE OF MIXING REGION

We consider next the collapse of a small [in the sense of (4.11)] mass of fluid that has been stirred--for example, by turbulence--in such a way as to conserve its mass but alter its potential energy with respect to a horizontal plane through its original center of gravity, say $z = -d$. Our definition of d then implies

$$\iiint (z+d)\rho_0(z)dV = 0, \quad (5.1)$$

conservation of mass implies

$$\iiint [\rho_0(z-\psi_0) - \rho_0(z)]dV \mp \iiint \rho_0'\psi_0 dV = 0, \quad (5.2)$$

and the potential energy is given by

$$E_0 = g \iiint (z+d)\rho_0(z-\psi_0)dV \mp g \iiint (z+d)\rho_0'(z)\psi_0 dV \quad (5.3a)$$

$$= Q\rho_0(-d)N^2(-d), \quad (5.3b)$$

where Q is the quadrupole moment of the region.

Considering now the second integral in (3.9), we expand $G(z,\zeta)$ about $\zeta = -d$ to obtain

$$\begin{aligned} \int G(z,\zeta)N^2(\zeta)\Psi_0(\zeta)d\zeta &= G(z, -d)\int N^2(\zeta)\Psi_0(\zeta)d\zeta \\ &+ G_\zeta(z, -d)\int(\zeta+d)N^2(\zeta)\Psi_0(\zeta)d\zeta \end{aligned} \quad (5.4)$$

and reduce (3.1c) to

$$\Psi_0(\zeta) \doteq \iint \psi_0(x, y, \zeta) dx dy \quad (5.5)$$

by virtue of our assumption that the dimensions of the mass are small. Substituting (5.5) into (5.4), we find that the first integral on the right-hand side vanishes while the second reduces to $QN^2(-d)$ by virtue of (5.2), (5.3), and the Boussinesq approximation. Substituting the resulting approximation into (3.9), we obtain

$$\Psi \sigma^{-1} \psi_0 = QN^2(-d) \kappa^2 \sigma^{-3} G_1(z, -d) \quad (5.6)$$

We apply this last result to a collapsing wake in the lee of a small moving obstacle on the hypothesis that the fluid in the wake is mixed, and perhaps also augmented by turbulent entrainment, over a distance x_0 behind the obstacle, at which point the turbulent wake begins to collapse and releases the potential energy $UE'(x_0)$ per unit time. The resulting, asymptotic (as $t \rightarrow \infty$) disturbance then is given by

$$\psi - \psi_0 \sim -Q'(x_0) N^2(-d) \int_{-\infty}^t d\tau \tau^{-\frac{1}{2}} \bar{x}^{-\frac{1}{2}} \bar{y}^{-\frac{1}{2}} \kappa^2 \sigma^{-2} G_1(x, -d) \quad (t \rightarrow \infty) \quad (5.7)$$

Carrying out the integration with respect to τ and invoking the fact that (as in Sec. IV above) the inverse-Laplace transform of the result is dominated by the pole at $\sigma = i\omega$, we obtain

$$\begin{aligned} \psi - \psi_0 &\sim k^2(-d) Q'(x_0) \bar{x}_{x+Ut-x_0}^{-1} \bar{y}_y^{-1} \{(\kappa^2/i\omega^3) G_1(z)\} \\ &\equiv \psi_q(x+Ut-x_0, y, z), \end{aligned} \quad (5.8)$$

where: k is given by (4.7); G_1 is given by (4.5); $Q'(x_0)$ is the cross-sectional quadrupole moment of the wake, is defined as in (5.3b), and has the dimensions of (length)⁴; the subscript q implies quadrupole. Similarly,

$$\begin{aligned} \phi &\sim Uk^2(-d)Q'(x_0)\mathfrak{F}_{x+Ut-x_0}^{-1}\mathfrak{F}_y^{-1}[\alpha^{-2}[G_{1z}(z)-\delta(z+d)]] \\ &= \phi_q(x+Ut-x_0, y, z) \end{aligned} \quad (5.9)$$

and

$$\eta \sim k^2(-d)Q'(x_0)\mathfrak{F}_{x+Ut-x_0}^{-1}\mathfrak{F}_y^{-1}[i(\beta^2/\alpha^3)G_{1z}(z)]|_{z=0} = \eta_q(x+Ut-x_0, y) \quad (5.10)$$

VI. CONSTANT-N MODEL

We now consider the specific model of a fluid in which N (and, hence, also k) is constant. This is a realistic model for those laboratory configurations in which the effects of lateral boundaries may be neglected. It is not a realistic model for typical oceanic configurations, but it does provide an extreme complement to the thermocline model of the following sections. We give special consideration to the limiting case $D = \infty$, which is appropriate for oceanic applications.

The solution of (3.7) and (3.8) is given by

$$G(z, \zeta) = \frac{\sinh(\lambda z) \sinh[\lambda(\zeta + D)]}{\lambda \sinh(\lambda D)} \quad (z > \zeta) \quad (6.1)$$

wherein α and ζ must be interchanged if $z < \zeta$. We observe that G is a meromorphic function of λ^2 , and therefore of each of α , β and σ , for finite D , and has the Fourier-series representation

$$G(z, \zeta) = -2D \sum_{n=1}^{\infty} \frac{\sin(n\pi z/D) \sin(n\pi \zeta/D)}{(\lambda D)^2 + (n\pi)^2} \quad (6.2)$$

We consider first the limiting case $D \rightarrow \infty$, for which (6.1) reduces to

$$G(z, \zeta) = \lambda^{-1} e^{\lambda \zeta} \sinh \lambda z \quad (z > \zeta, D = \infty), \quad (6.3)$$

which has the branch points of λ , qua function of each of α , β and σ . Substituting (6.3), together with the complementary result for $z < \zeta$, into (4.5), we place the result in the form

$$G_1(z) = \frac{1}{2} e^{-\lambda_1 |z+d|} \operatorname{sgn}(z+d) - \frac{1}{2} e^{\lambda_1(z-d)}, \quad (6.4)$$

where λ_1 is given by (4.6). We may interpret (6.4) in terms of a source at $z = -d$ and an image at $z = d$.* We carry out a detailed analysis only for the first term and move the origin to $z = -d$ with the implicit understanding that z must be replaced by $z+d$ and the image solution incorporated in the final results. In brief, we consider a (dipole or quadrupole) source at the origin of an unbounded fluid in which N is constant,

$$G(z, \zeta) = -\frac{1}{2}\lambda^{-1}e^{-\lambda|z-\zeta|}, \quad (5.5)$$

and

$$G_1(z) = \frac{1}{2}e^{-\lambda_1|z|}\operatorname{sgn}z \quad (6.6)$$

Substituting (6.5) into (4.9) and invoking (4.6b), we obtain

$$\phi_d(x, y, z) = -(DU/8\pi^2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\alpha^2 + \beta^2)^{-\frac{1}{2}} (k^2 - \alpha^2)^{\frac{1}{2}} e^{i\chi(\alpha, \beta)} d\alpha d\beta, \quad (6.7)$$

where

$$\chi = \alpha x + \beta y + i\lambda_1|z| \quad (6.8a)$$

$$= \alpha x + \beta y + \alpha^{-1}(k^2 - \alpha^2)^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}|z| \quad (6.8b)$$

Similar results may be obtained for ψ_d , ϕ_q and ψ_q by substituting (6.6) into (4.8), (5.9) and (5.8), respectively. We recall that x now is measured in a reference frame moving with the source (x replaces $x + Ut$ in the development of Secs. II-IV above) and that ϕ_d is an asymptotic solution that is strictly valid only for $kx \rightarrow \infty$ (although experience suggests that the asymptotic approximation is likely to be qualitatively valid for only moderately large values of kx , say $kx > 1$).

We obtain stationary-phase approximations to ϕ_d , ψ_d , ϕ_q and ψ_q in the appendix to this analysis. Introducing the spherical polar coordinates R , θ and φ according to

*The image term in (6.4) also may be expressed as $+\frac{1}{2}e^{-\lambda_1|z-d|}\operatorname{sgn}(z-d)$.

$$x = R \cos \theta, \quad r = (y^2 + z^2)^{1/2} = R \sin \theta, \quad y = r \cos \varphi,$$

$$z = r \sin \varphi \quad (0 < \theta < \pi, \quad 0 < \varphi < 2\pi) \quad (6.9)$$

and letting $kR \rightarrow \infty$ with θ and φ fixed, we find that $\chi(\alpha, \beta)$ has two (no) points of stationary phase if $\theta < (>) \frac{1}{2}\pi$, reflecting the fact that internal gravity waves (for which the group velocity exceeds the phase velocity) appear only downstream of their source in a steady flow. Substituting the resulting approximations into (2.4b), we obtain the velocity fields

$$v_d \sim -(k^2 D U / 2\pi R) \hat{e}_{\cot \theta \sin \varphi} (\cos^2 \varphi + \sin^2 \theta \sin^2 \varphi)^{1/2} \sin(kR \sin \varphi)$$

$$(kR \rightarrow \infty, \quad 0 < \theta < \frac{1}{2}\pi), \quad (6.10)$$

$$\text{and } v_q \sim (k^3 Q' U / 2\pi R) \hat{e}_{\csc^3 \theta} (\cos^2 \varphi + \sin^2 \theta \sin^2 \varphi)^{3/2} \cos(kR \sin \varphi)$$

$$(kR \rightarrow \infty, \quad 0 < \theta < \frac{1}{2}\pi), \quad (6.11)$$

where

$$\hat{e} = \{-\sin \theta, \cos \theta \cos \varphi, \cos \theta \sin \varphi\} \quad (6.12)$$

is the unit vector in the direction of increasing θ ; both v_d and v_q are asymptotically transverse to a spherical surface with center at $R = 0$ (a well known property of internal gravity waves).

The maximum velocities given by the approximations (6.10) and (6.12) are achieved in the neighbourhood of $\theta = 0$; however, the approximations are not uniformly valid as $\theta \rightarrow 0$, partially in consequence of the restriction $kr \gg 1$ (implicit in the stationary-phase approximation) and partially in consequence of the slender-body approximation, which does not give an adequate description of the interference among the shorter waves (which are especially important in the neighbourhood of $\theta = 0$) that originate at various points of a source of finite cross section. Assuming $r \ll x$ in (6.10) and (6.11), but imposing the restriction $kr \gg 1$ (so that $1/kR \ll \theta \ll 1$), we obtain

$$\zeta_d \sim -(k^2 DU/2\pi)|y|zr^{-4} \underline{\underline{e}} \sin(kxz/r) \quad (kx \gg kr \gg 1), \quad (6.13)$$

and

$$\zeta_q \sim (k^3 Q'U/2\pi)x^2|y|^3r^{-7} \underline{\underline{e}} \cos(kxz/r) \quad (kx \gg kr \gg 1), \quad (6.14)$$

where

$$\underline{\underline{e}} = \{0, y, z\}. \quad (6.15)$$

The corresponding approximations to the lateral strains, as defined by (4.10) and (5.10), are (we omit the details but emphasize that the results calculated from ϕ_d and ϕ_q have been doubled to incorporate the effects of the respective image solutions at the free surface)

$$\eta_d \sim (k^2 dD/\pi)x|y|^3r^{-6} \sin(kxd/r) \quad (kx \gg kr \gg 1) \quad (6.16)$$

and

$$\eta_q \sim -(k^3 Q'/\pi)x^3|y|^5r^{-9} \cos(kxd/r) \quad (kx \gg kr \gg 1), \quad (6.17)$$

wherein $r = (y^2 + d^2)^{1/2}$. The maxima of η_d and η_q with respect to $|y|$ are given by

$$\eta_d = (kD/8\pi d^2)(kx)\sin(kx/\sqrt{2}) \quad \text{at } y = d \ll x \quad (6.18)$$

and

$$\eta_q = -0.045(Q'/\pi d^4)(kx)^3 \cos(2kx/3) \quad \text{at } y = \frac{1}{2}\sqrt{3}d \ll x. \quad (6.19)$$

The loci of constant phase for η_d and η_q are hyperbolae, corresponding to the intersections of the conical, stationary-phase surfaces, $kR\sin\varphi = \chi(\alpha_s, \beta_s)$, with the free surface; the loci corresponding to the approximations of (6.16) and (6.17) are

$$(kx/\chi)^2 - (y/d)^2 = 1. \quad (6.20)$$

It does not appear possible to obtain a simple, asymptotic approximation (for $kx \gg 1$) to (6.7) that is uniformly valid with respect to kr ; however, we can obtain an approximation that is valid at $y = 0$ (although still suffering from the aforementioned deficiency of the slender-body approximation) by first evaluating the Fourier integral over θ [Erdelyi et al, Ref. 8, Sec 1.5(27)], whence

$$\phi_d = -(DU/4\pi^2) \int_{-\infty}^{\infty} e^{i\alpha x} K_0[\{\alpha^2(y^2+z^2) - k^2 z^2\}^{1/2}] (k^2 - \alpha^2)^{1/2} d\alpha, \quad (6.21)$$

where the real part of the radical is non-negative, and K_0 is a modified Bessel function of the second kind. Differentiating (6.21) twice with respect to y , integrating with respect to x , setting $y = 0$ and $z = d$ (in the reference frame with origin at the source), and doubling the result to incorporate the effect of the image solution, we obtain

$$\eta_d(x, 0) = (D/2\pi^2 d) \int_{-\infty}^{\infty} |\alpha| K_1[d(\alpha^2 - k^2)^{1/2}] e^{i\alpha x} d\alpha. \quad (6.22)$$

The dominant contribution to the integral in (6.22) comes from the neighbourhood of $\alpha = k$, which yields

$$\eta_d \sim -(D/d^2) (2k/\pi^3 x)^{1/2} \sin(kx - \frac{1}{2}\pi) \quad (kx \gg 1, y = 0). \quad (6.23)$$

Similarly, we obtain

$$\eta_q \sim (6Q'/d^4) (2kx/\pi^3)^{1/2} \sin(kx - \frac{1}{2}\pi) \quad (kx \gg 1, y = 0). \quad (6.24)$$

VII. THERMOCLINE MODEL

We consider now the discrete spectrum of internal waves associated with a thermocline model, for which (by definition)

$$0 \leq N^2(z) \leq N^2(-h) \equiv N_h^2 \quad (7.1)$$

and

$$\int_{-D}^0 N^2(z) dz \equiv N_h^2 b \div \frac{g \Delta \rho}{\rho} \equiv g', \quad (7.2)$$

where $-h$ is the vertical coordinate of the thermocline, that is the plane in which $N(z)$ achieves its maximum value, N_h ; $\Delta \rho$ is the total increase in density across the thermocline ($\Delta \rho \ll \rho$ by hypothesis); and g' is a reduced gravitational acceleration. Setting

$$\sigma = i\omega \quad (7.3)$$

in (3.5) and (3.7), we obtain

$$\left[\partial_z^2 + (\kappa/\omega)^2 N^2(z) - \kappa^2 \right] G(z, \zeta) = \delta(z - \zeta) \quad (7.4)$$

Invoking the assumptions, (7.1) and (7.2) above, that $N^2(z) > 0$ and that the integral of $N^2(z)$ is bounded (a nontrivial restriction if $D = \infty$), we infer from Sturm-Liouville theory that there exists a discrete set of eigenvalues, say κ_n^* , and eigenfunctions, say $f_n(z)$, that satisfy

^{*}It would be more conventional to regard the wave speed, $c = \omega/\kappa$, as the eigenvalue for the Sturm-Liouville problem, but we find it more convenient for the subsequent development to introduce κ_n as the eigenvalue and to regard both ω and κ as prescribed.

$$\left\{ \partial_z^2 + (\kappa_n N/w)^2 - \kappa^2 \right\} f_n(z) = 0, \quad (7.5)$$

$$f_n(0) = f_n(-D) = 0, \quad (7.6)$$

and

$$\int_{-D}^0 (N/w)^2 f_m f_n dz = \delta_{mn}, \quad (7.7)$$

where δ_{mn} is the Kronecker delta. Expanding G in the f_n in the usual way, we obtain

$$G(z, \zeta) = \sum_n (\kappa^2 - \kappa_n^2)^{-1} f_n(z) f_n(\zeta) \quad (7.8)$$

Substituting (7.8) into (4.5), we obtain

$$G_1(z) = - \sum_n (\kappa^2 - \kappa_n^2)^{-1} f_n'(-d) f_n(z) \quad (w = U\alpha), \quad (7.9)$$

where $f_n'(-d) \equiv (df/d\zeta)_{\zeta=-d}$.

