The Square Pyramidal Number and other figurate numbers

Luciano Ancora

Contents

Introduction

The articles presented here they originated from my research, carried out in the eighties in the field of elementary number theory, which produced the second article, the most important, in which is proved the Archimedes theorem on the quadrature of the parabolic segment, without the aid of integral calculus, but using only the Square Pyramidal Number and criteria for convergence of numerical sequences. This proof has heuristic value, since there are no previous works that correlate the Parabola to the Square Pyramidal Number. It can be seen immediately on Internet by typing together the keywords of these two mathematical objects.

The first article has originated from a search, auxiliary to the previous one, to achieve the well-known formula that calculates the sum of the squares of the first n natural numbers. At that time Internet was still at the origins and the need to know the formula, to develop the second work, led me to resort to "do it yourself". I then realized, much later, that the method used in those notes was new.

Another novelty is in the third article of the collection, born almost like a game in the wake of the first, where it is estimated the so called "Squared Triangular Number". This work is my favorite, for its originality and immediacy.

The collection continues with other items that do not offer anything new, while containing some a certain didactic value.

Luciano Ancora

Chapter 1

The Square Pyramidal Number

In number theory, the Square Pyramidal Number is a number that figuratively represents the number of spheres stacked in a pyramid with a square base. The n-th number of this type is then the sum of the squares of the first n natural numbers.

The first few square pyramidal numbers are:
1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819 (sequence A000330 in OEIS).

The formula that calculates this sum, reported in the following proposition, was obtained by an algebraic process, in an indirect manner. George Polya, in his book *The mathematical discovery*, presents this solution saying it "rained from the sky", obtained algebraically by a trick, like a rabbit drawn out from the hat.

We will perform here the derivation of the formula for calculating the square pyramidal number in a *direct manner*, using a three-dimensional geometric model.

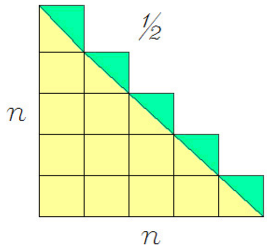
1.1 Proposition

The sum of the squares of the first n natural numbers is given by the square pyramidal number, expressed by the following formula:

$$P_n = \sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}$$

Proof

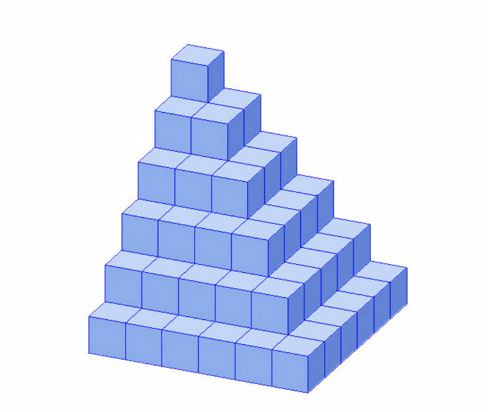
The idea that gave rise to our search is shown in the following figure:



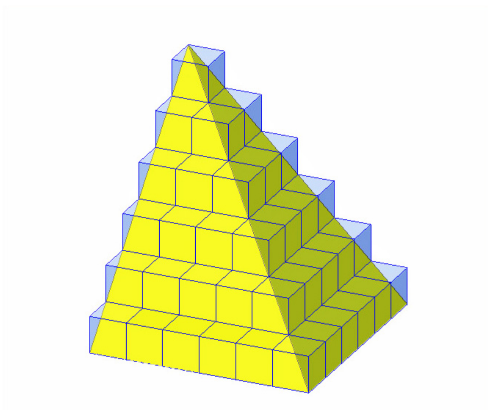
in which we see that the sum of the first n natural numbers is given by the Triangular Number:

$$\sum_{k=1}^{n} k = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$$

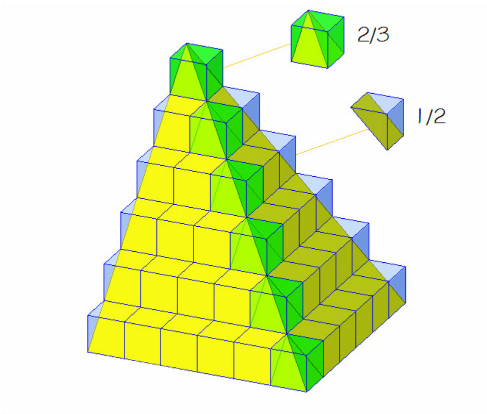
We develop this idea in 3D space building a geometric model that represents the sum of the squares of the first six natural numbers P_6, using cubic bricks of unit volume, as shown in the next figure:



We insert now in the building a pyramid (yellow), inscribed as follows:



Let V_6 the volume of the inscribed pyramid. To obtain the total volume of the building P_6 , just add to the volume V_6 of the yellow pyramid, the excess volume that is outside of the pyramid itself.



