Biruni's Measurement of the Earth

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Abstract

The peak from which Biruni measured the earth is pinpointed; his measurements are shown to tally with the geographical features of the landscape around the peak. Mathematics shows what measurements he is likely to have taken, and his famous formula is fully derived. Besides, an appendix is given for those who want to try measuring mountains by themselves and follow in Biruni's footsteps.

1. Introduction

Abu Rayhan (Muhammad ibn Ahmad) al-Biruni was born in Kath, capital of Khwarezm (the region of the Amu Darya delta), in modern Khiva, on 973 Sept 5 (362-12-03 AH), and died in Ghazni, Khorasan, on 1048 Dec 13 (440-07-04 AH).[1] He was a well-travelled Persian sage who spoke fluent Persian, Arabic, Turkish, Sanskrit, Greek, Hebrew, Syrian, and several other local languages in whose literatures he was also well versed. Not only did he possess 'all the well known books on astronomy written within the area extending from the Mediterranean Sea to the Bay of Bengal, including all the Greek, Indian, and Muslim authors from Spain to Egypt' (*Canon* 1:13, Ahmad 2009:171), but he also wrote prolifically on all branches of knowledge like cartography, geography, ethnography, history, philosophy, mathematics, astronomy, and science in general, a total of about 180 works, some sadly lost to us now (*EI2* 1:1237). Chief among these is his *Masudic Canon*, an encyclopaedia of astronomy he dedicated to Sultan Masud who, delighted with the sage's accomplishment, presented him with an elephant-load of silver! To everyone's amazement, Biruni refused it, pleading that he could not bear to abuse the generosity of someone who had already shown him more kindnesses than he felt he deserved (Minhaj Siraj *Tabaqat* 1975 2:343-4).

Of his many contributions to the good of humanity, it is hard to decide which is the most important and, certainly, it is not in the scope of a short paper like this to do justice to the vastness of his genius. So it is as a matter of personal choice that I have decided to deal here with just one of these: the fact that he measured the earth with an unprecedented precision, not equalled in the West until the 16th century (Norhudzaev 1973). You may find I am rather critical sometimes for, in Biruni's own teachings, it is the truth that we must seek and speak, even when it goes against us. His value for the earth's radius (6335.725 km) was the result of having solved a complex geodesic equation, which is the aim of this paper to explain. But to fully understand the whole story, we must know something about its background and about the early Islamic measures then (and herein) used: these were exactly those of ancient Mesopotamia, where 'each *mile* was a third of a *farsang*, or 4000 *cubits*, called black in Iraq, each of which equalled 24 *digits*' (Biruni *Instr. Astrol*. 208, tr. Wright 1934:119, Mercier 1994:178). That is, one farsang (5916 m) equalled three miles, one mile (1972 m) equalled 4000 cubits, and one (black) cubit (493 mm) equalled twenty-four digits. Note the connection with the Roman mile (1479 m) of 3000 cubits (each of 493 mm).

The story begins before Biruni, when Sultan al-Mamun ordered two teams of surveyors to measure the earth. They did so by departing from a place in the desert of Sinjad, nineteen farsangs from Mosul

[1] Some think that, according to the solar calendar, the Muslim date 440 AH should be taken to mean AD 1050 (*Encyclopaedia of Islam* 1979 1:1236, and *Encyclopaedia of World Biography* 1973 1:578).

and forty-three from Samarra, heading north and south respectively, and both determining that the length of one degree of latitude is somewhat between 56 and 57 Arabic miles (Biruni *Tahdid*, tr. Ali 1967:178-80). Among the several extant accounts of this survey, Habash al-Hasib (tr. Langermann 1985:108-28) quotes at length from a direct account from Khalid:

> 'The Commander of the Faithful al-Mamun desired to know the size of the earth. He inquired into this and found that Ptolemy mentioned in one of his books that the girth of the earth is so and so many thousands of stades. He asked the commentators about the meaning of *stade*, and they differed about the meaning of this. Since he was not told what he wanted, he directed Khalid ibn Abd al-Malik al-Marwarrudhi, Ali bin Isa al-Asturlabi [from his surname, evidently an instrument maker], and Ahmad ibn al-Bukhturi al-Dhari [from his surname, the Surveyor] with a group of surveyors and skilled artisans, including carpenters and brass makers, who were to maintain the instruments they needed. He led them to a place, which he chose in the desert of Sinjar. From there, Khalid and his party headed for the North Pole of the Little Bear, and Ali and Ahmad and their party headed to the South Pole. They proceeded until they found that the height of the Sun at noon had increased (or differed) by one degree from the noon height they had taken at the place from which they had separated, after subtracting from it the sun's declination along the path of the outward journey. They put arrows there. Then they returned to the arrows, testing the measurement a second time, and so found that one degree of the earth was 56 miles, of which one mile is 4000 black cubits. This is the cubit adopted by al-Mamun for the measurement of cloths, surveying of fields, and the distribution of way-stations.'

Another report is given by Ibn Yunus (*Hakimite Tables* 2), based on the accounts of Sind ibn Ali and Habash al-Hasib:

> 'Sind ibn Ali reports that al-Mamun ordered that he and Khalid ibn Abd al-Malik al-Marwarrudhi should measure one degree of the great circle of the earth's surface. "We left together," he says, "for this purpose." He gave the same order to Ali ibn Isa al-Asturlabi and Ali ibn al-Bukhturi, who took themselves to another direction. Sind ibn Ali said, "I and Khalid ibn Abd al-Malik travelled to the area between *Wamia* and *Tadmor*, where we determined a degree of the great circle of the earth's equator to be 57 miles. Ali ibn Isa and Ali ibn al-Bukhturi found the same, and these two reports containing the same measure arrived from the two regions at the same time."
>
> 'Ahmad ibn Abdallah, named Habash, reported in his treatise on observation made at Damascus by the authors of the *Mumtahan* [Verified tables] that al-Mamun ordered the measurement of one degree of the great circle of the earth. He said that for this purpose they travelled in the desert of Sinjar until the noon heights between the two measurements in one day changed by one degree. Then they measured the distance between the two places, which was 56¼ miles of 4000 cubits, the black cubits adopted by al-Mamun'.[1]

Biruni's take on the matter (tr. Ali 1967:178-80) is that the figure that eventually became generally accepted as the length of 1º of latitude is 56⅔ miles (111.747 km), which is quite close to the actual value (110.95 km) for the latitudes involved (35º to 36º N). 360 times this number yields the earth's girth (20400 mls), and from it the radius is easily deduced (6402.612 km). Mamun's teams had got a nearly perfect hit!

Years later, Biruni wished to repeat the experiment, but was hindered by lack of support. 'Who is going to help me in this venture?' He says. 'It requires strong command over huge tracts of desert, and extreme caution is needed from the dangerous treacheries of those spread over it. I once chose for this project the localities between Dahistan, in the vicinity of Jurjan, and the land of the Turks, but the findings were not encouraging and then the patrons who financed the project lost interest in it' (Biruni *Tahdid*, Ali's tr., p. 183).

[1] Ibn Yunus, *Hakimite Tables*, Chapter 2, from a Manuscript in Paris, Bibliotheque Nationale, MS Arabe 2495, fols. 44r-v. Later in this manuscript, Yunus explains that 'it is possible to keep the direction by means of three bodies spread out along the meridian, one of them hiding the others in line of sight. One advances by fixing the nearest one by sight, then the second, the third, and so on.'

Instead of being put off by difficulties, he thought up a new method for measuring the earth that 'did not require walking in deserts' (Ibid, p. 183, note 24). It only involved measuring the height of a mountain and taking the dip of the horizon from its top. That's four measurements in all, as we will see. It also involved his having to work out a mathematical equation that related these four measurements and, of course, finding a suitable mountain to yield the size of the earth by this method! Let us now turn to the first of these requirements.