Referring to Secs. IV and V above, we seek the far field ($kx \gg 1$) of a moving source. Substituting (7.9) into (4.8)-(4.10) and (5.8)-(5.10), invoking the Fourier integral

$$\mathcal{F}_y^{-1}(\beta^2 + a^2)^{-1} = \frac{1}{2a} e^{-a|y|} \quad (Ra \geq 0), \quad (7.10)$$

and setting $w = U\alpha$, we obtain

$$\begin{aligned} \langle \phi_d, \psi_d \rangle = & -(D/4\pi) \sum_n \int_{-\infty}^{\infty} \gamma_n^{-1} e^{i\alpha x - \gamma_n |y|} f_n(-d) \\ & \cdot \left[(i\alpha U/\kappa_n^2) f_n'(z) [1 - |\alpha|^{-1} \gamma_n e^{-(|\alpha| - \gamma_n)|y|}], f_n(z) \right] d\alpha \end{aligned} \quad (7.11)$$

$$\{\phi_q, \psi_q\} = \{k^2(-d)Q'(x_0)/4\pi\} \sum_n \int_{-\infty}^{\infty} (\alpha^2 - \gamma_n)^{-1} e^{i\alpha x - \gamma_n |y|} f'_n(-d) \cdot [-uf'_n(z), (ik_n^2/\alpha)f_n(z)] d\alpha \quad (|y| > 0), \quad (7.12)$$

$$\eta_d = -(D/4\pi) \sum_n \int_{-\infty}^{\infty} \kappa_n^{-2} f'_n(-d) f'_n(0) e^{i\alpha x} (\gamma_n e^{-\gamma_n |y|} - |\alpha| e^{-|\alpha y|}) d\alpha, \quad (7.13)$$

and

$$\eta_q = \{k^2(-d)Q'(x_0)/4\pi\} \sum_n \int_{-\infty}^{\infty} (i\gamma_n/\alpha^3) f'_n(-d) f'_n(0) e^{i\alpha x - \gamma_n |y|} d\alpha, \quad (7.14)$$

where

$$\gamma_n = (\alpha^2 - \kappa_n^2)^{\frac{1}{2}} \quad (\Re \gamma_n \geq 0), \quad (7.15)$$

and $f'_n(0) \equiv df_n/dz$ at $z = 0$.

VIII. THIN THERMOCLINE APPROXIMATION

We carry the development of the preceding section further for a thin thermocline, for which $N^2(z)$ differs significantly from zero only in a small neighbourhood of $z = -h$, where it exhibits a single, sharp peak. We also neglect bottom effects by setting $D = \infty$. This model is perhaps more realistic than, but in any event complements, that of Sec. VI.

The dispersion relation for the dominant mode of a thin thermocline may be expressed in terms of the thermocline parameters N_h and b , as defined by (7.1) and (7.2), and the depth of the thermocline on the basis of the assumptions

$$N_h b / U \equiv k_h b \ll 1 \text{ and } b/h \ll 1. \quad (8.1a, b)$$

Setting $N^2 = 0$ for $|z+h| \gg b$ and invoking the boundary conditions (7.6) and the requirement that $f(z)$ be continuous across $z = -h$ as $b \rightarrow 0$, we choose the solutions above and below the thermocline in the form

$$f(z) = f_h \begin{cases} -\text{csch} \kappa h \sinh \kappa z \\ e^{\kappa(z+h)} \end{cases} \quad (z \begin{matrix} > \\ < \end{matrix} -h), \quad (8.2)$$

where $f_h \equiv f(-h)$. Integrating (7.5) across the thermocline and remarking that both $f'' - \kappa^2 f$ and N^2 vanish except in the immediate neighbourhood of $z = -h$, where f'' is discontinuous, $f = f_h$, and the integral of N^2 is given by (7.2), we obtain

$$\begin{aligned} 0 &= \int_{-\infty}^0 \{f'' - \kappa^2 f + (\kappa_1 N/w)^2 f\} dz = \lim_{\epsilon \rightarrow 0^+} \left\{ f' \right\}_{-h-\epsilon}^{-h+\epsilon} \\ &+ (\kappa/w)^2 f_h \int_{-h-\epsilon}^{-h+\epsilon} N^2 dz = -\kappa (\coth \kappa h + 1) f_h + g' (\kappa_1/w)^2 f_h. \end{aligned} \quad (8.3)$$

Solving (8.3) for ω^2 , we obtain the dispersion relation [Ref. 9, (5.3.7)]*

$$\omega_1^2 = \frac{1}{2} g' \kappa (1 - e^{-2\kappa h}) \quad (8.4a)$$

$$\rightarrow g' h \kappa^2 (1 - \kappa h + \dots) \quad (\kappa h \rightarrow 0) \quad (8.4b)$$

$$\sim \frac{1}{2} g' \kappa \quad (\kappa h \rightarrow \infty). \quad (8.4c)$$

Similarly

$$|\alpha_1| = |\omega|/U = k_h \left[\frac{1}{2} \kappa b (1 - e^{-2\kappa h}) \right]^{\frac{1}{2}} \quad (8.5a)$$

$$\rightarrow k_h (bh)^{\frac{1}{2}} \kappa (1 - \frac{1}{2} \kappa h + \dots) \quad (\kappa h \rightarrow 0) \quad (8.5b)$$

$$\sim k_h \left(\frac{1}{2} b \kappa \right)^{\frac{1}{2}} \quad (\kappa h \rightarrow \infty) \quad (8.5c)$$

and

$$c_{g1} = \frac{d|\omega_1|}{d\kappa} = U \frac{d|\alpha_1|}{d\kappa} = \frac{1}{2} \left(\frac{g'}{2\kappa} \right)^{\frac{1}{2}} \left[\frac{1 - (1 - 2\kappa h)e^{-2\kappa h}}{(1 - e^{-2\kappa h})^{\frac{1}{2}}} \right] \quad (8.6a)$$

$$\rightarrow (g' h)^{\frac{1}{2}} (1 - \kappa h + \dots) \quad (\kappa h \rightarrow 0) \quad (8.6b)$$

$$\sim \frac{1}{2} (g'/2\kappa)^{\frac{1}{2}} \quad (\kappa h \rightarrow \infty). \quad (8.6c)$$

We note that (8.6b) and (8.6c) intersect at $\kappa h = \frac{1}{2}$ and serve as rough approximations (with maximum errors of 20 percent) for $\kappa h < \frac{1}{2}$ and

*The preceding derivation is an abbreviated form of a technique used by Lighthill, (Ref. 10) and Drazin & Howard, (Ref. 11). This technique also may be applied to the higher modes, but the results are rather unwieldy. Moreover, the contribution of the dominant mode to the free-surface disturbance will dominate the contributions of the higher modes if (8.1a) is satisfied (see end of Sec. IX below).

$\kappa h > \frac{1}{2}$, respectively, in the subsequent, stationary-phase approximations. Substituting (8.2) into (7.7), we obtain

$$f_h^2 = \frac{1}{2}\kappa(1 - e^{-2\kappa h}). \quad (8.7)$$

We emphasize that (8.4)-(8.7) hold only for the dominant mode* ($n = 1$).

We use the approximations of (8.2) and (8.5) to obtain asymptotic approximations to the lateral, free-surface strains, η_d and η_q , on the basis of these hypotheses: (a) the contributions of the higher modes ($n \geq 2$) are negligible compared with that of the dominant mode (we omit the subscript 1 with this understanding) and (b) $|\alpha| \ll \kappa$, for which a sufficient condition is $U \gg (g'h)^{\frac{1}{2}}$. The latter hypothesis permits the approximation

$$\gamma \doteq i\kappa \quad (0 \leq \alpha \ll \kappa) \quad (8.8)$$

in place of (7.15) and the neglect of $|\alpha|e^{-|\alpha y|}$ compared with $\gamma e^{-\gamma|y|}$ in (7.13). Invoking these approximations, substituting (8.2) and (8.7) into (7.13), setting $\alpha = 0$, and choosing κ , rather than α , as the variable of integration (thereby regarding α as the eigenvalue for prescribed κ in the Sturm-Liouville problem), we obtain

$$\eta_d \sim -(D/2\pi)R \int_0^\infty i\kappa^2 (d\alpha/d\kappa) D(\kappa) e^{-\kappa H + i(\alpha x - \kappa|y|)} d\kappa, \quad (8.9)$$

where

$$D(\kappa) = \begin{cases} (1 - e^{-2\kappa h})^{-1} (1 + e^{-2\kappa d}) & (d \geq h) \\ -1 & (d < h) \end{cases} \quad (8.10)$$

and

$$H = h + |d - h|. \quad (8.11)$$

*The oscillations of $f_n(z)$ across the thermocline do not permit the approximation $i(z) \doteq f_h$ for $n > 1$ in the integrands of (7.7) and (8.3)

We obtain the corresponding approximation to η_d by replacing $D\kappa^2$ by $\kappa^2(0)Q'(x_0)\kappa^4/i\alpha^3$ in (8.9). The discontinuity at $d = h$ corresponds to the discontinuity in $f'(z)$ at $z = -h$ and is an intrinsic characteristic of the thin-thermocline approximation.

Carrying out a stationary-phase approximation to (8.9), we obtain

$$\eta_d \sim -D(2\pi x)^{-\frac{1}{2}} \left[\kappa^2 (d\alpha/d\kappa) |d^2\alpha/d\kappa^2|^{-\frac{1}{2}} D(\kappa) e^{-\kappa H + i(\alpha x - \kappa|y| + \frac{3}{4}\pi)} \right], \quad (8.12)$$

where κ is determined by

$$d\alpha/d\kappa = c_g/U = |y|/x \quad (H \ll |y| < (gh')^{\frac{1}{2}} U^{-1} x). \quad (8.13)$$

There is no point of stationary phase, and η_d is $O(x^{-1})$ rather than $O(x^{-\frac{1}{2}})$ as $x \rightarrow \infty$, if $|y| > (gh')^{\frac{1}{2}} x/U$. A saddle-point, rather than a stationary-phase, approximation must be used if $|y|/H$ is not large; κ then must be determined by replacing $|y|$ by $|y| - iH$ in (8.13) and is complex.

The maximum value of $|\eta_d|$ corresponds roughly to the maximum of $\kappa^2 \exp(-\kappa H)$, that is $\kappa H \approx 2$, which yields a value of κ that increases from $1/h$ to $2/h$ as d increases from 0 to h and then remains at $2/d$ for $d > h$. We may refine these estimates, at least for $d < 4h$, by utilizing the asymptotic approximation (8.5c), the substitution of which into (8.12) on the assumption that κ is real ($|y| \gg H$) yields

$$|\eta_d| \sim D(2\pi U x)^{-\frac{1}{2}} (g'H)^{\frac{1}{2}} \kappa^{\frac{3}{2}} |D(\kappa)| e^{-\kappa H}. \quad (8.14)$$

We find that the maximum value of (8.14) occurs at $\kappa h = 1.0$ for $d \ll h$, $\kappa h = 2.1$ for $d = h$, and $\kappa h = 2.25$ for $d > h$, so that $\kappa H = 2$ provides an adequate basis for an estimate, namely (we take $D \approx 1$)

$$|\eta_d|_{\max} = 0.2 D g^{\frac{1}{2}} (U x)^{-\frac{1}{2}} H^{-\frac{3}{4}} \quad (d < 4h) \quad (8.15)$$

at

$$|y|/x \approx (g'H)^{\frac{1}{2}}/4U \quad (d < 4h). \quad (8.16a)$$

If $d > 4h$, we must use (8.6b) in place of (8.6c), the principal effect of which is to replace $H^{-9/4}$ by $d^{-2}h^{-1/4}$ in (8.15) and (8.16a) by

$$|y|/x \doteq (g'h)^{1/2} u^{-1} [1 - (2h/d)] \quad (d > 4h). \quad (8.16b)$$

The counterparts of (8.12) and (8.14) for η_q are

$$\eta_q \sim k^2 (-d) Q'(x_0) (2\pi x)^{-1/2} [k^4 a^{-3} (da/dk) |d^2 a/dk^2|^{-1/2} \cdot D(\kappa) e^{-\kappa H + i(\alpha x - \kappa |y| + \frac{1}{2}\pi)}] \quad (8.17)$$

$$\text{and } |\eta_q| \sim [N(-d)/N_h]^2 Q'(x_0) (u/\pi x)^{1/2} b^{-1} (8/g')^{1/2} \kappa^{11/4} D(\kappa) e^{-\kappa H}. \quad (8.18)$$

The maximum value of $|\eta_q|$ occurs at $\kappa H \doteq 11/4$, where the deviation of $D(\kappa)$ from unity is small, so that

$$|\eta_q|_{\max} = 1.0 [N(-d)/N_h]^2 Q'(x_0) g'^{-1/2} u^{1/2} b^{-1} H^{-11/4} x^{-1/2} \quad (8.19)$$

$$\text{at } |y|/x \doteq 0.2 (g'u)^{1/2} / u \quad (d < 5.5h) \quad (8.20a)$$

$$\text{or } |y|/x \doteq (g'h)^{1/2} u^{-1} [1 - 2.75(h/d)] \quad (d > 5.5h). \quad (8.20b)$$

We also note that

$$\left| \frac{\eta_q}{\eta_d} \right| = \left[\frac{N(-d)}{N_h} \right]^2 \left[\frac{Q'(x_0)}{bD} \right] \frac{u}{c_g}. \quad (8.21)$$

We use this last result to compare the lateral surface strains produced by the dipole effect of a small, prolate ellipsoid of radius a and length l and its wake on the hypothesis that the wake is (or has the same potential energy as a wake that is) fully mixed and of radius a ; then $D \doteq 2\pi a^2 l/3$ and $Q' = \pi a^4/4$. Substituting these results into (8.21), we obtain

$$\left| \frac{\eta_g}{\eta_d} \right| \approx \frac{3}{8} \left(\frac{a^2}{bL} \right) \left[\frac{N(-d)^2}{N_h} \right] \frac{u}{c_g} \quad (8.22a)$$

$$\approx \left(\frac{a^2}{bL} \right) \left[\frac{N(-d)}{N_h} \right]^2 \frac{u}{(g'H)^{\frac{1}{2}}} \quad (8.22b)$$

The factors a^2/bL might lie between 10^{-2} and 10^{-1} for a typical submarine, u/c_g might lie between 10 and 10^2 , and $[N(-d)/N_h]^2$ is less than unity and might be as small as 10^{-2} if the submarine is well outside of the thermocline. It follows that, within the limitations of the hypotheses implicit in our model, the dipole effect is likely to dominate the wake effect. Both effects achieve their maxima if the submarine is in the thermocline ($d \approx h \approx H$) and fall off rapidly with increasing d/h .

IX. WKB APPROXIMATION

The WKB approximation to the solution of the Sturm-Liouville problem posed by (7.5) and (7.6) may be expected to yield qualitatively accurate results for all but the dominant mode ($n = 1$ below), although the implicit assumption that $N^2(z)$ is a slowly varying function renders it quantitatively accurate only for those modes for which $\kappa_n b \gg 1$. It is generally inadequate for even a qualitative description of the dominant mode of a thin thermocline, for which $\kappa_1 b \ll 1$ and $N^2(z)$ varies rapidly near $z = -h$. It is consistent with the WKB approximation to neglect the effects of both upper and lower boundaries (the implicit restrictions are $\kappa_n h \gg 1$ and $\kappa_n |D-h| \gg 1$, respectively; the violation of the former restriction is likely to be qualitatively significant only for the dominant mode, while the latter restriction is almost always satisfied in a real ocean). Keeping these remarks in mind, we rewrite the Sturm-Liouville problem of (7.5)-(7.7) in the form

$$\left\{ \frac{d^2}{dz^2} + \kappa_n^2 w(z) \right\} f_n(z) = 0, \quad (9.1)$$

$$f_n(\pm\infty) = 0, \quad (9.2)$$

and
$$\int_{-\infty}^{\infty} w f_m f_n dz = \delta_{mn}, \quad (9.3)$$

where
$$w(z) = w^{-2} N^2(z) - 1 \quad (9.4)$$

is the weighting function, and κ_n is the eigenvalue. The results presented in (7.8) through (7.15) remain valid for this revised formulation.