This excess is:
- 2/3 for each unit cube placed on the central edge of the pyramid;
- 1/2 for each unit cube forming the steps of the building.

Calculating one has:

$$P_6 = V_6 + \frac{2}{3} \times 6 + \frac{1}{2} \times (2+4+6+8+10) = V_6 + \frac{2}{3} \times 6 + (1+2+3+4+5)$$
$$= V_6 + \frac{2}{3} \times 6 + \sum_{k=1}^{5} k$$

Applying the induction principle, we can write that, in general:

$$P_n = V_n + \frac{2n}{3} + \sum_{k=1}^{n-1} k = \frac{n^3}{3} + \frac{2n}{3} + \frac{n^2+n}{2} - n$$

that is:

$$P_n = \sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}$$

what is the formula that we were looking.

1.2 Sums of squares of the first n even and odd numbers

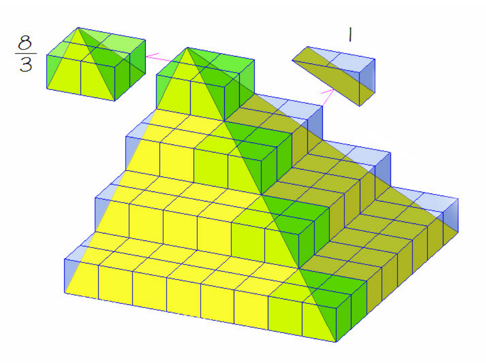
The method used in the previous proof, that we will say *method of the inscribed pyramid*, applies equally well to the case of sums of squares of the first n odd and even numbers.

Sum of squares of the first n even numbers

The sum of the squares of the first n even numbers is obtained from the well known formula:

$$\sum_{k=1}^{n}(2k)^2 = \frac{2(2n^3+3n^2+n)}{3}$$

Proceeding with the introduced method you get to the end the following situation:



Here the exceeding volumes are:
- 8/3 for each block of cubes placed on the central edge of the building;
- 1 for each pair of cubes forming the steps.

The volume of building S_4 is calculated by adding to the volume V_4 of the inscribed pyramid, the total volume of exceeding parts:

$$S_4 = V_4 + \frac{8}{3} \times 4 + 2 \times (2+4+6)$$

We can then write that in general:

$$S_n = V_n + \frac{8n}{3} + 2 \times (n^2 - n) = \frac{(2n)^3}{6} + \frac{8n}{3} + 2 \times (n^2 - n)$$

that is:

$$\sum_{k=1}^{n}(2k)^2 = \frac{2(2n^3+3n^2+n)}{3}$$

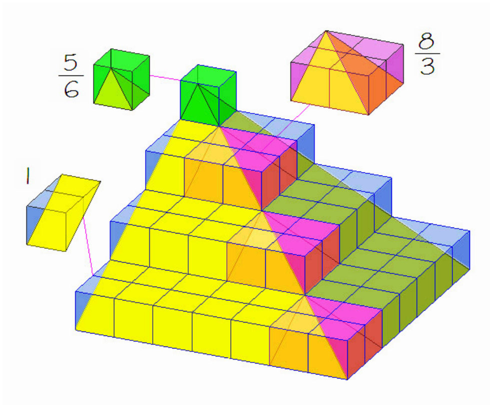
which is the formula that we were looking.

Sum of squares of the first n odd numbers

The sum of the squares of the first n odd numbers is obtained from the formula:

$$\sum_{k=1}^{n}(2k-1)^2 = \frac{4n^3-n}{3}$$

Even here, proceeding with the same method, are obtained at the end the following figure:



The exceeding volumes are:
- 8/3 for each block of cubes placed on the central edge;
- 1 for each pair of cubes forming the steps of the building;
- 5/6 for the cube on te top.

The volume of building S'_4 is calculated by adding to the volume V'_4 of the inscribed pyramid, the total volume of excess parts:

$$S'_4 = V'_4 + \frac{8}{3} \times 3 + \frac{5}{6} + 2 \times (1+3+5)$$

We can then write that in general:

$$\begin{aligned} S'_n &= V'_n + \frac{8(n-1)}{3} + \frac{5}{6} + 2 \times (n-1)^2 \\ &= \frac{(2n-1)^3}{6} + \frac{8(n-1)}{3} + \frac{5}{6} + 2 \times (n-1)^2 \end{aligned}$$

that is:

$$\sum_{k=1}^{n}(2k-1)^2 = \frac{4n^3-n}{3}$$

which is the formula that we were looking.

The two formulas just derived are obtainable algebraically, from each other, in a very simple way. This without taking anything away from our geometric proofs, which retain the merit of being autonomous and direct.

1.3 Squares in a square

Square pyramidal numbers also solve the problem of counting the number of squares in an $n \times n$ grid. We count the squares in a chessboard (8×8).

The squares with unit side are $8^2 = 64$.
The squares with side greater than 1 are neatly counted moving on the rows (or columns) one column (or row) at a time. You get:
7^2 squares with side 2
6^2 squares with side 3
5^2 squares with side 4
4^2 squares with side 5
3^2 squares with side 6
2^2 squares with side 7
1 square with side 8

Therefore, the total number of visible squares in a chessboard is given by the sum of squares of the first 8 natural numbers:

$$1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 = 204$$

In general, in a $n \times n$ square board, visible squares are given by the square pyramidal number P_n.

Links

`http://youtu.be/r9XcoQNeBGQ`

References

George Polya (1981), Mathematical Discovery - Vol. II, Paperback

Chapter 2

Quadrature of the Parabola whith the Square Pyramidal Number

We perform here a new proof of the Archimedes theorem on the quadrature of the parabolic segment, executed without the aid of integral calculus, but using only the Square Pyramidal Number and criteria for convergence of numerical sequences.

The translation of the discussion in the numerical field will happen using as unit of measurement of the areas involved in the proof, equivalent triangles, suitably identified in the grid of construction of the parabolic segment.

2.1 Introduction

The Quadrature of the Parabola is one of the first works composed by Archimedes. It has as subject the quadrature of the parabolic segment, namely the construction (with ruler and compass) of a polygon equivalent to it. For parabolic segment Archimedes means the area between a straight line and a parabola, conceived as a section of a right cone. The work opens with an introduction to the basic properties of the parabola; then move to perform the quadrature of the parabola in a mechanical way, with considerations on the lever equilibrium; finally we get to the geometric proof of the quadrature, performed applying the exhaustion method.

Our proof revisits in a modern key the work of Archimedes, using the same figure that he uses in the proposition 16, where it is proved the fundamental result that the triangle ABC is triple of the parabolic segment. Archimedes uses a triangle ABC rectangle in B, having shown, in the previous proposition 15, that the result for such a situation generalizes to a parabolic segment with base not perpendicular to the axis. In the next proposition 17 Archimedes infers from this result the other, more known, that the parabolic segment is 4/3 of the inscribed triangle.

2.2 Proposition 16

Let AB the base of a parabolic segment, and draw through B the straight line BC parallel to the axis of the parabola to meet the tangent at A in C. I say that the area of the parabolic segment il one-third of the ABC triangle area.

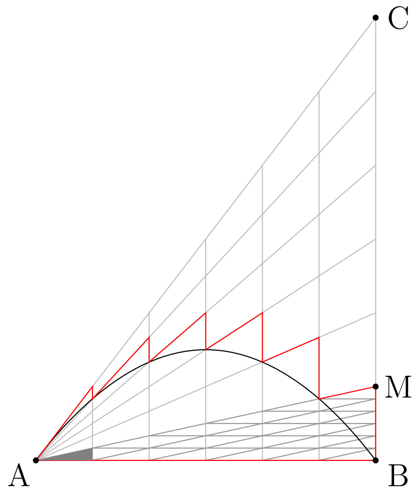


Figure 2.1: Proof

Proof

Split the segments AB and BC into six equal parts and lead, from split points on AB parallels to BC, and from points on BC lines joining with A. The parabola passes through the points of intersection of the grid, as drawn, because, for one of its properties, it cut the vertical lines of the grid in the same ratio in which the vertical lines cuts the segment AB.

Consider the sawtooth figure that circumscribes the parabolic segment. The area of this figure exceeds the area of the segment of a quantity that is equal to the overall area of the teeth. If we increase the number of divisions n on AB and BC, the excess area tends to zero as n tends to infinity. In other words: the area of the sawtooth figure converges to the area of the parabolic segment, as n tends to infinity.

In the graph, the sawtooth figure is divided into six vertical stripes composed: the first of 6 equivalent triangles, and the other strips, respectively, of 5, 4, 3, 2, 1 trapezoids, equivalent to each other in each strip. Now consider the triangle (shown in gray) with a vertex at the point A. We will use this triangle as the measurement unit of the areas in the counts that follow:

- The triangle ABM contains: $1 + 3 + 5 + 7 + 9 + 11 = 6^2$ gray triangles (the sum of the first n odd numbers is n^2).