We proceed on the assumptions that $N^2(z)$ satisfies (7.1) and has only a single peak ($N = N_h$ at $z = -h$) and that $|w| < N_h$ (waves for which $|w| > N_h$ are not propagated); then $w(z)$ has only two zeros, say z_l and z_u , such that

$$w(z_l) = w(z_u) = 0 \quad (z_l < -h < z_u) \quad (9.5a)$$

and $w(z) > 0 \quad (z_l < z < z_u)$. (9.5b)

We also define $\arg w^{\frac{1}{2}} = 0$ for $w > 0$ and infer $\arg w^{\frac{1}{2}} = \frac{1}{2}\pi$ for $w < 0$ from the requirement $\Re \sigma > 0$ (or, equivalently, $\Im \omega < 0$) and the facts that $N'(z_l) > 0$ and $N'(z_u) < 0$. We then may pose the WKB phase integral in the forms

$$P(z) \equiv \int_{z_l}^z w^{\frac{1}{2}} dz \begin{cases} = P(z_u) + iQ_u(z) & (z > z_u) \\ > 0 & (z_l < z < z_u) \\ = -iQ_l(z) & (z < z_l) \end{cases}, \quad (9.6)$$

where

$$Q_l(z) = \int_z^{z_l} (-w)^{\frac{1}{2}} dz \quad (9.7a)$$

and

$$Q_u(z) = \int_{z_u}^z (-w)^{\frac{1}{2}} dz. \quad (9.7b)$$

Invoking the fact that $w \rightarrow -1$ outside of the thermocline, we obtain

$$Q_l(z), Q_u(z) \sim |z+h| \quad (|z+h| \gg b). \quad (9.8)$$

The WKB solution of (9.1) and (9.2) is given by the following (we omit the details but note that the problem is analogous to that of the harmonic oscillator in quantum mechanics)

$$f_n(z) = C_n |w(z)|^{-\frac{1}{2}} \begin{cases} (-)^{n-1} \exp\{-\kappa_n Q_u(z)\} & (z > z_u) \\ 2\cos\{\kappa_n P(z) - \frac{1}{2}\pi\} & (z_l < z < z_u) \\ \exp\{-\kappa_n Q_l(z)\} & (z < z_l) \end{cases} \quad (9.9)$$

except in the neighbourhoods of $z = z_l$ and $z = z_u$, where Airy-integral representations must be invoked. The corresponding approximations to the eigenvalues are given by

$$\kappa_n = (n - \frac{1}{2})\pi \left[\int_{z_l}^{z_u} w^{\frac{1}{2}} dz \right]^{-1} \quad (n = 1, 2, \dots) \quad (9.10)$$

The normalization of (9.3) implies

$$C_n = [\pi + (2n-1)^{-1}]^{-\frac{1}{2}} \kappa_n^{\frac{1}{2}} \quad (9.11)$$

We calculate κ_n on the basis of the parabolic approximation

$$N^2(z) \doteq N_h^2 \left[1 - \left(\frac{z+h}{s} \right)^2 \right] \quad (z_l < z < z_u) \quad (9.12)$$

If we assume that (9.12) is valid for all $|z-h| < s$ and that $N^2 = 0$ in $|z+h| > s$, (7.2) implies $b = 4s/3$. If we assume that $N^2 = N_h^2 \exp\{-(z+h)^2/s^2\}$, for all z and is approximated by (9.12) in $z_l < z < z_u$, (7.2) implies $b = \pi^{\frac{2}{3}} s$.] Substituting (9.12) into (9.10), we obtain*

$$\kappa_n = (2n-1)s^{-1} N_h \omega(N_h^2 - \omega^2)^{-1} \quad (\omega < N_h) \quad (9.13a)$$

$$= (2n-1)s^{-1} \kappa_\alpha (k^2 - \alpha^2)^{-1} \quad (\alpha < k), \quad (9.13b)$$

where, here and throughout this section, $k \equiv k(-h)$.

*This result is exact if $N(z)$ is described exactly by (9.12), for which (9.1) is Hermite's equation, and the $f_n(z)$ are Hermite functions.

We proceed on the hypothesis that

$$|\alpha| \ll |\kappa_n|, \quad (9.14)$$

by virtue of which we may approximate (7.15) by

$$\gamma_n \doteq i\kappa_n, \quad (9.15)$$

where $\arg \gamma_n$ is determined by the requirements $\Re \gamma_n \geq 0$ and $\Im \alpha < 0$, and neglect the term in $\exp(-|\alpha y|)$ in (7.11) and (7.13). We also use the approximations of (9.8) outside of the thermocline. Invoking these approximations, substituting (9.9) into (7.13), and restricting the range of integration to that of the propagated waves ($|\alpha| < k$; waves for which $|\alpha| > k$ are not propagated and are negligible for $kx \gg 1$)

$$\eta_d \sim -(D/2\pi) \sum_{n=1}^{\infty} A_n R \int_0^k i\kappa_1^2 e^{i\alpha x - (2n-1)i\kappa_1(|y|-1)H} d\alpha, \quad (h, |d-h| \gg s), \quad (9.16)$$

where $A_n = (2n-1) [\pi + (2n-1)^{-1}]^{-1} [\text{sgn}(n-d)]^n, \quad (9.17)$

κ_1 is given by (9.13b), and H is given by (8.11). Introducing the change of variable

$$\alpha = k \sin \zeta \quad (9.18)$$

and the parameter

$$\mu_n = (2n-1)(kxs)^{-1} (|y|-1)H \quad (-\frac{1}{2}\pi < \arg \mu < 0), \quad (9.19)$$

we rewrite (9.16) in the form

$$\eta_d \sim -(kD/2\pi s^2) \sum_{n=1}^{\infty} A_n R \int_0^{\frac{1}{2}\pi} i \sin^2 \zeta \sec^3 \zeta e^{ikx \sin \zeta (1 - \mu_n \sec^2 \zeta)} d\zeta. \quad (9.20)$$

The integrand of (9.20) has a saddle point at the point determined parametrically by

$$\sin \zeta = \alpha/k = v_n \mu_n = (1-v_n^2)^2(1+v_n^2)^{-1}, \quad (9.21a,b)$$

where v_n is a complex number. The contribution of this point dominates the asymptotic approximation (as $kx \rightarrow \infty$) to the integral (after an appropriate deformation of the path of integration) if $|\mu_n| < 1$ and $|1-\mu_n|$ is not small. Carrying out the saddle-point approximation, we obtain

$$\eta_d \sim (kD/s^2) \sum_{n=1}^{\infty} A_{dn}(kx, \mu_n),$$

$$A_{dn} = \frac{1}{2}(\pi kx)^{-\frac{1}{2}} A_n R \left\{ v_n^{\frac{3}{2}} (1-v_n^2)^{-\frac{3}{2}} (1+v_n^2)^{\frac{1}{2}} (3+v_n^2)^{-\frac{1}{2}} e^{i(X_n - 3\pi/4)} \right\}. \quad (9.22a,b)$$

where
$$\chi_n = 2kx v_n^3 (1+v_n^2)^{-1}. \quad (9.23)$$

Similarly, starting from (7.14), we obtain

$$\eta_q \sim [N(-d)/N_n]^2 [Q'(x_0)/s^4] \sum_{n=1}^{\infty} A_{qn}(kx, \mu_n),$$

$$A_{qn} = \frac{1}{2}(\pi kx)^{-\frac{1}{2}} (2n-1)^2 A_n R \left\{ v_n^{\frac{1}{2}} (1-v_n^2)^{-\frac{7}{2}} (1+v_n^2)^{\frac{1}{2}} (3+v_n^2)^{-\frac{1}{2}} e^{i(X_n - 3\pi/4)} \right\} \quad (9.24a,b)$$

The largest terms in the modal summations of (9.22) and (9.24) are those, if any, for which $|\mu_n|$ is small and, from (9.21b),

$$v_n = 1 - (|\mu_n|)^{\frac{1}{2}} + O(|\mu_n|). \quad (9.25)$$

Substituting (9.25) into (9.23) and retaining only the dominant terms in each of the real and imaginary parts, we obtain

$$\chi_n \approx kx - i(2n-1)^{\frac{1}{2}} (kx/s)^{\frac{1}{2}} \{ (\gamma^2 + H^2)^{\frac{1}{2}} - |y| \}^{\frac{1}{2}} \equiv kx - i\chi_{ni} \quad (9.26)$$

Substituting (9.25) into (9.21) and (9.13b), we obtain

$$|\alpha/\kappa_n| = (2n-1)^{-1} k s |2\mu_n|^{1/2}, \quad (9.27)$$

so that the approximation (9.25) is consistent with the restriction (9.14).

We use the approximations (9.25) and (9.26) to obtain the estimates

$$|A_{dn}| \doteq 2^{-\frac{9}{4}} \pi^{-\frac{1}{2}} [\pi + (2n-1)^{-1}]^{-1} (2n-1)^{\frac{1}{2}} (kx)^{\frac{1}{2}} s^{\frac{3}{4}} (y^2 + H^2)^{-\frac{3}{8}} e^{-X_{ni}} \quad (9.28)$$

$$\text{and } |A_{qn}| \doteq 2^{-\frac{13}{4}} \pi^{-\frac{1}{2}} [\pi + (2n-1)^{-1}]^{-1} (2n-1)^{\frac{5}{4}} (kx)^{\frac{5}{4}} s^{\frac{7}{4}} (y^2 + H^2)^{-\frac{7}{8}} e^{-X_{ni}}. \quad (9.29)$$

Assuming $|y| \gg H$, we find that $|A_{dn}|$ has its maximum at

$$|y|/H = (2/9)(2n-1)(H/s)kx \gg 1 \quad (9.30)$$

and similarly for $|A_{qn}|$, with 2/9 replaced by 2/49. These maxima are fairly sharp (in $|y|/H$) and therefore can be achieved by only a single mode at any given point. The corresponding maxima in $|\eta_d|$ and $|\eta_q|$, neglecting all modes except that for which (9.30) and its counterpart for $|A_{dq}|$ are satisfied, are

$$|\eta_d|_{\max} = 0.025(2n-1)^{-\frac{1}{2}} (kD/s^{\frac{1}{2}} H^{\frac{3}{2}}) (kx)^{-\frac{1}{2}} \quad (9.31)$$

$$\text{and } |\eta_q|_{\max} = 0.15(2n-1)^{-\frac{1}{2}} [N(-d)/N_h]^2 [Q'(x_0)/s^{\frac{1}{2}} H^{\frac{7}{2}}] (kx)^{-\frac{1}{2}} \quad (9.32)$$

Comparing these maxima with those of (8.15) and (8.19) for $b = 4s/3$ and $n = 2$ (typically the most important of the higher modes), we

conclude that the contributions of the higher modes, relative to those of the dominant mode, to $|\eta_d|$ and $|\eta_q|$ are not likely to exceed $0.05(H/s)^{3/2}$ and $0.12(s/H)^{3/4}$, respectively. The former ratio could be larger than unity, but only for H/s such that all contributions to $|\eta_d|$ would be very small; the latter ratio is certainly small--typically between 10^{-1} and 10^{-2} .

APPENDIX TO ANALYSIS B
STATIONARY-PHASE APPROXIMATIONS

We require stationary-phase approximations to integrals of the form

$$I = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) e^{iX(\alpha, \beta)} d\alpha d\beta, \quad (A1)$$

where
$$X = \alpha x + \beta y + \alpha^{-1}(k^2 - \alpha^2)^{\frac{1}{2}}(\alpha^2 + \beta^2)^{\frac{1}{2}}z, \quad (A2)$$

$z > 0$, and $R = (x^2 + y^2 + z^2)^{\frac{1}{2}} \rightarrow \infty$.

Considering first the β -integration, we find that X has a point of stationary phase at

$$\beta = \beta_s(\alpha) = -\alpha | \alpha | y (k^2 z^2 - \alpha^2 r^2)^{-\frac{1}{2}}, \quad (A3)$$

at which point

$$X(\alpha, \beta_s(\alpha)) = \alpha x + (k^2 z^2 - \alpha^2 r^2)^{\frac{1}{2}} \text{sgn} \alpha, \quad (A4)$$

and

$$X_{\beta\beta} = (\alpha z)^{-2} (k^2 - \alpha^2)^{-1} (k^2 z^2 - \alpha^2 r^2)^{\frac{3}{2}} \text{sgn} \alpha \quad (A5)$$

Carrying out the stationary-phase approximation to the β -integral, we obtain

$$I \sim (2\pi)^{-\frac{3}{2}} \int_{-\infty}^{\infty} (fX_{\beta\beta}^{-\frac{1}{2}})_{\beta=\beta_s(\alpha)} \exp\{iX(\alpha, \beta_s(\alpha)) + \frac{1}{2}i\pi \text{sgn} \alpha\} d\alpha. \quad (A6)$$

The integral in (A6) has two points of stationary phase at

$$\alpha = \alpha_s = \pm kxz/rR = \pm k\cos\theta\sin\varphi \quad (A7)$$

$$\text{and } \beta = \beta_s = -\alpha_s yx/r^2 = \mp k\cos^2\theta\csc\theta\sin\varphi\cos\varphi \quad (A8)$$

if $x > 0$ and no points of stationary phase if $x < 0$; R , θ and φ are polar coordinates, defined by (6.9) above. We also obtain

$$\chi(\alpha_s, \beta_s) = \pm kzR/r = \pm kR\sin\theta \quad (A9)$$

$$\chi_{r1} = \mp R^3/krz, \quad (A10)$$

$$\text{and } \chi_{\beta\beta} = \pm (r^7R/kx^2z)(y^2R^2+z^2r^2)^{-\frac{1}{2}}, \quad (A11)$$

where the upper and lower signs correspond to $\alpha_s \gtrless 0$. Assuming that $f(-\alpha, \beta)$ is the complex conjugate of $f(\alpha, \beta)$, we find that the contributions of the two points to the stationary-phase approximation to I are complex conjugates, with the end result

$$I \sim (kxz/\pi r^3 R^2)(y^2R^2+z^2r^2)^{\frac{1}{2}} \{f(\alpha_s, \beta_s)e^{ikzR/r}\} \quad (x > 0) \quad (A12a)$$

$$= (k/rR)\cot\theta\sin\varphi(\cos^2\varphi+\sin^2\theta\sin^2\varphi)^{\frac{1}{2}} \{f(\alpha_s, \beta_s)e^{ikR\sin\theta}\} \quad (x > 0). \quad (A12b)$$

Comparing (6.7) and the corresponding representations of ψ_d , ϕ_q , and ψ_q to (A1), we obtain

$$f\{\phi_d, \psi_d\} = \frac{1}{2}D[-U(\alpha^2+\beta^2)^{-\frac{1}{2}}(k^2-\alpha^2)^{\frac{1}{2}}, \text{sgnz}] \quad (A13)$$

and

$$f\{\phi_q, \psi_q\} = \frac{1}{2}ik^2Q'a\{U(\alpha^2+\beta^2)^{\frac{1}{2}}(k^2-\alpha^2)^{\frac{1}{2}}, -(\alpha^2+\beta^2)\text{sgnz}\}. \quad (A14)$$

Substituting (A13) and (A14) into (A12b), we obtain

$$\{\phi_d, \psi_d\} \sim (rD/2\pi R)\{-U, \cot\theta \sin\varphi\}(\cos^2\varphi + \sin^2\theta \sin^2\varphi)^{\frac{1}{2}} \\ \cdot \cos(kR \sin\varphi) \quad (kR \rightarrow \infty, 0 < \theta < \frac{1}{2}\pi) \quad (A15)$$

and $\{\phi_q, \psi_q\} \sim (k^2 Q_1/2\pi R)\{-U \csc\varphi, \cot\theta\} \sec\theta \csc^2\theta \\ \cdot (\cos^2\varphi + \sin^2\theta \sin^2\varphi)^{\frac{3}{2}} \sin(kR \sin\varphi) \quad (kR \rightarrow \infty, 0 < \theta < \frac{1}{2}\pi) \quad (A16)$

Substituting (A15) and (A16) into (2.4b), we obtain (6.10) and (6.11).

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SUPPORTING ANALYSIS C

THE INTERACTION OF INTERNAL WAVES AND GRAVITY WAVES

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We wish to study the effects on rather small surface waves, and specifically on their height and slope, of the existence of internal waves in the region of ocean through which the surface waves are moving. Surface waves damp exponentially in depth in a distance comparable to their wavelength, so for small surface waves, we can assume that the dimensions of the internal wave are very large compared to those of the surface waves, and therefore the internal wave can be well represented by a horizontal depth independent current. We describe this with a velocity field $\underline{U}(x,y,t)$, which is a function only of time, of the horizontal coordinates x , y , and with no vertical component. Furthermore, we may expect the times and horizontal distances over which U varies to be much greater than those over which the surface waves of interest vary.

We shall ignore viscous and other dissipative effects for the surface waves, that is, we shall assume that damping is unimportant over horizontal distances comparable to the region occupied by the internal wave. Typically, we shall be interested in dimensions of surface waves which dissipate in distances considerably longer than that. On the other hand, since the size of the effects we are interested in will be characterized by the parameter U/c_g , where c_g is the group velocity of the surface wave, we are also most interested in slow (i.e., short wavelength) gravity waves. Yet these are also the waves that dissipate most quickly. We must therefore strike a balance between the two requirements.