- The triangle ABC contains: $6 \times 6^2 = 6^3$ gray triangles.

- In general, for any number n of divisions of AB and BC, the triangle ABC contains n^3 gray triangles.

- The circumscribed sawtooth figure $A(cir.)$ contains (for the equivalence of trapezes view above):

$$A(cir.) = 1\times 6+3\times 5+5\times 4+7\times 3+9\times 2+11\times 1 = 91 \text{ triangles.} \quad (1)$$

We see that in (1) the sequence of the addends is formed by products in which: the first factors give the sequence of the first six odd numbers and the second the backward sequence of the first six natural numbers.

We represent now (1) with the following scheme (that also plays the dislocation of the trapezoids in the figure and their contents, making it more intelligible the count):

1						=	1^2
1	3					=	2^2
1	3	5				=	3^2
1	3	5	7			=	4^2
1	3	5	7	9		=	5^2
1	3	5	7	9	11	=	6^2
							91

Figure 2.2: Scheme

The schematized counting shows that the total number of gray triangles in the $A(cir.)$ figure is given by the sum of the squares of the first 6 natural numbers (there is also a diagonal path in the scheme, which leads to the same conclusion).

It therefore appears that, the area of the sawtooth figure, expressed in gray triangles, is given by the *Square Pyramidal Number* P_6:

$$A_{(cir.)} = P_6 = \sum_{k=1}^{6} k^2$$

The generalization to any number of divisions n, it follows from the possibility to extend the scheme of Figure 2.2 to the number n, adding rows containing successive sequences of odd numbers, until the n-th. The result is, in general, that:

$$A_{n(cir.)} = 1 \times n + 3 \times (n-1) + 5 \times (n-2) + ... + (2n-1) \times 1$$

and the area of the umpteenth sawtooth figure, which circumscribes the parabolic segment, will be expressed by the square pyramidal number P_n:

$$A_{n(cir.)} = P_n = \sum_{k=1}^{n} k^2$$

This circumstance, together with the general result obtained for the area of triangle ABC (which is equal to n^3), we can reduce the proof to the simple check of the following relationship:

$$\lim_{n\to+\infty} \frac{\sum_{k=1}^{n} k^2}{n^3} = \frac{1}{3} \qquad (2)$$

where the first member numerator is the n-th square pyramidal number P_n.
You know that the sum in the numerator of (2) is:
$P_n = n^3/3 + n^2/2 + n/6$
after which the limit (2) follows from the fact that the ratio of two polynomials of the same degree in the variable n tends (as n tends to infinity) to the ratio between respective leading coefficients (coefficients of terms of maximum degree).

But (2) states that: the area (measured in gray triangles) of the circumscribed figure is one-third the area of the triangle ABC, as n tends to infinity. Follows the statement in the proposition 16.

The proof "from below"

So that a proof could be called "complete" requires two estimates, one from above and one from below, ie, with a figure out and an inside the parabolic segment.

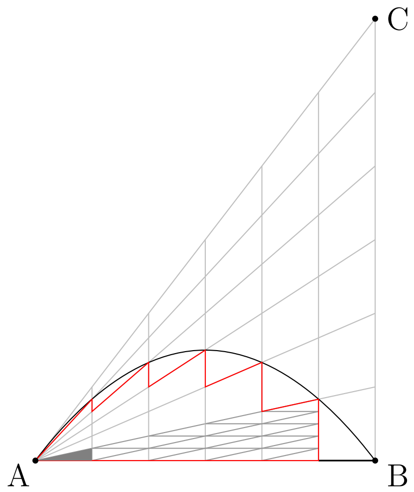


Figure 2.3: Proof from below

Inscribing a sawtooth figure $A(ins.)$ in the parabolic segment, as in the figure, one can see that it contains:

$$A_{(ins.)} = 1 \times 5 + 3 \times 4 + 5 \times 3 + 7 \times 2 + 9 \times 1 = 55 \text{ gray triangles.}$$

But the number 55 appears to be the 5-th square pyramidal number; therefore, following the same reasoning made above, we can write:

$$A_n(ins.) = 1 \times (n-1) + 3 \times (n-2) + ... + (2n-3) \times 1$$

and, for the umpteenth area of the inscribed figure:

$$A_{n(ins.)} = P_{n-1} = \sum_{k=1}^{n-1} k^2$$

Thus, the proof "from below" follows from the equality:

$$\lim_{n \to +\infty} \frac{\sum_{k=1}^{n-1} k^2}{n^3} = \frac{1}{3} \tag{3}$$

which is also true, since the sum in the numerator of (3) is:
$P_{n-1} = n^3/3 + n^2/2 + n/6 - n^2$
which is still a third-degree polynomial in n with leading coefficient equal to 1/3.