Finally, we shall assume incompressible irrotational flow in the region of ocean occupied by the surface waves. Irrotational flow is described by a velocity potential ϕ which satisfies

$$\nabla^2 \phi = 0 \quad (1)$$

There are three boundary conditions on the solution to the equation:

(1) At the surface, the vertical velocity of the fluid $\partial\phi/\partial z$ is the same as the time derivative of the height of the surface $h(x,y,t)$, so that

$$\frac{dh}{dt} = \frac{\partial h}{\partial t} + (\nabla\phi \cdot \nabla)h = \frac{\partial\phi}{\partial z} \quad (2)$$

and Bernoulli's equation relates h to the derivatives of ϕ :

$$\frac{\partial\phi}{\partial t} + \frac{1}{2} (\nabla\phi)^2 = -gh \quad (3)$$

(2) The second boundary condition is the assumption that at large depths z approaches the imposed velocity \underline{u}

$$\underline{u}(x,y,z,t) \xrightarrow{z \rightarrow -\infty} \underline{u}(x,y,t)$$

or if ϕ is the velocity potential for \underline{u} so that

$$\phi(x,y,z,t) \xrightarrow{z \rightarrow -\infty} \phi(x,y,t)$$

(3) The final boundary condition is the initial condition that for times far in the past the imposed flow vanishes and the wave approaches a freely propagating wave ϕ_0 ,

$$\phi(x,y,z,t) \xrightarrow{t \rightarrow -\infty} \phi_0(x,y,z,t) \quad (4)$$

$$\phi(x,y,t) \xrightarrow{t \rightarrow -\infty} 0$$

The effect of the imposed flow on the propagation of surface waves is expected to be small since the velocity of the surface current is, in general, much smaller than that of the surface wave in open sea conditions. The hydrodynamic equations adumbrated above may therefore

be expanded in powers of \underline{U} and only the linear term retained. We begin by writing

$$\phi = \xi + \phi_0 + \phi_1 \quad (5)$$

where ϕ_0 is the velocity potential the initial wave would have had in the absence of the surface current and ϕ_1 is the correction due to its presence.

The equations governing ϕ_1 are obtained by substituting Eq. (5) in Eqs. (1) through (4) and retaining only terms linear in ϕ_1 and ξ . Noting that ξ and ϕ_0 separately satisfy Eqs. (1) through (4), we have

$$\nabla^2 \phi_1 = 0 \quad (6)$$

with the boundary conditions

$$\phi_1(x, y, z, t) \xrightarrow[t \rightarrow -\infty]{} 0 \quad (7)$$

$$\phi_1(x, y, z, t) \xrightarrow[z \rightarrow -\infty]{} 0 \quad (8)$$

and

$$\begin{aligned} \frac{\partial^2 \phi_1}{\partial t^2} + g \frac{\partial \phi_1}{\partial z} + 2 \frac{\partial}{\partial t} [(\nabla \xi + \nabla \phi_1) \cdot \nabla \phi_0] + \frac{1}{2} (\nabla \phi_1 + \nabla \xi) \cdot \nabla (\nabla \phi_0)^2 \\ + \nabla \phi_0 \cdot \nabla [\nabla \phi_0 \cdot (\nabla \xi + \nabla \phi_1)] = 0 . \end{aligned}$$

As a consequence of the small amplitude assumption, terms which are quadratic in ϕ_0 are negligible in comparison with those linear in ϕ_0 . Further, since ϕ_1 is a small correction to ϕ_0 , terms like $\phi_1 \phi_0$ may be neglected in comparison with ϕ_1 .

The boundary condition at the surface then becomes

$$\frac{\partial^2 \phi_1}{\partial t^2} + g \frac{\partial \phi_1}{\partial t} = -2 \frac{\partial}{\partial t} [\nabla \xi \cdot \nabla \phi_0] = -2 \frac{\partial}{\partial t} [u \cdot \nabla \phi_0] . \quad (9)$$

The problem is now to solve Eq. (6) with the boundary conditions (7) to (9). The general solution of Laplace's equation (Eq. (6)) which satisfies the boundary condition (8) may be written

$$\phi_1(x, z, t) = \int \frac{d^2 k \, d\omega}{(2\pi)^3} e^{i(k \cdot x - \omega t)} e^{kz} a_1(k, \omega) .$$

Since ϕ_0 , the unperturbed flow, also satisfies Eqs. (6) and (8), it has a similar Fourier decomposition with Fourier transform $a_0(k, \omega)$. Here and in the following, \underline{x} will mean a two-dimensional vector in the xy plane. Equation (9) then becomes

$$a_1(k, \omega) = \frac{2i\omega}{gk - \omega^2} F(k, \omega)$$

where $F(k, \omega)$ is the Fourier transform of $(u \cdot \nabla \phi_0)_{z=0}$ given in terms of a_0 and the Fourier transform of u by

$$F(k, \omega) = i \int \frac{d^2 q \, d\nu}{(2\pi)^3} \underline{u}(k-q, \omega-\nu) \cdot \underline{g} a_0(q, \nu) . \quad (10)$$

This solves the problem of determining the perturbation ϕ_1 to a small amplitude surface wave ϕ_0 caused by an arbitrary surface current \underline{u} . The quantity of chief interest, however, is not the velocity potential ϕ but rather the height h . If we write δh for the change in height caused by the surface current, then we have from Eq. (3)

$$\delta h = -\frac{1}{g} \left[\frac{\partial \phi_1}{\partial t} + \underline{u} \cdot \nabla \phi_0 \right] .$$

Here we have retained only terms linear in u , have neglected terms in ϕ_0^2 , $\phi_0 \phi_1$, and have used the fact that there is no vertical displacement from the velocity potential ξ in accordance with assumptions (1) through (4). For $\partial \phi_1 / \partial t$ we have

$$\begin{aligned}
\frac{\partial \phi_1}{\partial t} &= \int \frac{d^2 k}{(2\pi)^3} \frac{d\omega}{(2\pi)} \left[\frac{2\omega^2}{gk - \omega^2} \right] F(k, \omega) e^{i(k \cdot x - \omega t)} \\
&= \int \frac{d^2 k}{(2\pi)^3} \frac{d\omega}{(2\pi)} \left[-2 + \frac{2gk}{gk - \omega^2} \right] F(k, \omega) e^{i(k \cdot x - \omega t)} \\
&= -2 \underline{u} \cdot \nabla \phi_0 + 2 \int \frac{d^2 k}{(2\pi)^3} \frac{d\omega}{(2\pi)} \frac{gk F(k, \omega)}{gk - \omega^2} e^{i(k \cdot x - \omega t)}.
\end{aligned}$$

Thus for δh we have

$$\delta h(x, t) = \frac{\underline{u} \cdot \nabla \phi_0}{g} + \int \frac{d^2 k}{(2\pi)^3} \frac{d\omega}{(2\pi)} \frac{2k}{\omega^2 - gk} F(k, \omega) e^{i(k \cdot x - \omega t)}$$

or in terms of the Fourier transforms of ϕ_0 and U (cf. Eq. (10))

$$\begin{aligned}
\delta h(x, t) &= \frac{\underline{u} \cdot \nabla \phi_0}{g} + i \int \frac{d^2 k}{(2\pi)^3} \frac{d\omega}{(2\pi)} \frac{2k}{\omega^2 - gk} \int \frac{d^2 q}{(2\pi)^3} \frac{d\nu}{(2\pi)} \underline{u}(k-q, \omega-\nu) \cdot \underline{g} \\
&\quad a_0(q \cdot \nu) e^{i(k \cdot x - \omega t)}. \tag{11}
\end{aligned}$$

In general, the nonlinear effects on the unperturbed wave will be of the order of magnitude

$$(\nabla \phi)^2 / \frac{\partial \phi}{\partial t} \sim \frac{h_{\max}}{2\pi\lambda}$$

typically a number like 1/50. These effects are therefore large compared with the effect of the surface current which might optimistically be of the order of several percent. For a calculation of the total wave, these nonlinear effects cannot be neglected. If, however, one is, as here, mainly interested in the change in the wave structure due to the surface current as calculated from Eq. (11), then the change arising from the linear part of the wave will be larger by the factor

($h_{\max}/2\pi\lambda$) than that coming from the nonlinear correction. To calculate δh from Eq. (11), we can therefore replace a_0 with the value appropriate for a plane wave with wave vector \underline{k}_0 , frequency $\omega_0 = (g k_0)^{1/2}$, and amplitude A :

$$a_0(\underline{k}, \omega) = (2\pi)^3 A \delta^{(2)}(\underline{k} - \underline{k}_0) \delta(\omega - \omega_0).$$

One finds, then, that

$$\delta h(\underline{x}, t) = \frac{\underline{U} \cdot \nabla \phi_0}{g} + i \int \frac{d^2 k}{(2\pi)^3} \frac{d\omega}{\omega^2 - gk} \frac{2kA}{\omega^2 - gk} \underline{U}(\underline{k} - \underline{k}_0, \omega - \omega_0) \cdot \underline{k}_0 e^{i\underline{k} \cdot \underline{x} - \omega t} \quad (12)$$

Writing

$$\underline{U}(\underline{k} - \underline{k}_0, \omega - \omega_0) = \int_{-\infty}^{+\infty} e^{i(\omega - \omega_0)t'} \underline{U}(\underline{k} - \underline{k}_0, t') dt',$$

the ω integration can be performed by evaluating

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega^2 - gk} = I(t-t'). \quad (13)$$

In order that ϕ_1 vanish for large negative times the poles in the denominator of the integrand in Eq. (13) must be displaced slightly into the lower-half-complex- ω -plane. One then has

$$I(t-t') = \begin{cases} \frac{i}{2\sqrt{kg}} [e^{-i\sqrt{kg}(t-t')} - e^{i\sqrt{kg}(t-t')}] & t < t' \\ \frac{i}{2\sqrt{kg}} [e^{-i\sqrt{kg}(t-t')} - e^{i\sqrt{kg}(t-t')}] & t > t' \end{cases}$$

Making these substitutions in Eq. (12) and displacing the \underline{k} integration by an amount \underline{k}_0 , one has

$$\delta h(\underline{x}, t) = \left\{ i \frac{\underline{k}_0 \cdot \underline{U}(\underline{x}, t)}{g} + \frac{1}{g} \int_{-\infty}^t J(\underline{x}, t, t') dt' \right\} A e^{i\underline{k}_0 \cdot \underline{x} - \omega_0 t}$$

where

$$J(x, t, t') = \int \frac{d^2k}{(2\pi)^2} \sqrt{g|k+k_0|} e^{+i\omega_0(t-t')} k_0 \cdot \underline{U}(k, t') e^{ik \cdot x}$$

$$\left[e^{-i\sqrt{g|k+k_0|}(t-t')} - e^{i\sqrt{g|k+k_0|}(t-t')} \right].$$

Now, the wavelength of the internal wave is much longer than that of the surface wave and $\underline{U}(k, t)$ will be sharply peaked about $k=0$. We may therefore expand $\sqrt{g|k+k_0|}$ about $k=0$.

$$\sqrt{g|k+k_0|} = \sqrt{g k_0} \left(1 + \frac{1}{2} \frac{k \cdot k_0}{k_0^2} + \dots \right).$$

Writing $c_g = \frac{1}{2} \frac{g}{k_0} (g/k_0)^{\frac{1}{2}}$ for the group velocity of the surface wave, we have

$$J(x, t, t') = \int \frac{d^2k}{(2\pi)^2} [\omega_0 + c_g \cdot k] k_0 \cdot \underline{U}(k, t')$$

$$\left[e^{ik \cdot [x - c_g(t-t')]} - e^{2i\omega_0(t-t')} e^{ik \cdot [x + c_g(t-t')]} \right].$$

Since the surface current varies slowly in time as well as space [assumption (4)], the second term will be small compared to the first. Therefore

$$\underline{J}(x, t, t') = [\omega_0 - i c_g \cdot \nabla] \int \frac{d^2k}{(2\pi)^2} k_0 \cdot \underline{U}(k, t') e^{ik \cdot [x - c_g(t-t')]}$$

$$= [\omega_0 - i c_g \cdot \nabla] k_0 \cdot \underline{U}(x - c_g(t-t'), t').$$

If we denote the height of the unperturbed wave by $h_0(x, t)$, so that

$$h_0(x,t) = + \frac{i A \omega_0}{g} e^{i(k_0 \cdot x - \omega_0 t)}$$

then we may write

$$\frac{\delta h}{h_0} = \frac{\hat{k}_0 \cdot \underline{u}(x,t)}{2 c_g} + \int_{-\infty}^t dt' [-i \hat{k}_0 - \frac{1}{2} \hat{k}_0 \cdot \nabla] \hat{k}_0 \cdot \underline{u}[x - c_g(t-t'), t'].$$

The height of a wave can change at a given point and at a given time not only because the amplitude of the wave changes, but also because the phase changes. Since it is the change in amplitude which is of chief interest as far as the identification of the current is concerned, it is important to separate these two effects.

The general wave can be written

$$h(x,t) = A(x,t) e^{iX(x,t)}$$

where A and X are real. For small δA , δX perturbations away from unperturbed values A_0 and X_0 we have

$$\frac{\delta h}{h_0} = \frac{\delta A}{A_0} + i \delta X$$

Thus

$$\frac{\delta A}{A} = \text{Re} \left(\frac{\delta h}{h} \right) = \frac{\hat{k}_0 \cdot \underline{u}(x,t)}{2 c_g} - \frac{1}{2} \hat{k}_0 \cdot \nabla \int_{-\infty}^t \hat{k}_0 \cdot \underline{u}[x - c_g(t-t'), t'] dt'$$

$$\text{and } \delta X = \text{Im} \left(\frac{\delta h}{h} \right) = - \hat{k}_0 \cdot \int_{-\infty}^t \underline{u}(x - c_g(t-t'), t') dt'$$

The first term in the amplitude enhancement is an instantaneous effect and very small. The second term is a time integrated effect and depends on the gradient of the flow.

In addition to the change in amplitude, the change in wavelength, frequency, and mean square slope are also of interest. The change in the wave number δk and the change in the frequency $\delta\omega$ may be obtained from the change in phase δX through the relations

$$\delta k = \nabla(\delta X), \quad \delta\omega = -\partial(\delta X)/\partial t.$$

The mean square slope in m^2 is given by the time average of $(\nabla h)^2$. The time average is taken over a period of the surface wave, a time which is short compared with the characteristic variation time of the surface current. In this case the time average may be expressed in terms of the complex waves by

$$m^2 = \frac{1}{2} \nabla h \cdot \nabla h^*.$$

In terms of the change in amplitude and phase, one can then easily find for the change in mean square slope

$$\frac{\delta m^2}{m_0^2} = \left(2 \frac{\delta A}{A_0} + \frac{k_0 \cdot \nabla \delta X}{k_0^2} \right), \quad (14)$$

or, since

$$k_0 \cdot \nabla \delta X = k_0 \cdot \delta k = k_0 \delta k,$$

this can be written as

$$\frac{\delta m^2}{m_0^2} = 2 \frac{\delta(kA)}{k_0 A_0}.$$

Inserting expressions for δX and $\delta A/A$ into Eq. (14), one has, finally,

$$\frac{\delta m^2}{m_0^2} = \frac{k_0 \cdot \underline{u}(x, t)}{c_g} - 3 (k_0 \cdot \nabla) \int_{-\infty}^t \underline{u}[x - c_g(t-t'), t'] dt'.$$

In order to investigate the magnitude of possible enhancements of amplitude and slope, let us assume that the surface flow \underline{u} has the form of a wave propagating with a phase velocity C in a direction specified by a unit vector \underline{n} :

$$\underline{u}(\underline{x}, t) = \underline{u}(\underline{n} \cdot \underline{x} - Ct)$$

In this case we have

$$\frac{\delta A}{A} = \frac{\underline{k}_0 \cdot \underline{u}(\underline{x}, t)}{2 c_g} - \frac{1}{2} (\underline{k}_0 \cdot \underline{v}) \int_{-\infty}^t dt' \underline{k}_0 \cdot \underline{u}[\underline{n} \cdot (\underline{x} - \underline{c}_g t) + (\underline{n} \cdot \underline{c}_g - C)t']$$

and a similar expression for the mean square slope change. Suppose now we follow a crest in the internal wave which for simplicity we assume to occur when the phase of \underline{u} vanishes. Then \underline{x} is related to t by

$$\underline{x} = C t \underline{n}$$

and we have

$$\frac{\delta A}{A} = \frac{\underline{k}_0 \cdot \underline{u}(0)}{2 c_g} - \frac{1}{2} (\underline{k}_0 \cdot \underline{n}) \int_{-\infty}^t \underline{k}_0 \cdot \underline{u}'[(C - \underline{c}_g \cdot \underline{n})(t-t')] dt',$$

where \underline{u}' denotes the derivative of \underline{u} with respect to its argument.

Surface waves which have these components of the group velocity in the direction of propagation of the internal wave equal to the internal waves's phase velocity may experience a large enhancement from the second term. For these waves

$$\underline{c}_g \cdot \underline{n} = C$$

and the enhancements in slope and amplitude are

$$\frac{\delta m^2}{m^2} = \frac{\hat{k}_o \cdot \underline{U}(0)}{c_g} - 3 (\hat{k}_o \cdot \hat{n}) (\hat{k}_o \cdot \underline{U}(0)) T,$$

and

$$\frac{\delta A}{A} = \frac{\hat{k}_o \cdot U(0)}{2 c_g} - \frac{1}{2} (\hat{k}_o \cdot \hat{n}) (\hat{k}_o \cdot U(0)) T$$

where T is the time that the surface and internal waves have been interacting. If T is long enough this time-integrated effect will be appreciable and will dominate the instantaneous first term.

As a particular example, let us take for \underline{U} a sine wave with wave number \underline{k} , frequency Ω , and the surface current in the direction of propagation.

$$\underline{U} = \hat{k} U_o \sin (\underline{k} \cdot \underline{x} - \Omega t).$$

If we denote by θ the angle between \hat{k} and \hat{k}_o , then $\delta A/A$ and $\delta m^2/m^2$ may be written

$$\frac{\delta m^2}{m^2} = \frac{U_o \cos \theta}{c_g} - 3 (TKU_o) \cos^2 \theta \left[\frac{\sin T K(c_g \cos \theta - C)}{T K(c_g \cos \theta - C)} \right].$$

and

(15)

$$\frac{\delta A}{A} = \frac{U_o \cos \theta}{2 c_g} - \frac{1}{2} (TKU_o) \cos^2 \theta \left[\frac{\sin T K(c_g \cos \theta - C)}{T K(c_g \cos \theta - C)} \right]$$

The most favorable case is for waves traveling in the same direction as the internal waves with $c_g = C$. For these waves, the dominant effect is

$$\frac{\delta A}{A} = -\frac{1}{2} TKU_o \quad \frac{\delta m^2}{m^2} = -3 TKU_o.$$

For waves which travel at an angle with respect to the internal wave but have their component of c_g in the internal wave direction equal to C , there is the same enhancement decreased by a factor $\cos^2\theta$.

Waves whose velocity component in the direction of internal wave propagation is greater or less than C will eventually pull ahead or lag behind the internal wave. This is indicated mathematically by the decrease of the bracketed factor in Eq. (15) for large T if $c_g \cdot \cos \theta \neq C$.

There will thus be a strong time-integrated amplitude and slope enhancement for the special class of waves which ride along with the internal wave. This effect is proportional to the gradient of the surface current and to the time of interaction. This time, in turn, will at best be the minimum of the characteristic times of decrease of the internal wave and the surface wave due to dissipative effects. If the lesser of these times is long enough, there may be an appreciable enhancement.

SUPPORTING ANALYSIS D

RADAR SCATTERING FROM THE OCEAN SURFACE

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I. INTRODUCTION

We shall be concerned with the scattering of electromagnetic radiation from the ocean surface, with emphasis on frequencies in the microwave region. In order to simplify our calculations we have chosen to study a model in which the electromagnetic wave is taken to be a scalar field. We believe that such a model brings out all the essential physical phenomena involved in the scattering process, except possibly in scattering from the sea at low angles of elevation. When the scattering occurs near Brewster's angle, the scattering of vertically polarized waves should be strongly suppressed, a phenomenon which our scalar model cannot reproduce. In any case our approximation scheme for studying scalar waves breaks down at small angles because of the phenomena of shadowing and multiple reflection.

The actual extension of the theory presented here to the full problem of vector electromagnetic waves is perfectly straightforward. The resulting formulas for polarization, etc. will be given in a future, more detailed report. In the following discussion we also ignore Doppler effects arising from the fact that the ocean surface is constantly in motion. These effects, which are not believed to be important in the present context, will also be treated in the later report.

II. SCATTERING FROM THE OCEAN SURFACE

At any instant the air-water interface is given by the surface $z = h(x,y)$, the origin being so chosen that if we average over many instants, $\langle h(x,y) \rangle = 0$. We assume that the illuminating radar pulse is of such a short duration that during the time it is actually incident on the sea surface, $h(x,y)$ changes by a negligible amount. Since radar pulses are, typically, microseconds long, this approximation should be excellent. Our problem, then, divides into two parts: (a) given a plane electromagnetic wave, of wave vector k , incident on a fixed sea surface $z = h(x,y)$, find the wave scattered in any direction and (b) find the average power scattered in any direction where the average is taken over many values of $h(x,y)$ corresponding either to many different times or many illuminated patches on the sea surface. Since we are neglecting instantaneous motions of the sea surface, we can say nothing about possible doppler shifting of the frequency of the scattered wave.

For frequencies in the microwave region, the index of refraction of sea water is well represented by

$$n = n_0 (1 + i\sigma/\omega n_0^2 \epsilon_0)$$

where $n_0 = 80$, $\sigma = 3\text{mhos/meter}$, and ϵ_0 is the dielectric constant of vacuum. The quantity $\sigma/\omega n_0^2 \epsilon_0 = 10^7/\nu$ describes the relative importance of conduction and displacement currents in the equation of motion. At microwave frequencies, $\nu \approx 10^{10}$ cps, and the imaginary part of the index of refraction is totally negligible. Therefore we may safely think of our problem as that of computing the scattering of electromagnetic waves from the interface between two purely dielectric media with indices of refraction 1 and 80 respectively.

Boundary value problems of this kind can, unfortunately, be solved exactly only if the boundary surface is quite simple: plane, elliptic, etc. Therefore we are forced to resort to approximate methods. In general, there are two sorts of boundary which admit simple approximate solutions. In the first case, suppose that the radius of curvature of the boundary surface is everywhere large compared to the wavelength of the incident radiation. We can then apply geometrical optics to compute the intensity of scattered radiation. In the second case, suppose that the deviation, Δ , of the surface from one which has a known solution is everywhere small compared to the radar wave length λ . Then, by a simple form of perturbation theory, the scattering can be computed correct to order Δ/λ (we will shortly show how this is done). Therefore if the sea surface $h(x,y)$ can be written as $h = h_0 + h_1$, where h_1 is everywhere small compared to λ , and the radius of curvature of h_0 is everywhere large compared to λ , we can combine the above two approximation methods to get a decent solution. Whether or not this can be done clearly depends on the detailed nature of the sea surface.

At any instant the sea surface, $h(x)$, can be written as a Fourier integral $h(\vec{x}) = \int d\vec{k} a(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$. We can obviously make the decomposition $h = h_0 + h_1$ where

$$h_0 = \int_{k < k_c} d\vec{k} a(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$$

$$h_1 = \int_{k > k_c} d\vec{k} a(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$$

and k_c is for the moment arbitrary. We can then show that if k_c is properly chosen the surface h_0 has a mean radius of curvature which is large compared with the radar wavelength, while the mean magnitude of h_1 is everywhere small compared to the radar wavelength. Since the sea surface is a random process we can really talk only about the

mean values of the coefficients $a(\vec{k})$. All such information is contained in the correlation function

$$\begin{aligned}\rho(\vec{x}) &= \langle h(\vec{x})h(0) \rangle \\ &= \int d\vec{k} A(\vec{k}) e^{i\vec{k}\cdot\vec{x}}\end{aligned}$$

which has been determined experimentally to have the form

$$\begin{aligned}\rho(\vec{x}) &= C \int_{k_1}^{k_2} d\vec{k} \frac{1}{k^4} e^{i\vec{k}\cdot\vec{x}} \\ C &= \frac{2}{\pi} \times 10^{-3}.\end{aligned}$$

The cutoff k_1 corresponds to the gravity waves moving with the wind velocity and the cutoff k_2 corresponds to very short (say, mm) capillary waves. The corresponding functions for the surfaces h_0 and h_1 are then

$$\begin{aligned}\rho_0(\vec{x}) = \langle h_0(\vec{x})h_0(0) \rangle &= C \int_{k_1}^{k_c} d\vec{k} \frac{1}{k^4} e^{i\vec{k}\cdot\vec{x}} \\ \rho_1(x) = \langle h_1(\vec{x})h_1(0) \rangle &= C \int_{k_c}^{k_2} d\vec{k} \frac{1}{k^4} e^{i\vec{k}\cdot\vec{x}}.\end{aligned}$$

The mean square height H^2 of the surface h_1 is then

$$H^2 = \rho_1(0) = 2\pi C \int_{k_c}^{k_2} dk/k^3 = \pi C (k_c^{-2} - k_2^{-2}) = 2 \times 10^{-3} k_c^{-2}$$

and the mean square radius of curvature, R^2 , of the surface h_0 is given by

$$\begin{aligned}
1/R^2 &= (\nabla^2)^2 \rho_c(x) \\
&= 2\pi C \int_{k_1}^{k_c} k dk \\
&= \pi C (k_c^2 - k_1^2) = 2 \times 10^{-3} k_c^2.
\end{aligned}$$

We should like to satisfy simultaneously the conditions $H/\lambda \ll 1$ and $\lambda/R \ll 1$, where λ is the radar wavelength. According to the above equations $H/\lambda = 0.007 \lambda_c/\lambda$, and $\lambda/R = 0.29 \lambda/\lambda_c$, so that if we choose $\lambda_c \approx 6\lambda$, we have $H/\lambda \approx \lambda/R \approx 0.05$. With such small values for the expansion parameters, we feel safe in computing the scattering from the surface h_0 by geometrical optics and in computing the extra effect of the surface h_1 by perturbation theory. We emphasize that our ability to make both expansion parameters small simultaneously is a stroke of good luck depending on the detailed statistical structure of ocean waves. It probably is not possible for other sorts of random surface.

We now have to show how this approximate calculation is carried out in detail. In order to demonstrate the ideas involved we study the scattering of a plane scalar wave from a surface $h(x,y)$ which can be decomposed into two surfaces h_0 and h_1 in the manner just described. Once we have solved this problem it is not hard to fold in the complications due to the vector nature of the electromagnetic field.

In a medium of varying dielectric constant $n(\vec{x})$, we assume the wave function ψ to satisfy

$$[\nabla^2 + k^2 n(\vec{x})] \psi(\vec{x}) = 0, \quad k = \omega/c.$$

This means that if $n(\vec{x})$ has a discontinuity on a surface, then ψ and its normal derivative must be continuous across that surface. In the

case at hand, $n(\bar{x})$ takes on either of two constant values 1, or n , the dielectric constant of seawater, jumping from one value to the other at the surface $z = h(x,y)$. We then want to find the solution to this equation when a unit amplitude plane wave, $\psi_{in} = e^{i\bar{k}\cdot\bar{x}}$, $\bar{k} = k(\cos\theta, 0, -\sin\theta)$ is incident on the sea surface from above. (See Fig. 1).

Corresponding to the division of the surface $z = h(x,y)$ into a part, $h_0(x,y)$, with small curvature, and a part with small amplitude, $h_1(x,y)$, we can write $n(x) = n_0(x) + n_1(x)$. $n_0(x)$ takes on the values n and 1 and describes the air-sea interface $z = h_0(x,y)$. $n_1(x)$ takes on the values 0, $\pm(n-1)$ and is nonzero only in a small region around the surface $z = h_0(x,y)$ as is described in Fig. 2.

Let us suppose that the solution to the scattering problem for the surface $z = h_0(x,y)$ is known and let it be called ψ_0 . Let us also define $\delta\psi$ by $\psi = \psi_0 + \delta\psi$ where ψ is the desired solution for the surface $z = (h_0 + h_1)(x,y)$. We can combine the two equations

$$[\nabla^2 + k^2 n_0(\bar{x})] \psi_0 = 0$$

$$[\nabla^2 + k^2 (n_0(x) + n_1(x))] (\psi_0 + \delta\psi) = 0$$

to give

$$[\nabla^2 + k^2 n_0(\bar{x})] \delta\psi = -k^2 n_1(\bar{x}) \psi_0.$$

This equation, in turn, can be put into integral form if we introduce the Green's function, $G_0(\bar{x}, \bar{x}')$, which is a solution of

$$(\nabla_{\bar{x}}^2 + k^2 n_0(\bar{x})) G_0(\bar{x}, \bar{x}') = \delta(\bar{x} - \bar{x}').$$

This allows us to write

$$\delta\psi(\bar{x}) = -k^2 \int dx' G_0(\bar{x}, \bar{x}') n_1(\bar{x}') \psi_0(\bar{x}').$$

Since n_1 is nonzero only in a volume which goes to zero as h_1 goes to zero, $\delta\psi$ is of order h_1 . To this order, therefore, we can replace ψ , within the integral, by ψ_0 :

$$-\delta\psi(x) = -k^2 \int d\bar{x}' G_0(\bar{x}, \bar{x}') n_1(\bar{x}') \psi_0(\bar{x}').$$

To the same order, we can actually replace the volume integral by a surface integral over the surface $z = h_0(x, y)$. The equations of motion satisfied by ψ_0 and G_0 imply that at this boundary surface, both these functions and their normal derivatives are continuous. Therefore the effect on the integral of their variation over the small volume in which n_1 is nonzero is higher than first order in h_1 . We can therefore write

$$\delta\psi(\bar{x}) = -k^2(n-1) \int dS G_0(\bar{x}, \bar{x}'(S)) \bar{n}_1(x'(S)) \psi_0(\bar{x}'(S))$$

where the surface integral is taken over $z = h_0(x, y)$, $\bar{x}'(S)$ is the three-dimensional position vector of the element of surface, and \bar{n}_1 is the normal distance between the surface $z = h_0$ and $z = h_0 + h_1$ (taken positive or negative according as $z = h_0 + h_1$ lies above or below $z = h_0$). Finally, it is convenient to convert this into an integral over the plane surface $z = 0$, taking $\bar{\rho} = (x, y)$ to be the position vector in that surface

$$\delta\psi(\bar{x}) = -k^2(n-1) \int d\bar{\rho} h_1(\bar{\rho}) G_0(\bar{x}, \bar{x}(\bar{\rho})) \psi_0(\bar{x}(\bar{\rho})),$$

where $\bar{x}(\bar{\rho}) = (\bar{\rho}, h_0(\bar{\rho}))$ and h_1 is exactly the quantity earlier called h_1 , the vertical distance between the two surfaces $z = h_0$ and $z = h_0 + h_1$. The geometry of the transformation is best explained by Fig. 3. Therefore, if we know ψ_0 and G_0 on the surface $z = h_0$, we can calculate $\delta\psi$, correct to order h_1 .

According to our assumption, the radius of curvature of h_0 is everywhere large compared to the wavelength of the illuminating radiation, so that scattering can safely be computed via geometrical optics.

In particular, we need to know ψ_0 on the surface $z = h_0$. Geometrical optics means that, if we neglect multiple scattering and shadowing, the field at a point on the surface can be computed by replacing the curved surface by its local tangent plane and imagining the given incident wave to be scattering from it. It is easy to show that if the incident wave is $e^{i\mathbf{k}\cdot\bar{\mathbf{x}}}$, the total field at a plane boundary between regions with dielectric constants n and 1 is

$$\psi = \frac{2\cos\alpha}{\cos\alpha + \sqrt{n-\sin^2\alpha}} e^{i\mathbf{k}\cdot\bar{\mathbf{x}}}$$

where $\cos\alpha = \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}$, $\hat{\mathbf{n}}$ being the unit normal to the boundary.

We also need to know $G_0(\bar{\mathbf{x}},\bar{\mathbf{x}}')$ with $\bar{\mathbf{x}}'$ on the surface. From the equation satisfied by G_0 , it is clear that $G_0(\bar{\mathbf{x}},\bar{\mathbf{x}}')$ represents the total field generated at $\bar{\mathbf{x}}'$ by placing a unit source at $\bar{\mathbf{x}}$, given the boundary specified by $n_0(\bar{\mathbf{x}})$. If $\bar{\mathbf{x}}'$ is near the surface, and if we neglect multiple scattering and shadowing, the geometrical optics approximation to G_0 is gotten by replacing the curved surface by its local tangent plane. The solution for G_0 in the presence of a plane boundary is well known. If we set $\bar{\mathbf{x}} = \hat{\mathbf{k}}'R$, take $\bar{\mathbf{x}}$ on the boundary, and let $R \rightarrow \infty$, it becomes

$$G_0(\hat{\mathbf{k}}'R, \bar{\mathbf{x}}) = -\frac{e^{-i\mathbf{k}\hat{\mathbf{k}}'\cdot\bar{\mathbf{x}}}}{4\pi R} \frac{2\cos\alpha'}{\cos\alpha' + \sqrt{n - \sin^2\alpha'}}$$

α' being the angle between $\hat{\mathbf{k}}'$ and the local surface normal.