2.3 Proposition 15

The model chosen for our proof provides an opportunity to show in a different way as stated by Archimedes in the proposition 15, in which its proof is generalized to a parabolic segment with base not perpendicular to the axis.

Our proof can refer (without changing anything in the text) to a figure more general obtained by shifting arbitrarily the segment BC on its straight line, like this:

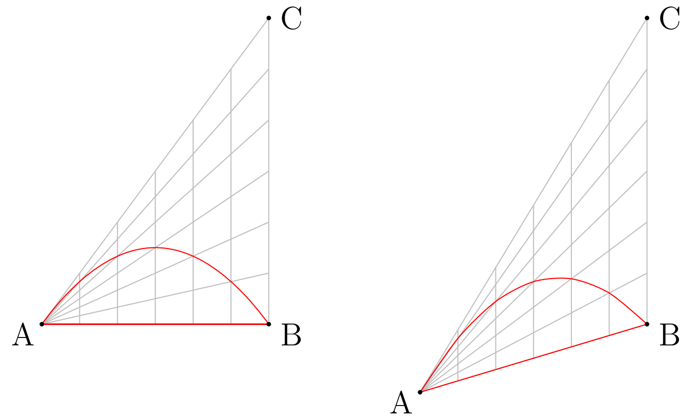


Figure 2.4: Proposition 15

In fact, the transformation does not affect any of equivalence relations between trapezoids and triangles used in the proof, which are the essence of the proof itself.

2.4 Proposition 17

The immediate consequence of the proposition 16 is the proposition 17, with which Archimedes proves the fundamental theorem on the area of the parabolic segment (but using a mechanical method):
The area of the parabolic segment is 4/3 of the triangle having the same base and equal height.

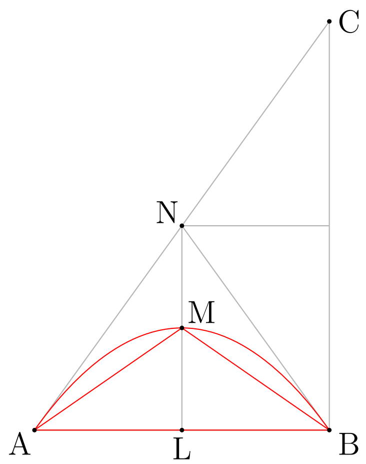


Figure 2.5: Proposition 17

For a property of the parabola:

$LM = MN$, then: $A_{PS} = ABC/3 = 4ABM/3$.

The same thing when you consider the half ALM of the parabolic segment ABM. In fact we have: $ALM = AMN$, and the parabolic segment with base AM has area equal to 1/3 of AMN (prop. 15) and therefore of ALM. Doubling follows the statement.

Since the proposition 17 is a corollary of 16, having demonstrated geometrically 16, then it is also geometrically proved the fundamental theorem.

Links

See animation of the proof at `http://youtu.be/6S-vGLJR0iM`

References

L. Ancora, Quadratura della parabola con il numero piramidale quadrato, in "Archimede" (Le Monnier), 4/2014

Chapter 3

The Squared Triangular Number

In number theory, the sum of the first n cubes is given by the so called Squared Triangular Number, expressed by the formula:

$$\sum_{k=1}^{n} k^3 = \Big(\sum_{k=1}^{n} k\Big)^2$$

As seen from the formula, this number is the square of the n-th triangular number and gives the sum of cubes of the first n natural numbers.

The first few squared triangular numbers are:
0, 1, 9, 36, 100, 225, 441, 784, 1296, 2025 (sequence A000537 in OEIS).

The above identity is sometimes called Nicomachus's theorem, named after the Greek mathematician of the Hellenistic age, Nicomachus of Gerasa, which proved it arithmetically. Many mathematicians have studied this equality, demonstrating it in many different ways. The idea of visually demonstrate the Nicomachus's identity is not new. Roger B. Nelsen, in his work *Proofs without Words* (1993) presents seven different versions. The advantage of visual demonstrations is to provide sometimes, as in the present work, a graphic evidence of the solution.