We now can write down our expression for the field $\delta\psi(\bar{\mathbf{x}})$ when $\bar{\mathbf{x}}$ is far away from the surface:

$$R\delta\psi(\hat{\mathbf{k}}'R) = -\frac{k^2(n-1)}{4\pi} \int d\bar{\rho} h_1(\bar{\rho}) T(\alpha(\bar{\rho})) T(\alpha'(\bar{\rho})) e^{i\mathbf{k}(\hat{\mathbf{k}}-\hat{\mathbf{k}}')\cdot\bar{\mathbf{x}}(\bar{\rho})}$$

where

$$T(\alpha) = 2\cos\alpha / (\cos\alpha + \sqrt{n-\sin^2\alpha}) .$$

We are particularly interested in the field scattered back along the direction of the incident beam, in which case $\hat{k}' = -\hat{k}$, and

$$R\delta\psi(-\hat{k}R) = \frac{k^2(n-1)}{4\pi} \int d\bar{\rho} h_1(\bar{\rho}) T^2(\alpha(\bar{\rho})) e^{2i\bar{k}\cdot\bar{x}(\bar{\rho})}.$$

It turns out to be convenient, for purposes of computing the average backscattered power, to recast the expression for $\delta\psi$ in a slightly different form. First of all, we note that the reflection coefficient is a function of $\cos\alpha = -\hat{n}\cdot\hat{k}$ where \hat{n} is the local surface normal. In turn, \hat{n} is a simple function of $\bar{\nabla}_{\bar{\rho}} h_0(\bar{\rho})$, so that we can write $T = T(\bar{\nabla}_{\bar{\rho}} h_0(\bar{\rho}))$. If we introduce the Fourier decomposition of $h_1(\bar{\rho})$, $\bar{h}_1(\bar{\ell})$, we then have

$$\delta\psi(-\hat{k}R) = \frac{k^2(n-1)}{4\pi R} \int d\bar{\ell} \bar{h}_1(\bar{\ell}) \int d\bar{\rho} T^2(\bar{\nabla}_{\bar{\rho}} h_0(\bar{\rho})) e^{i[(2\bar{k}+\bar{\ell})\cdot\bar{\rho} - 2k\sin\theta h_0(\bar{\rho})]}$$

Since the surface $h_0(\bar{\rho})$ is one which satisfies the criteria of geometric optics, we can evaluate the integral over $\bar{\rho}$ by the method of stationary phase. This means that the only important contributions come from those points $\bar{\rho}$ where

$$\bar{\nabla}_{\bar{\rho}} ((2\bar{k}+\bar{\ell})\cdot\bar{\rho} - 2k\sin\theta h_0(\bar{\rho})) = 0$$

or

$$\bar{\nabla}_{\bar{\rho}} h_0(\bar{\rho}) = (2\bar{k}+\bar{\ell})/2k\sin\theta.$$

Therefore we have

$$\delta\psi(-\hat{k}R) \approx \frac{k^2(n-1)}{4\pi R} \int d\bar{\ell} \bar{h}_1(\bar{\ell}) T^2\left(\frac{2\bar{k}+\bar{\ell}}{2k\sin\theta}\right) \times \int d\bar{\rho} \exp[i((2\bar{k}+\bar{\ell})\cdot\bar{\rho} - 2k\sin\theta h_0)].$$

The virtue of this expression is that the arguments of T no longer depend on the specific surface, so that the averaging process is simplified.

To compute the backscattered power we need $\langle |\psi_0^s + \delta\psi|^2 \rangle$, where ψ_0^s is the backscattered field from the surface h_0 , and the average is over the various possible forms of the sea surface. Since different Fourier coefficients of the sea surface are statistically independent, and since $\delta\psi$ depends on h_1 while ψ_0^s does not, the cross terms of the form $\psi_0^s \delta\psi^*$ vanish upon taking the average. Therefore, the average backscattered power is the sum of two terms, $\langle |\psi_0^s|^2 \rangle$ and $\langle |\delta\psi|^2 \rangle$, which we shall compute separately.

It is convenient to define a scattering cross-section in order to eliminate the distance of the observation point from the sea surface. The energy density at any point is just $|\psi|^2$. If a finite patch of sea surface, of area A , is illuminated and we observe at $\bar{x} = \hat{k}'R$, R very large, then all the energy at \bar{x} is flowing in the direction \hat{k}' . If the antenna subtends a solid angle $\Delta\Omega$, the total received power is then $|\psi|^2 R^2 \Delta\Omega$ and the received power per unit illuminated area is $|\psi|^2 R^2 \Delta\Omega/A$. We shall define the quantity $\sigma = |\psi|^2 R^2/A$, so that antenna power is $\sigma A \Delta\Omega$.

Let us first compute $\sigma_1 = \langle |\delta\psi|^2 \rangle R^2/A$. If we make the standard assumptions about the Gaussian nature of the sea surface, and make the definitions

$$\langle h_0(\bar{x})h_0(0) \rangle = C_0(\bar{x})$$

$$\langle \tilde{h}_1(\bar{z})\tilde{h}_1^*(\bar{z}') \rangle = \bar{\rho}_1(\bar{z}) \delta(\bar{z}-\bar{z}')$$

we find that

$$\sigma_1 = (k^2(n-1)/4\pi)^2 \int d\bar{z} \bar{\rho}_1(\bar{z}) T^4\left(\frac{2\bar{k}+\bar{z}}{2k\sin\theta}\right) \times \\ \int d\bar{r} \exp[i(2\bar{k}+\bar{z})\cdot\bar{r} - (2k\sin\theta)^2(C_0(0)-C_0(\bar{r}))]$$

where we should now remember that when we write \vec{k} we mean $\vec{k} = (k_x, k_y) = (k \cos \theta, 0)$. The largest contribution to the integral over \vec{r} comes in the neighborhood of $\vec{r}=0$, where we write

$$C_0(r) = h^2(1 - ar^2 + \dots)$$

so that

$$\begin{aligned} & \int d\vec{r} \exp[i(2\vec{k} + \vec{l}) \cdot \vec{r} - (2k \sin \theta)^2 (C_0(0) - C_0(r))] \approx \\ & \int d\vec{r} \exp[i(2\vec{k} + \vec{l}) \cdot \vec{r} - (2k \sin \theta)^2 a r^2] = \\ & \frac{\pi}{(2k \sin \theta)^2} \exp[-(2\vec{k} + \vec{l})^2 / 4a(2k \sin \theta)^2] = f(2\vec{k} + \vec{l}) \end{aligned}$$

We note that in the limit $a \rightarrow 0$, $f(2\vec{k} + \vec{l}) \rightarrow (2\pi)^2 \delta(2\vec{k} + \vec{l})$. In fact a is rather small:

$$\begin{aligned} h^2 a &= -(1/2) \nabla^2 C_0(0) = (C/2) \int_{k_c}^{k_1} dk/k^2 = \pi C \log_e (k_1/k_c) \\ &= 2 \times 10^{-3} \log_e (\lambda_{\text{wind}}/6\lambda_{\text{radar}}) \end{aligned}$$

where λ_{wind} is the wavelength of those ocean waves whose velocity equals the wind velocity. For a 10 m/s wind and a 10 cm radar wavelength, we have $h^2 a = 0.9 \times 10^{-2}$. This means that in terms of the dimensionless variable $|2\vec{k} + \vec{l}| / 2k \sin \theta$ the width of f is about 0.2 in a typical situation. This width decreases slowly with wind velocity.

If we ignore the width of f , replacing it with a delta function, we have the simple formula

$$\sigma_1 = \frac{k^4 (n-1)^2}{4} T^4(0) \tilde{\rho}_1(-2\vec{k}).$$

If we include the effect of the wind broadening of f we see that σ_1 is proportional to the average value of $T^4 \tilde{\rho}_1(\mathcal{L})$ over a circle of radius $\approx 0.4k\sin\theta$ in \mathcal{L} -space, centered about $\mathcal{L} = -2\mathbf{k} = (-2k\cos\theta, 0)$. In the undisturbed ocean we know that $\tilde{\rho}_1(\mathcal{L}) = 2 \times 10^{-3} \mathcal{L}^{-4} / \pi$ and therefore could actually perform the average, if necessary. If we evaluate the zero-width approximation to σ_1 in the limit $n \rightarrow \infty$ (appropriate for the sea surface since $n = 80$), we get

$$\sigma_1 = (10^{-3}/2\pi) \tan^4 \theta$$

for the backscattering cross-section from the undisturbed sea.

We now must compute $\sigma_0 = \langle |\psi_0|^2 \rangle R^2/A$. Since ψ_0 is the field generated by that part of the sea surface for which the approximations of geometrical optics are correct we can adopt the classical results for scattering light from a Gaussianly rough surface:

$$\sigma_0 = (\pi h^2 \alpha (2\sin\theta)^4)^{-1} \exp(-\cot^2 \theta / \alpha h^2)$$

for backscattering. We note that σ_0 falls off exponentially as θ decreases from $\pi/2$. In fact we can easily see that for $\theta < 80^\circ$, σ_1 dominates σ_0 , while for $\theta \geq 80^\circ$ the reverse is true.

At this point we may reasonably summarize our results: We have found two basic regimes in radar backscattering; one occurs when the angle of elevation is large, nearly 90° , the other occurs for moderate elevation angles. In the first case, the backscattering cross-section is a function of αh^2 , while in the other it is determined by $\tilde{\rho}_1$, the ocean wave power spectrum, at some appropriate wave number. The quantity αh^2 , is just the mean square slope of the ocean waves, which in turn is an integral over the complete wave power spectrum. Therefore, the difference between the two regimes is that in one case we measure $\tilde{\rho}_1$, at a point in wave number space while in the other we measure what amounts to an average of $\tilde{\rho}_1$ over all wave number space. This distinction will turn out to be most important in the applications.

Finally, we would like to point out that the scattering of polarized radiation from the sea surface may be calculated by much the same methods, although the formulas are much more complicated. We shall refrain from writing them down here since nothing essentially new in the physics of radar backscattering is introduced.

III. THE EFFECT OF INTERNAL WAVES ON RADAR BACKSCATTERING

As far as the applications considered in this report are concerned, we need to know the effect of an internal wave, over and above the random background, on the radar return from the sea surface. The ocean wave heights are in general described by the power spectrum $F(\mathbf{k})$ where

$$\langle h(\bar{x})h(0) \rangle = \int d\mathbf{k} F(\mathbf{k}) e^{i\mathbf{k}\cdot\bar{x}}$$

with $F(\mathbf{k}) = \frac{2 \times 10^{-3}}{\pi} k^{-4}$ for the standard wind-generated sea. Hartle and Zachariassen have shown that if an internal wave of phase velocity C , wave length L , and maximum surface water velocity V_0 is present, then the power spectrum is changed by

$$\frac{\delta F(\mathbf{k})}{F(\mathbf{k})} = - \frac{T 2\pi \frac{V_0}{L} \cos^2 \phi}{L} \frac{\sin(2\pi TL^{-1}(C_g \cos \phi - C))}{2\pi TL^{-1}(C_g \cos \phi - C)}$$

where ϕ is the angle between \mathbf{k} and the direction of propagation of the internal wave, $C_g = \frac{1}{2}\sqrt{g/k}$ is the group velocity of the surface waves with wave number k , and T is the time during which these same surface waves have been acted on by the internal wave. It is convenient to introduce $J = TC/L$, which is just T measured in internal wave periods, and $\epsilon = V_0/C$ so that

$$\frac{\delta F(\mathbf{k})}{F(\mathbf{k})} = - \epsilon 2\pi J \cos^2 \phi \frac{\sin(2\pi J(C_g \cos \phi / C - 1))}{2\pi J(C_g \cos \phi / C - 1)}$$

In practical cases, ϵ turns out to be very small. We note that for small J , $\delta F/F$ is uniformly distributed over k -space, while for $J \gg 1$,

$\delta F/F$ is very sharply concentrated around the curve $C_g \cos \phi = C$. This turns out to have a profound effect on the size of the radar return for different values of J .

We recall that if the angle of elevation, θ , of the radar beam from the horizontal is not too large, the backscattered power is proportional to the average of $F(\vec{k})$ over a circle in \vec{k} -space centered at $\vec{k}_0 = (-2k_{\text{rad}} \cos \theta, 0)$ and with radius ak_{rad} , where a depends on wind velocity but might typically be 0.1. Let us suppose that we have cleverly chosen k_{rad} and θ so that \vec{k}_0 lies on the curve $C_g \cos \phi = C$. We now want to compute the ratio $\delta P/P$ where P is the radar return from the undisturbed ocean and $P+\delta P$ is the radar return from the ocean in the presence an internal wave. Let $\Omega(|\vec{k}-\vec{k}_0|)$ be 1 for $|\vec{k}-\vec{k}_0| < ak_{\text{rad}}$ and zero otherwise. Then

$$\frac{\delta P}{P} = \frac{\int d\vec{k} \delta F(\vec{k}) \Omega(|\vec{k}-\vec{k}_0|)}{\int d\vec{k} F(\vec{k}) \Omega(|\vec{k}-\vec{k}_0|)}.$$

There are two interesting regimes in which we want to calculate this ratio. First of all, if $2\pi J(C_g \cos \phi / C - 1)$ is small through the region where Ω is non-zero, we have $\delta F = -\epsilon 2\pi J \cos^2 \phi F$ and

$$\frac{\delta P}{P} \approx -\epsilon 2\pi J \cos^2 \phi.$$

On the other hand, if J is very large, $\delta F/F \rightarrow -\epsilon \cos^2 \phi \delta(C_g \cos \phi / C - 1)$. With our assumption that \vec{k}_0 is centered on the curve $C_g \cos \phi = C$, we have

$$\frac{\delta P}{P} = -\frac{2\epsilon \cos^2 \phi}{a}$$

Because a is small, this ratio can easily be as much as 10ϵ .

IV. STATISTICAL CONSIDERATIONS

Thus far, we have shown how to calculate the backscattered power averaged over a statistical ensemble of ocean surfaces. The question of just what "averaged over an ensemble" means in terms of physical measurements remains. This will be our next topic.

One way of performing an average is to look repeatedly at different pieces of the ocean surface which are far enough apart to be statistically independent. The minimum distance between two such pieces is determined, of course, by the correlation length of those properties of the ocean surface which are important in the scattering process. Since the specular part of the scattering cross-section depends only on the mean square slope, the correlation length relevant for specular reflections is clearly that for slopes, which turns out to be some tens of centimeters. For Bragg scattering (scattering from waves of a definite wavelength, the dominant process in backscattering at moderate elevation angles) the high-frequency part, h_1 , of the elevation determines the scattering. To find the correlation length relevant for Bragg scattering we must, therefore, study the correlation function

$$\rho_1(\bar{x}-\bar{y}) = \langle h_1(\bar{x})h_1(\bar{y}) \rangle = \frac{2 \times 10^{-3}}{\pi} \int_{k_c}^{k_2} d\bar{k} k^{-4} e^{i\bar{k} \cdot (\bar{x}-\bar{y})}$$

Because it contains a factor k^{-4} , the integral on the right is rapidly convergent and receives most of its contribution from the region $k_c \leq k \leq 2k_c$. When $|\bar{x}-\bar{y}|$ is small compared to $\lambda_c = 2\pi k_c^{-1}$, the exponential is essentially constant over this region and $\rho(\bar{x}-\bar{y})$ is of order $2 \times 10^{-3} k_c^{-2}$. However, if $|\bar{x}-\bar{y}|$ is large compared to λ_c , the exponential factor oscillates rapidly and the integral is very small.

Thus, we may take the correlation length for Bragg scattering to be something like λ_c which, as pointed out above, will be a few radar wavelengths.

Actually, the above correlation lengths are so small that they are of little interest except in very special cases. To see this we have to understand what happens when a radar looks at the ocean surface.

Suppose a radar illuminates an area A of the ocean surface. Referring to the above numbers, we see that in general A will be very large compared to the relevant coherence length squared. Imagine now, dividing A into patches whose linear dimension is of the order of a coherence length. We can write the backscattered wave as $\sum \psi_i$ where ψ_i is that part of the backscattered wave which comes from the i -th patch. Then setting $\psi_i = a_i e^{i\phi_i}$, we have

$$P = \sum_{i,j} a_i a_j e^{i(\phi_i - \phi_j)}$$

for the returned power. If we now average P over an ensemble of statistically independent areas A , the averaged power is

$$\langle P \rangle = \sum_{i,j} \langle a_i a_j e^{i(\phi_i - \phi_j)} \rangle = \sum_i \langle a_i^2 \rangle$$

which follows from the fact that the phases $e^{i\phi_i}$ are random.

There are now two questions: (i) What do we mean by independent areas? and (ii) How many areas are needed to determine $\langle P \rangle$ to a given accuracy? The answer to the first question is almost trivial. In order that the phases $e^{i\phi_i}$ be uncorrelated, the two areas must be non-overlapping. We are assuming that the time difference between measurements is less than the decorrelation time of the phases. To answer the second question, we need the variance of P . Here we appeal to the well-known fact that for a sum of terms with random phases, such as in the last equation, the variance is always of the same order as the square of the average, i.e.,

$$\sqrt{\langle (P - \langle P \rangle)^2 \rangle} = \langle P \rangle.$$

This means, of course, that to measure $\langle P \rangle$ to, say, one part in ten, we need one hundred separate areas.

Notice that the above conclusions are independent of the size of the area illuminated by the radar (so long as its linear dimensions are large compared to the correlation length). Thus, contrary to one's first impression, the accuracy of a measurement of P does not improve as the size of A increases. Also, it is clear that the magnitude of the small correlation lengths does not enter in a critical way.