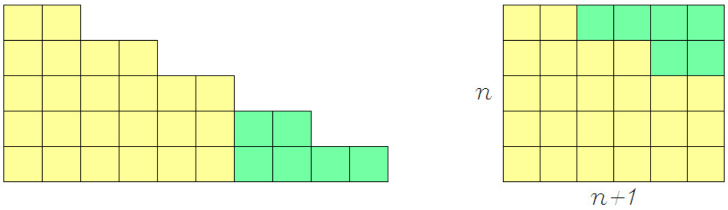
3.1 Proposition

The sum of the cubes of the first n natural numbers is given by the Squared Triangular Number, ie by the square of the n-th triangular number:

$$\sum_{k=1}^{n} k^3 = T_n^2 = \Big(\sum_{k=1}^{n} k\Big)^2$$

Proof

The idea that gave rise to our search is shown in the following figure:



in which we see that the sum of the first n even numbers is given by:

$$\sum_{k=1}^{n}(2k) = n(n+1) = n^2 + n$$

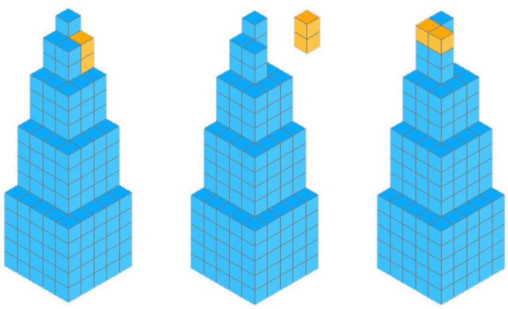
Exploiting this idea, we will demonstrate the proposition using the manipulation of a three-dimensional geometric model, as shown in the following images sequence.

We start from a three-dimensional model, built with cubic bricks of unit volume, that represents the sum of the cubes of the first 5 natural numbers. In an attempt to obtain a figure equivalent to this model, which gives evidence of the identity to prove, we perform an inductive transformation, moving the unit cubes as shown in sequence.

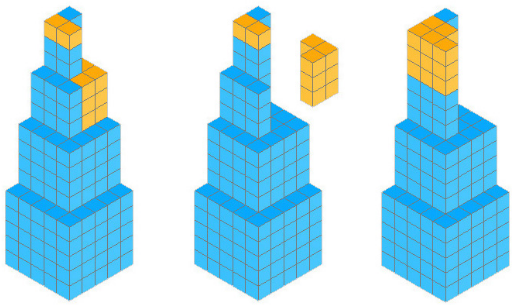
The inductivity of the process lies in the fact that, in each cube of the sum, the unit cubes to move are neatly arranged in $1 + 2 + 3 + ... + (k-1)$ columns, each of height k.

The final result of the transformation is always, for any n, a pseudo-parallelepiped whose base is a geometrical representation of the triangular number T_n, and whose height is the number T_n itself, which remains unchanged during the transformation.

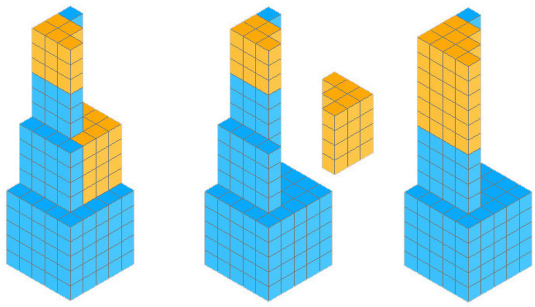
[1]



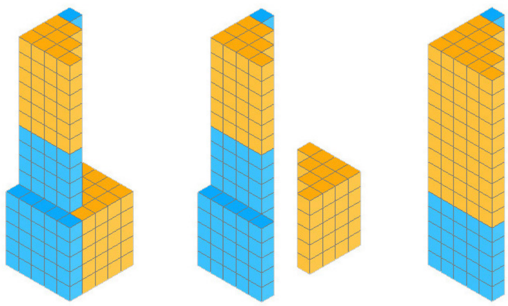
[2]



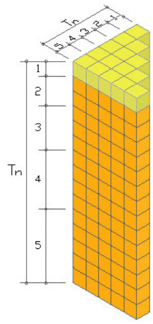
[3]



[4]



[5]



Therefore, as you see in the figure [5] of the sequence, the total number of unit cubes, which gives the sum of the cubes of the first n natural numbers, is given by:

$$\sum_{k=1}^{n} k^3 = T_n^2 = \left(\sum_{k=1}^{n} k\right)^2$$

that is, the identity that one wanted to prove.

3.2 Sum of cubes of the first n even and odd numbers

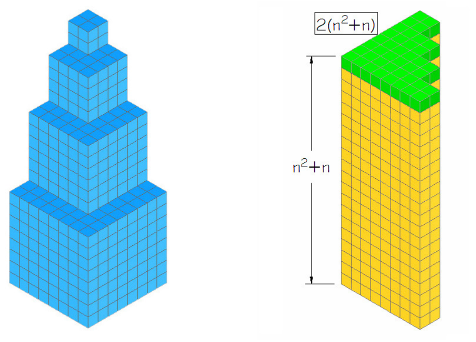
The method used in the previous proof, that we will say *method of successive transformations*, applies equally well to cases of sums of cubes of the first n odd and even numbers.