This perhaps surprising situation arises because a radar is a coherent source of radiation. Suppose, on the contrary, that the source of radiation were incoherent. If this were the case, the equation given on the previous page for the backscattered power should be replaced by

$$\bar{r} = \sum_{i,j} a_i a_j e^{i[(\phi_i + \phi'_i) - (\phi_j + \phi'_j)]}$$

where ϕ'_i is the phase of the incident radiation, assumed to vary rapidly with time and index i (this is to incorporate the assumption that the incident radiation is incoherent). Averaging over a time long compared to the coherence time of the ϕ'_i gives

$$\langle P \rangle_{\text{time average}} = \sum_i a_i^2.$$

The point is now, that the average of $\sum_i a_i^2$ over an ensemble of areas A gives $\langle P \rangle$ as before, but the variance in $\sum_i a_i^2$ is not of order $\langle P \rangle$ but rather of order $\langle P \rangle (L^2/A)^{-1/2}$, where L is the larger of the coherence length of the radiation and the relevant coherence length of the ocean. Thus, for an incoherent source the accuracy of a measurement does increase with A .

The reader may wonder why, in the case of a coherent source, we did not average the power equation over time in order to obtain a result similar to that just described. The reason is that the time scale involved is vastly different. If an incoherent source has a coherence length L , its coherence time is very small, being on the order of L/c , where c is the velocity of light. With a coherent source the corresponding time is the coherence time of the ocean surface. This is on the order of L'/v , where L' is a coherence length for the ocean and v is a typical wave velocity. It is the large ratio $c/v \approx 10^9$ that makes coherent and incoherent sources so different.

Finally, it should be pointed out that measurement of Bragg scattering has some statistical properties which are different from those of specular reflections. Suppose, for example, that we make many measurements of specular reflection from a single patch of ocean surface using various wavelengths of incident radiation and angles of incidence, but completing all the measurements within one coherence time. The statistics of the measurement have not been improved in this case. No matter what angle or wavelength we use to measure specular reflection we are always measuring the same quantity, namely the mean square slope. Thus we might as well have carried out all the measurements at the same wavelength and angle, gaining no improvement in statistics. Bragg scattering is different, however. By carrying out the measurement at different angles and wavelengths we are measuring different Fourier coefficients of the correlation function ρ_1 . Since these Fourier components are statistically independent each measurement gives new information and the statistics can be improved. As an example, suppose we wish to measure $\rho_1(0)$ which is the Fourier component integrated over ℓ -space. According to our formulas in Section III, scattering at moderate elevation angles samples the Fourier components of ρ_ℓ over an area of order $k_{\text{radar}}^2 m^2$ in ℓ -space, where m^2 is the mean-square slope of the ocean surface. Since there are m^{-2} such areas available, we can make m^{-2} independent measurements whose sum (which gives $\rho_1(0)$) will have a variance of m^2 times the variance of a single measurement. Since m^2 is of order 10^{-2} this is a non-trivial increase in statistical accuracy.

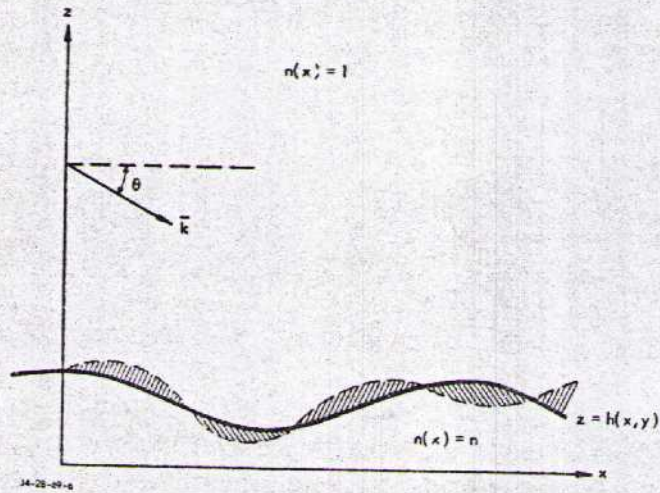


FIGURE 1. Scattering Configuration

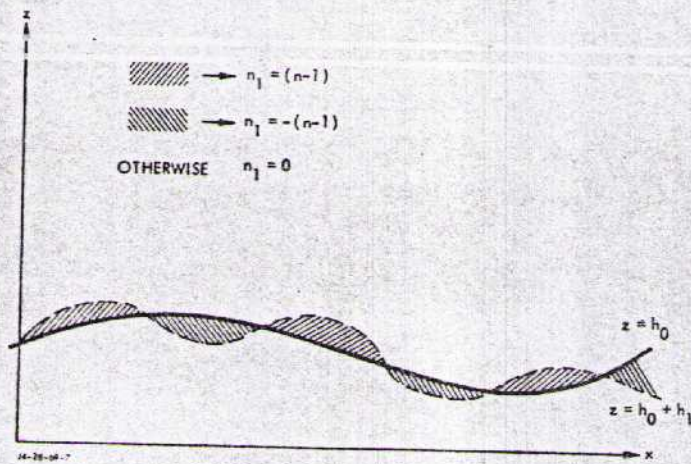


FIGURE 2. Partitioning of Scattering Configuration

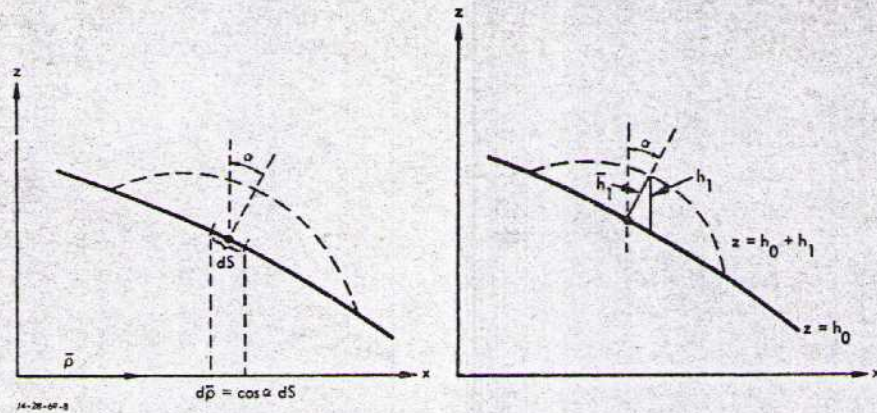


FIGURE 3. Transformation of Coordinates

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SUPPORTING ANALYSIS E

WAVELENGTH DEPENDENCE OF RADIOMETRIC SIGNALS

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To detect ripples on the surface of the sea by passive electromagnetic means several different wavelengths may be employed. The question naturally arises as to which wavelength range is most suitable. The answer depends not only on the physics of the interaction of electromagnetic waves with the surface of the ocean but also on the technology of detectors. In this section a comparison of the infrared and centimeter wavelength ranges will be made.

Several physical effects contribute to the detection of surface ripples by electromagnetic waves. Only one of these will be considered here. A detector pointed at the sea surface receives reflected radiation from different portions of the sky due to the presence of the ripple. Since the radiance of the sky varies with elevation, the presence of the ripples will lead to an average variation in the received radiance and the detection of the ripples.

Quantitatively, the spectral radiance (power per unit area per unit solid angle per unit wavelength) $W_{\text{det}}(\lambda, \underline{N}_d)$ received from a direction given by a unit vector \underline{N}_d at wavelength λ consists of two parts: (1) the reflected radiance of the sky at a direction \underline{N}_s related to \underline{N}_d by the law of reflection and (2) the emitted radiance of the sea itself. In terms of the spectral radiance of the sea W_{sea} , that of the sky W_{sky} , the reflectivity ρ of sea water, the zenith direction \underline{z} , and the normal to the sea surface \underline{N} we write

$$W_{\text{det}}(\lambda, \underline{N}_d) = \rho(\lambda, \underline{z} \cdot \underline{N}) W_{\text{sky}}(\lambda, \underline{z} \cdot \underline{N}_s) + [1 - \rho(\lambda, \underline{z} \cdot \underline{N})] W_{\text{sea}}(\lambda) \quad (1)$$

The angle of incidence is related to the angle of reflection by

$$\underline{N}_s = (-\underline{I} + \underline{N} \underline{N}) \underline{N}_d \quad (2)$$

where \underline{I} is the unit dyadic. For small sea slopes, \underline{N} may be written as $\underline{z} + \underline{\xi}$, Eq. (1) expanded in powers of $\underline{\xi}$ and averaged over the distribution of sea slopes. The averages of $\underline{\xi}$ and $\underline{\xi\xi}$ may be expressed in terms of the r.m.s. sea slope, m , by

$$\begin{aligned} \langle \underline{\xi} \rangle &= -m^2 \underline{z} \\ \langle \underline{\xi\xi} \rangle &= \frac{1}{2} m^2 [\underline{I} - \underline{z}\underline{z}] \end{aligned} \quad (3)$$

Expressed in terms of the zenith angle θ we then have (see Appendix) for the average variation in received radiance

$$\begin{aligned} \delta W_{\text{det}}(\lambda, \theta) &= m^2 \left[\rho \left(\frac{\partial^2 W_{\text{sky}}}{\partial \theta^2} + \cot \theta \frac{\partial W_{\text{sky}}}{\partial \theta} \right) + \frac{\partial W_{\text{sky}}}{\partial \theta} \frac{\partial \rho}{\partial \theta} \right. \\ &\quad \left. + k (W_{\text{sky}} - W_{\text{sea}}) \left(\frac{\partial^2 \rho}{\partial \theta^2} + \cot \theta \frac{\partial \rho}{\partial \theta} \right) \right] \end{aligned} \quad (4)$$

The sky radiance contrast arises from two sources--absorption and elastic scattering. Elastic scattering is important only for wavelengths $\leq 2\mu$ because of the wavelength dependence of the elastic cross section and the size distribution of the scattering particles. In that region, the sky contrast arises because on the average the reflected radiation originates one mean free path length away and there are more scatterers at low elevations near the horizon than at high elevations near the zenith. This effect can lead to strong radiance contrasts at low elevation angles (see Fig. 1) of the order

$$\begin{aligned} (\text{elevation angle}) &\leq \left(\frac{\text{scale height}}{\text{for scatterer}} \right) / (\text{mean free path}) \\ &\leq 1 \text{ km} / 10 \text{ km} \quad (1\mu) \end{aligned}$$

Observations at such small angles from airplanes are different because at typical airplane heights the observation distance is comparable with the mean free path. We will not consider the sky constant from elastic scattering further.

The important factor contributing to the sky radiance contrast is absorption. On the average the reflected radiance originates one attenuation distance $L(\lambda)$ away at a height $L(\lambda) \sin$ (elevation angle). Since the temperature of the atmosphere and hence the radiance varies with height this will lead to a greater radiance at the horizon than at zenith. The resulting contrast will be small for those wavelengths where the absorption is large and large when the absorption is small.

Figure 2 shows the experimental sky radiance for various angles in the infrared range. No contrast is observed in regions of strong absorption (e.g., 5μ and 15μ) while the maximum contrast is obtained in the region about 10μ . In figure 3 the sky radiance (and its equivalent temperature) from this data at 10μ is plotted as a function of angle. Also plotted on the same graph is the spectral radiance for 1.54 cm normalized to the same height at $\theta = 90^\circ$. Several features are clear. Because atmospheric absorption is stronger at 10μ than 1.5 cm the temperature contrast is smaller in the infrared region than in the microwave. However, because the dependence of radiance on temperature in the infrared is exponential ($hc/\lambda kT \sim 5$) while in the microwave it is linear ($hc/\lambda kT \sim 1/50$), there is not a great difference in the radiance contrasts.

These curves are the first elements which enter into a calculation of $\delta W_{\text{det}}/W_{\text{det}}$. The second element is the reflectivity. This is estimated from the standard Fresnel formulae. For the $\lambda = 1.5$ cm the curves of these quantities are already in hand in Fig. 4

The largest value of δW_{det} is obtained at high angles. At a typical large angle of $\theta = 75^\circ$ we find by crudely estimating the derivatives of these curves

$$\frac{\delta W_{\text{det}}}{W_{\text{det}}} \approx 0.6 m^2, m \text{ in radians, } \lambda = 1.5 \text{ cm.}$$

Taking a temperature resolution of 0.2°K we have approximately for $\lambda = 1.54$ cm.

$$\frac{\delta W_{\text{det}}}{\delta W_{\text{inst}}} \approx 100 \text{ m}^2, \text{ m in radians}$$

where δW_{inst} is the instrumental radiance resolution.

For the 10μ wavelength the index of refraction was computed and plotted in Fig. 4 using an index of refraction of 1.3. The curves of W_{det} for horizontal and vertical polarizations are given in Fig. 5. Crudely estimating Eq. (4) for $\theta = 75^\circ$ one finds

$$\frac{\delta W_{\text{det}}}{W_{\text{det}}} \approx 0.6 \text{ m}^2, \text{ m in radians}, \lambda = 10\mu$$

The similarity of this number with that obtained for $\lambda = 1.54 \text{ cm}$ reflects the similarity of the sky contrasts at the two wavelengths.

If we take $\delta T = 0.01^\circ \text{K}$ for the temperature resolution in the infrared we find

$$\frac{\delta W_{\text{inst}}}{W_{\text{det}}} = \left(\frac{hc}{\lambda kT_{\text{det}}} \right) \frac{\delta T}{T_{\text{det}}} \approx 2 \times 10^{-1}$$

Thus for $\lambda = 10\mu$

$$\frac{\delta W_{\text{det}}}{\delta W_{\text{inst}}} \approx 3000 \text{ m}^2, \text{ m in radians.}$$

The conclusion is then that the infrared is favored over microwave radiometer by roughly a factor of 30. The basic reason for this is that the window at 10μ is sufficiently transparent that the sky radiance contrasts are almost the same at the two wavelengths, while the resolution of the infrared is better by roughly a factor of 30.

APPENDIX

DERIVATION OF EQUATION (4)

Let \underline{n} = normal to sea surface
 \underline{n}_o = direction to observer
 \underline{n}_s = direction seen on sky

Snell's law is expressed by

$$\underline{n} \times \underline{n}_o = - \underline{n} \times \underline{n}_s$$

Forming the vector product of both sides with \underline{n} ,

$$\underline{n} \times (\underline{n} \times \underline{n}_o) = - \underline{n} \times (\underline{n} \times \underline{n}_s)$$

whence

$$\underline{n}(\underline{n} \cdot \underline{n}_o) - \underline{n}_o = - \underline{n}(\underline{n} \cdot \underline{n}_s) + \underline{n}_s$$

Since

$$\underline{n} \cdot \underline{n}_o = \underline{n} \cdot \underline{n}_s$$

we have

$$\underline{n}_s = - \underline{n}_o + 2\underline{n}(\underline{n} \cdot \underline{n}_o) = (-\underline{I} + 2\underline{nn}) \cdot \underline{n}_o$$

The detected radiance is equal to the incident radiance

$$W_{\text{det}} = W_{\text{sky}} (\underline{z} \cdot \underline{n}_s)$$

where \underline{z} is the zenith vector

$$= W_{\text{sky}} (\underline{z} \cdot (-\underline{I} + 2\underline{nn}) \cdot \underline{n}_o)$$

Now for small sea slopes $\underline{n} = \underline{z} + \underline{\epsilon}$

$$\underline{\epsilon} = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta - 1)$$

or

$$\underline{\varepsilon} \approx (\theta \cos \varphi, \theta \sin \varphi, -\theta^2/2) + O(\theta^3)$$

Therefore, accurate to second order in θ , we have

$$\begin{aligned} W_{\text{det}} &= W_{\text{sky}} [\underline{z} \cdot [-\underline{I} + 2\underline{z}\underline{z} + 2(\underline{z}\underline{\varepsilon} + \underline{\varepsilon}\underline{z}) + 2\underline{\varepsilon}\underline{\varepsilon}] \cdot \underline{n}_0] \\ &= W_{\text{sky}} [\underline{z} \cdot \underline{n}_0 + 2\underline{z}[(\underline{z}\underline{\varepsilon} + \underline{\varepsilon}\underline{z}) + \underline{\varepsilon}\underline{\varepsilon}] \cdot \underline{n}_0] \\ &= W_{\text{sky}} (\underline{z} \cdot \underline{n}_0) + W'_{\text{sky}} [2\underline{z} \cdot (\underline{z}\underline{\varepsilon} + \underline{\varepsilon}\underline{z}) \cdot \underline{n}_0 + 2\underline{z} \cdot (\underline{\varepsilon}\underline{\varepsilon}) \cdot \underline{n}_0] \\ &\quad + 2 W''_{\text{sky}} [\underline{z} \cdot (\underline{z}\underline{\varepsilon} + \underline{\varepsilon}\underline{z}) \cdot \underline{n}_0]^2 + \dots \end{aligned}$$

The following averages will be needed:

$$\langle \underline{\varepsilon} \rangle = -\frac{1}{2} \langle \theta^2 \rangle \underline{z}$$

$$\langle \underline{\varepsilon}\underline{\varepsilon} \rangle = a\underline{I} + b\underline{z}\underline{z}$$

$$a = \frac{1}{2} \langle \theta^2 \rangle$$

$$a + b = 0 \quad b = -\frac{1}{2} \langle \theta^2 \rangle$$

$$\langle \underline{\varepsilon}\underline{\varepsilon} \rangle = \frac{1}{2} \langle \theta^2 \rangle [\underline{I} - \underline{z}\underline{z}]$$