Sum of cubes of the first n even numbers

The sum of the cubes of the first n even numbers is obtained from the known formula:

$$\sum_{k=1}^{n} (2k)^3 = 2(n^2 + n)^2$$

Proceeding with the introduced method is obtained:



that is, a pseudo-parallelepiped having its base formed by:

$4 \times (1 + 2 + 3 + ... + n) = 2(n^2 + n)$

unit cubes, and height (which remains unchanged) amounting to:

$(n^2 + n)$ unit cubes.

Therefore, the volume of the figure, ie the sum they were looking, is:

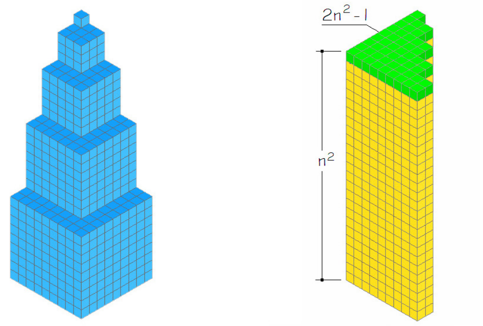
$$\sum_{k=1}^{n} (2k)^3 = 2(n^2 + n)^2$$

Sum of cubes of the first n odd numbers

The sum of the cubes of the first n odd numbers is obtained from the formula:

$$\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$$

Even here, proceeding with the transformation, is obtained:



From which, by calculating, the number of unit cubes is:

$$\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$$

which is the formula they were looking.

The two formulas just derived are obtainable algebraically, from each other, in a very simple way. This without taking anything away from our geometric proofs, which retain the merit of being autonomous and direct.

3.3 Rectangles in a square

The squared triangular number also count the number of rectangles with horizontal and vertical sides formed in an $n \times n$ grid.

This is achieved by "combining" all possible vertical stripes of the grid, having width $1, 2, ..., n$, counted by T_n, with the same number of horizontal stripes. But there is noteworthy that this count includes all the squares (rectangles with equal sides). Limiting only to the properly so called "rectangles" should subtract the squares, that are counted with the square pyramidal number (see sequence A052149 in OEIS).

Links

http://youtu.be/XM2plLoJkRk

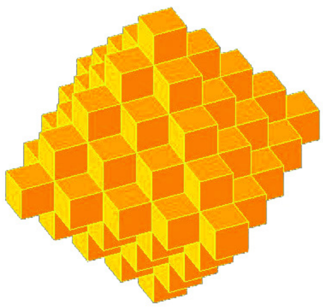
References

Nelsen, Roger B. (1993), Proofs without Words, Cambridge University

Chapter 4

The Centered Octahedral Number

Strolling among the figurate numbers, I met the Centered Octahedral Number, which is a construction formed by concentric layers of cubes accreting [1] around a central cube, as in the following figure:



This number, denoted by C_n, is given by the formula:

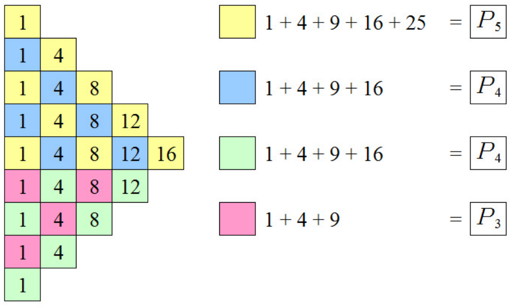
$$C_n = \frac{(2n+1) \times (2n^2 + 2n + 3)}{3} \qquad (1)$$

The first centered octahedral numbers ($n = 0, 1, 2, ...$) are:
1, 7, 25, 63, 129, 231, 377, 575, 833, 1159 (sequence A001845 in OEIS).

I wondered if, as it happens for the Octahedral Number [2], this number can be expressed in terms of square pyramidal numbers $P_n(n = 1, 2, 3, ...)$. You get the answer immediately, by looking at the following diagram, which shows, layer by layer, the concentric accretions around the central cube of the layer:

[1] The centered octahedral number was born in mineralogy, from the study of the "accreting" of octahedral crystals around a central source.

[2] The octahedral number O_n represents an octahedron, or two square-based pyramids with a common base. It is easy to see, by observing its construction, that the n-th octahedral number is obtained by summing the $(n-1)$-th to the n-th square pyramidal number.



On the right is shown the breakdown of the scheme in terms of sums P. Also this scheme can be generalized by induction, so we can write:

$$C_n = 2P_n + P_{n+1} + P_{n-1} \tag{2}$$

But the second member of the (2) is equivalent to the sum of two consecutive octahedral numbers [note 2], then is also valid the relationship:

$$C_n = O_n + O_{n+1} \tag{3}$$

as you can easily verify.

The formula (1) can be obtained from (2) and (3) by substituting the formulas of the sums P and O.

Links

1 - `http://youtu.be/p8ySArn-7KQ`

Chapter 5

A relationship between figurate numbers

Proposition

The Square Pyramidal Number can be decomposed into the sum of two Tetrahedral Numbers less a Triangular Number, in the following way:

$$P_n = 2Te_n - T_n \tag{1}$$

You can see it easily in two different ways:

A) You can put in a column two sequences of tetrahedral numbers:
1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, ...(A000292in OEIS)
1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, ...
and a sequence of triangular numbers:
1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, ...(A000217in OEIS)
and subtract the latter from the sum of the first two, obtaining the square pyramidal number:
1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, ...(A000330in OEIS)

B) Or, replacing the second member of (1) with the solving formulas:

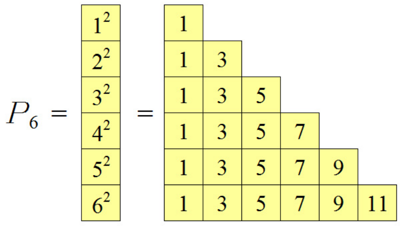
$$P_n = 2\frac{n(n+1)(n+2)}{6} - \frac{n(n+1)}{2} = \frac{2n^3+3n^2+n}{6} \tag{2}$$

But doing so we perform simple checks. I believe that there are many relations of the type proposed in (1). They may search with a calculation program that analyzes correspondences between the values of sequences of figured numbers, as we did in the first test. However, the possible results of such research should be explained and proved for each n.

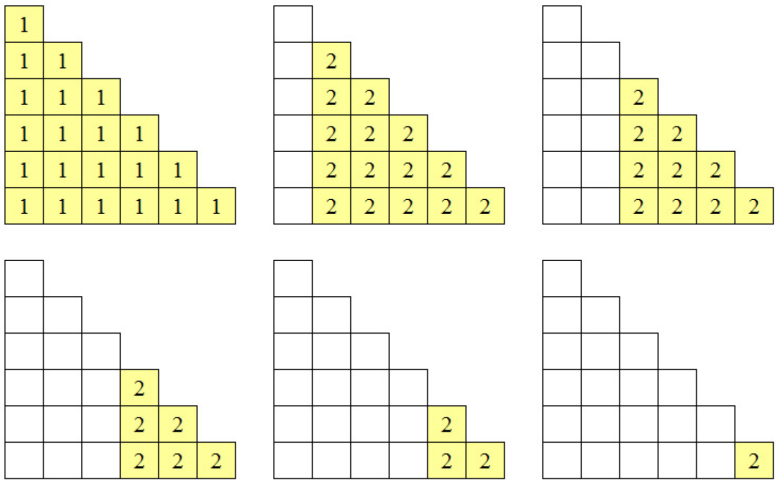
Proof

I show you how I found out the proposition, giving at the same time the proof of it.

Consider the scheme used in ch.2 to represent the square pyramidal number P_6:



This scheme can be seen as the superposition of six homogeneous "layers" :



which represent, in terms of triangular numbers, the following quantities: $T_6, 2T_5, 2T_4, 2T_3, 2T_2, 2T_1$. Therefore, we can write :

$$P_6 = T_6 + 2(T_5 + T_4 + T_3 + T_2 + T_1) = T_6 + 2\sum_{k=1}^{5} T_k$$

But the sum of the first five triangular numbers is the tetrahedral number Te_5, then:

$$P_6 = T_6 + 2Te_5 = T_6 + 2(Te_6 - T_6) = 2(Te_6 - T_6)$$

Even here, the generalization follows from the fact that the passage, from a number n to the next, is an inductive process which is realized by adding:

- a line with the sequence of $n+1$ odd numbers, in the first figure;
- two layers, T_n and $T_n + 1$, in the second figure.

One can therefore say that, in general, the square pyramidal number P_n can be expressed as:

$$P_n = 2Te_n - T_n$$

what is the relation (1) that we proposed.

Follow from (1) the other relationships:

$$T_n = 2Te_n - P_n \tag{3}$$

and

$$Te_n = \frac{P_n + T_n}{2} \tag{4}$$

Is famous the history of algebraic derivation (ch.1) of the formula to calculate P_n. I was wondering if, having made the chronological checks for the three component of formula (2), the previous proof can be regarded as "another way" to get the formula for the sums P_n.