Using these results, the first bracket in the expansion of W_{det} becomes on averaging $-2 \langle \theta^2 \rangle (\underline{z} \cdot \underline{n}_0)$. The second bracket is equivalent to

$$\begin{aligned} 2[\underline{\varepsilon}(\underline{I} + \underline{z}\underline{z}) \cdot \underline{n}_0]^2 &= 2\underline{n}_0 \cdot (\underline{I} + \underline{z}\underline{z}) \cdot \underline{\varepsilon}\underline{\varepsilon} \cdot (\underline{I} + \underline{z}\underline{z}) \cdot \underline{n}_0 \\ &= \langle \theta^2 \rangle \underline{n}_0 \cdot (\underline{I} + \underline{z}\underline{z}) \cdot (\underline{I} - \underline{z}\underline{z}) \cdot (\underline{I} + \underline{z}\underline{z}) \cdot \underline{n}_0 \\ &= \langle \theta^2 \rangle \underline{n}_0 \cdot (\underline{I} - \underline{z}\underline{z}) \cdot \underline{n}_0 \\ &= \langle \theta^2 \rangle [1 - (\underline{n}_0 \cdot \underline{z})^2] = \langle \theta^2 \rangle \sin^2 \theta_0 \end{aligned}$$

where θ_0 is the angle of observation measured from the vertical. Thus,

$$W_{\text{det}} = W(\cos\theta_0) + \langle\theta^2\rangle [-2W' \cos\theta_0 + W'' \sin^2\theta_0]$$

Now

$$\begin{aligned} W' &= \frac{dW}{d\theta} \frac{1}{\sin\theta} & W'' &= + \frac{1}{\sin\theta} \frac{d}{d\theta} \left(\frac{dW}{d\theta} \frac{1}{\sin\theta} \right) \\ & & &= \frac{1}{\sin\theta} \frac{d^2W}{d\theta^2} - \frac{\cos\theta}{\sin^3\theta} \frac{dW}{d\theta} \end{aligned}$$

Thus,

$$W_{\text{det}} = W(\theta_0) + \langle\theta^2\rangle \frac{\partial^2 W}{\partial\theta^2} + \cot\theta \frac{\partial W}{\partial\theta}$$

This computation was made leaving out the reflectivity. Actually,

$$W_{\text{det}} = \rho(\underline{n} \cdot \underline{n}_0) W_{\text{sky}}(\underline{z} \cdot \underline{n}_s)$$

Expanding the reflectivity in the same manner as before,

$$\begin{aligned} \rho(\underline{n} \cdot \underline{n}_0) &= \rho(\underline{z} \cdot \underline{n}_0 + \underline{\epsilon} \cdot \underline{n}_0) \\ &= \rho(\underline{z} \cdot \underline{n}_0) + \rho'(\underline{\epsilon} \cdot \underline{n}_0) + \frac{1}{2} \rho''(\underline{\epsilon} \cdot \underline{n}_0)^2 + \dots \end{aligned}$$

Including the extra terms in the expression for W_{det}

$$\begin{aligned} W_{\text{det}} &= [\rho + \rho'(\underline{\epsilon} \cdot \underline{n}_0) + (\frac{1}{2}) \rho''(\underline{\epsilon} \cdot \underline{n}_0)^2] \left\{ W_{\text{sky}} \right. \\ &\quad + W'_{\text{sky}} [2\underline{z} \cdot (\underline{z}\underline{\epsilon} + \underline{\epsilon}\underline{z}) \cdot \underline{n}_0 + 2\underline{z} \cdot (\underline{\epsilon}\underline{\epsilon}) \cdot \underline{n}_0] \\ &\quad \left. + 2W''_{\text{sky}} [\underline{z} \cdot (\underline{z}\underline{\epsilon} + \underline{\epsilon}\underline{z}) \cdot \underline{n}_0]^2 \right\} \end{aligned}$$

EXTRA TERMS

$$\begin{aligned}
 W_{\text{det}} = W_{\text{sky}} + \langle \theta^2 \rangle & \left[\frac{\partial^2 W_{\text{sky}}}{\partial \theta^2} + \cot \theta \frac{\partial W_{\text{sky}}}{\partial \theta} \right] \\
 & + 2\rho' \langle (\underline{\epsilon} \cdot \underline{n}_0) [\underline{z} \cdot (\underline{z}\underline{\epsilon} + \underline{\epsilon}\underline{z}) \cdot \underline{n}_0] \rangle W'_{\text{sky}} \\
 & + \rho' \langle (\underline{\epsilon} \cdot \underline{n}_0) \rangle W_{\text{sky}} + (\frac{1}{2})\rho'' \langle (\underline{\epsilon} \cdot \underline{n}_0)^2 \rangle W_{\text{sky}}
 \end{aligned}$$

The first average is equivalent to

$$\begin{aligned}
 \underline{n}_0 \cdot \langle \underline{\epsilon}\underline{\epsilon} \rangle (\underline{I} + \underline{z}\underline{z}) \cdot \underline{n}_0 &= \langle (\theta^2)/2 \rangle \underline{n}_0 \cdot (\underline{I} - \underline{z}\underline{z}) (\underline{I} + \underline{z}\underline{z}) \cdot \underline{n}_0 \\
 &= \langle (\theta^2)/2 \rangle \underline{n}_0 \cdot (\underline{I} - \underline{z}\underline{z}) \cdot \underline{n}_0 \\
 &= \langle (\theta^2)/2 \rangle \sin^2 \theta_0
 \end{aligned}$$

The second average is simply $\langle (\theta^2)/2 \rangle \cos \theta_0$. The third average is

$$\underline{n}_0 \cdot \langle \underline{\epsilon}\underline{\epsilon} \rangle \underline{n}_0 = \langle (\theta^2)/2 \rangle \sin^2 \theta_0$$

Assembling these results and noting that $\langle \theta^2 \rangle$ is equivalent to the mean-square slope m^2 within the approximations used here, we obtain

$$\begin{aligned}
 W_{\text{det}} = W_{\text{sky}} + m^2 & \left[\frac{\partial^2 W_{\text{sky}}}{\partial \theta^2} + \cot \theta \frac{\partial W_{\text{sky}}}{\partial \theta} \right] \\
 & + \frac{m^2}{\sin^2 \theta} \left(\frac{\partial W_{\text{sky}}}{\partial \theta} \right) \left(\frac{\partial \theta}{\partial \rho} \right) \sin^2 \theta + (m^2/2) \cot \theta \left(\frac{\partial \rho}{\partial \theta} \right) W_{\text{sky}} \\
 & + (m^2/4) \sin^2 \theta \left[\frac{1}{\sin^2 \theta} \frac{\partial^2 \theta}{\partial \theta^2} - \frac{\cos \theta}{\sin^2 \theta} \frac{\partial \theta}{\partial \theta} \right] W_{\text{sky}}
 \end{aligned}$$

The net result is that the variation in radiance is given by

$$\delta W_{\text{det}} = m^2 \left\{ \rho \left[\frac{\partial^2 W_{\text{sky}}}{\partial \theta^2} + \cot \theta \frac{\partial W_{\text{sky}}}{\partial \theta} \right] + \left(\frac{\partial W_{\text{sky}}}{\partial \theta} \right) \left(\frac{\partial \rho}{\partial \theta} \right) + (k) W_{\text{sky}} \left[\frac{\partial^2 \rho}{\partial \theta^2} + \cot \theta \frac{\partial \rho}{\partial \theta} \right] \right\}$$

This expression accounts for the contribution to the variation in received radiance due to the first term in Eq. (1). The contribution of the second term can be obtained by inspection of the above expression (replacing W_{sky} by W_{sea} and noting that W_{sea} is independent of θ). Thus, the contribution of the second term is

$$\delta W_{\text{det}} = -(m^2/4) W_{\text{sea}} \left[\frac{\partial^2 \rho}{\partial \theta^2} + \cot \theta \frac{\partial \rho}{\partial \theta} \right]$$

Combining these results yields Eq. (4) of the text.

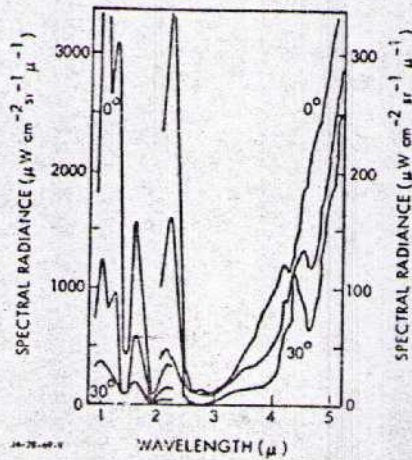


FIGURE 1. Spectral radiance of a clear sky, showing the dependence of the scattered radiation on elevation angle. Measurements made from Colorado Springs, Colorado, near noon in September; elevation angles 0° (top curve), 7.2° and 30° (lowest curve). Note that the ordinate scale for the short-wavelength set of curves is 10 times larger than the scale of the longer-wavelength curves. (From Bell et al [1960])

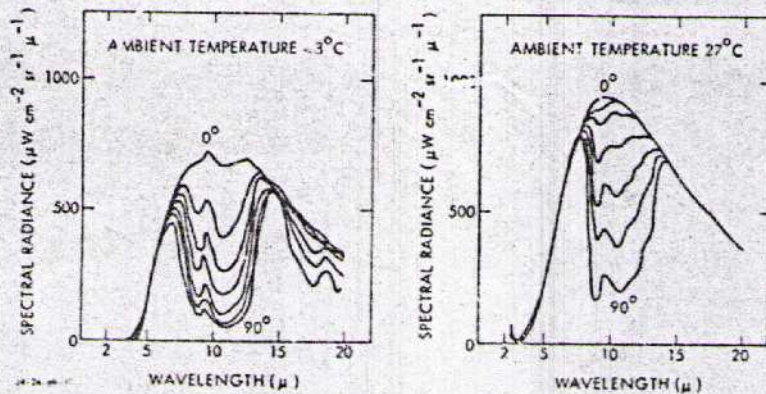


FIGURE 2. Spectral radiance of a clear sky (for angles of elevation 0° , 1.8° , 3.6° , 7.2° , 14.5° , 30° , and 90° above the horizon.) Ambient temperature 8°C ; measured at night in September from Elk Park Station, Colorado, 11,000 ft above sea level. Ambient temperature about 27°C ; measured in June from Cocoa Beach, Florida. (From Bell et al [1960])

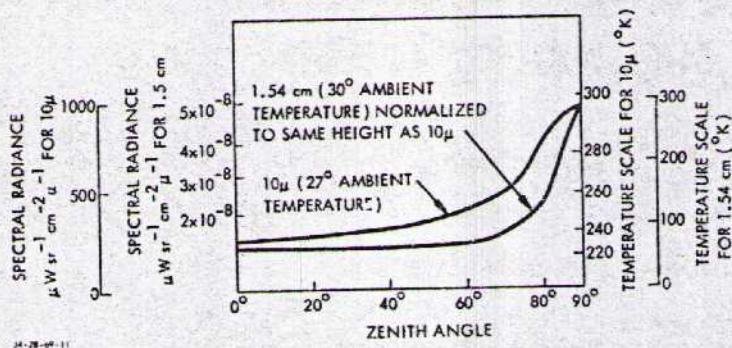


FIGURE 3. Sky Radiance

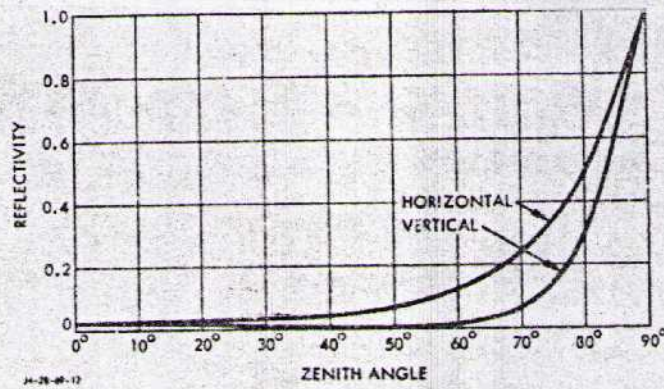


FIGURE 4. Reflectivity

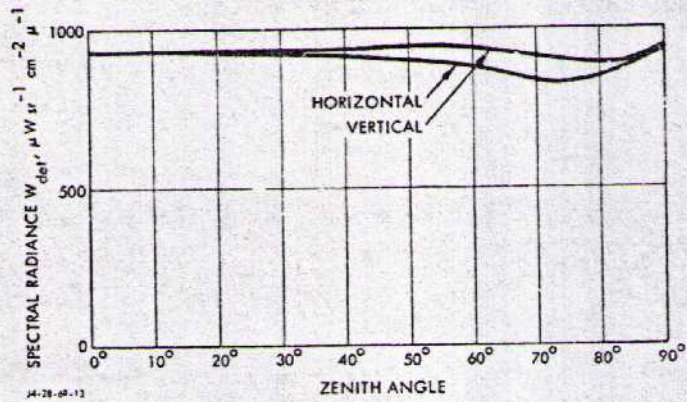


FIGURE 5. W_{det} for $10\ \mu$

SUPPORTING ANALYSIS F

DECISION THEORY APPLIED TO SENSOR EVALUATION

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Sophisticated signal processing is claimed by some of its advocates to make possible remarkable improvements in detection systems. While this is true in some cases, in others the claims are extravagant. This note is written to provide a basic criterion for detection that depends only on hardware capability and by which ultimate performance limits can be set: limits that can be approached but not surpassed by astute processing and presentation. The basis for this is the work of Harris* on decision theory. The derivation of one of his main results is abstracted here for completeness.

We will look at the detection problem as a binary decision between two signal sources denoted I and II, with the signals accompanied by additive Gaussian noise. For definiteness, take a two-dimensional data presentation with mean flux densities of $H_I(x,y)$ and $H_{II}(x,y)$ for the sources I and II. The likelihood that a set of flux readings R_1, R_2, \dots, R_n with dispersions $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$ for patches of area $\Delta x \Delta y$ is observed in response to source I is

$$L(I) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi} \sigma_i} \exp[-(R_i - H_{Ii} \Delta x \Delta y)^2 / 2\sigma_i^2]$$

where R_i is the flux measured at the display point (x_i, y_i) , and $H_{Ii} = H_I(x_i, y_i)$. Similarly, the likelihood of the same readings in response to source II is

$$L(II) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi} \sigma_i} \exp[-(R_i - H_{IIi} \Delta x \Delta y)^2 / 2\sigma_i^2]$$

This formulation assumes that the incremental area $\Delta x \Delta y$ is sufficiently large for the observed readings R_i to be regarded as statistically independent.

For the purpose of this discussion, the relative risks of error in deciding that source I is present when source II is really present, and vice versa, can be ignored. Under these conditions, the decision rule is to select the alternative with the larger likelihood. In terms of

$$\psi = 2 \log[L(I)/L(II)]$$

the procedure is to decide that I is present if $\psi > 0$, and to decide that II is present if $\psi < 0$. Specifically,

$$\psi = \sum_{i=1}^n (1/\sigma_i^2) [-2R_i(H_{IIi} - H_{Ii}) \Delta x \Delta y + (H_{IIi}^2 - H_{Ii}^2)(\Delta x)^2(\Delta y)^2]$$

Now suppose that source I is actually present; then

$$R_i = H_{Ii} \Delta x \Delta y + n_i$$

with

$$\langle n_i^2 \rangle = \sigma_i^2 = v_i \Delta x \Delta y$$

Here, n_i represents the additive Gaussian noise, and v_i is the noise variance per unit area. Substituting this into the expression for ψ yields

$$\psi_I = \sum_{i=1}^n \frac{(H_{IIi} - H_{Ii})^2 \Delta x \Delta y}{v_i} - \sum_{i=1}^n \frac{2n_i (H_{IIi} - H_{Ii})}{v_i}$$

The mean of ψ_I can be expressed as

$$\mu_I = \iint \frac{[H_{II}(x,y) - H_I(x,y)]^2}{v(x,y)} dx dy$$

The integral is taken over the area of presentation (e.g., the field of view of display). The variance of ψ_I is calculated to be

$$\sum_1^2 = 4\mu_I$$

The probability of correct decision (i.e., that $\psi_I > 0$) is

$$\rho = (1/\sqrt{2\pi}) \int_{-\mu/\Sigma}^{\infty} e^{-z^2/2} dz \quad (1)$$

The probability of a correct decision depends on the single parameter

$$\mu/\Sigma = \left(\frac{1}{2} \right) \left\{ \iint (S/N)^2 dx dy \right\}^{1/2} \quad (2)$$

when N denotes the background noise per unit area.

Well-matched processing and data presentation can take full advantage of the signals provided by detection equipment, but cannot improve the probability of a correct decision over that implied by Eq. (1) and (2).

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