2. Measuring mountains

He was well acquainted with the mathematical procedure for measuring mountain heights, having himself measured all sorts of distances and heights from his very youth. He did this by the usual method of taking the summit from two places, that is, by measuring the distance *d* between two places (in a straight line from the mountain) and the angles θ from them to the mountaintop. The formula he used that relates these angles to the mountain height *h* is this (see Figure 1):[1]

$$h = \frac{d \tan\theta_1 \tan\theta_2}{\tan\theta_2 - \tan\theta_1}$$

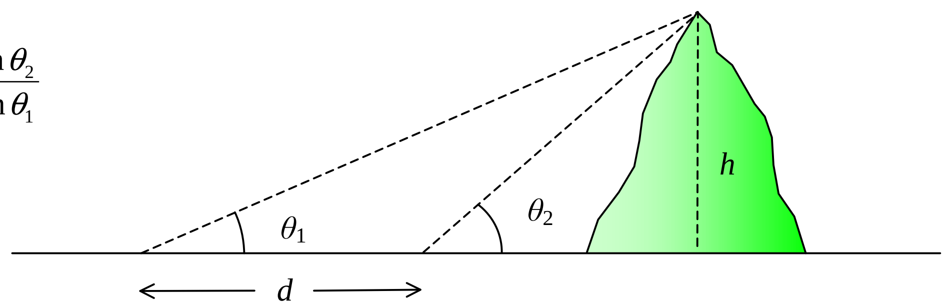


Figure 1: finding the height *h* of a mountain requires taking three measurements: the distance *d* between two level points that lie in a straight line from the mountain, and the angles θ from these points to the mountaintop.

Then, one day, while he was staying in the fort of Nandana[2] (at the southern end of the pass through the Salt Range, near Baghanwala in the Punjab),[3] he spotted 'a high mountain standing west of the fort [particularly suited for this project, for] it faces south to a wide flat plain whose flatness serves as the smooth surface of the sea' (*Canon* 5.7, ed. 1954-6 2:530; *Tahdid*, Ali's tr., p. 188).[4] He measured its height by the method described, and found it to be 652 cubits and 3′18″ (321.463 m) above the plain. This figure, which was expressed in the then customary mixture of decimal and sexagesimal systems, seems too precise to represent a physical reality. I originally thought it to be the result of averaging no fewer than ten measurements which yielded 652 as the average number of cubits and 3.3 as the average number of minutes, but now I think this is likely to be a self-fulfilling mathematical mirage that proves just the opposite: that he took a single measurement; but this raises the question: which one?

[1] Today we would use a slightly simpler formula, see the Appendix.
[2] Nandana Fort's coords are 32º43′33.52″ N, 73º13′45.16″ E, 404 m above the sea.
[3] This is the pass Alexander the Great took to descend from Taxila into the Indian Plain, just before his famous battle with Raja Poros in 326 BC (Stein 1932:31-46).
[4] Mercier (1994:183) says this peak is likely to be one situated 1.4 km south-southwest of the fort and standing 478 m above sea level, and about 265 m above the plain to the south. He mentions how Rizvi (1979), certainly having this peak in mind, gave its height as 547 m above see level, and 321½ m above the plain, in order to make it fit Biruni's report exactly. I haven't found such a place. Perhaps they meant the one situated 780 m southwest of the fort (32º43′14.60″ N, 73º13′26.00″ E) and standing 533 m above sea level. There is another candidate 3.8 km west of the fort (32º43′46.08″ N, 73º11′20″ E) and standing 677 m above sea level.

Biruni doesn't describe the instrument he used for his observations, but an astrolabe has been traditionally assumed. Assuming it was big enough to take angles good to half a degree and assuming that Biruni chose an integer (or at least a not very complex fractional number) as the number of cubits between his points for ease of computation leads to just one mathematical probability: that of his taking the angles 5½º and 7½º from two points exactly 1819 cubits apart. If his instrument was good to a quarter of a degree, then another possibility arises: that of taking the angles 5¼º and 6½º from two points 1373¼ cubits apart. If his instrument did not take angles directly, but rather computed them instead, for example, by comparing the length of and distance between two sticks aligned with the mountaintop and sticking out of a portable trough of water which provided the necessary horizontal line, then the accuracy thus computed (though not necessarily real) could reach even seconds of a degree. Allow for complex fractional cubit gaps between our angles, and several other possibilities are offered us, perhaps the most exotic of which is that of 1111⅓ cubits separating 4º40′ and 5º25′, or, for its smallness, 145⅞ cubits separating 4º35′ and 4º40′.[1]

Before measuring the mountain's height, he climbed to the top and, with his instrument, took the angle of the line of sight to the horizon as it dips below the horizontal: he found it to be 34′. All that remained was to find a formula relating the earth's radius R to the mountain height h and the dip angle θ he had taken. To this end he applied the *law of sines* most ingeniously as follows (see Figure 2):

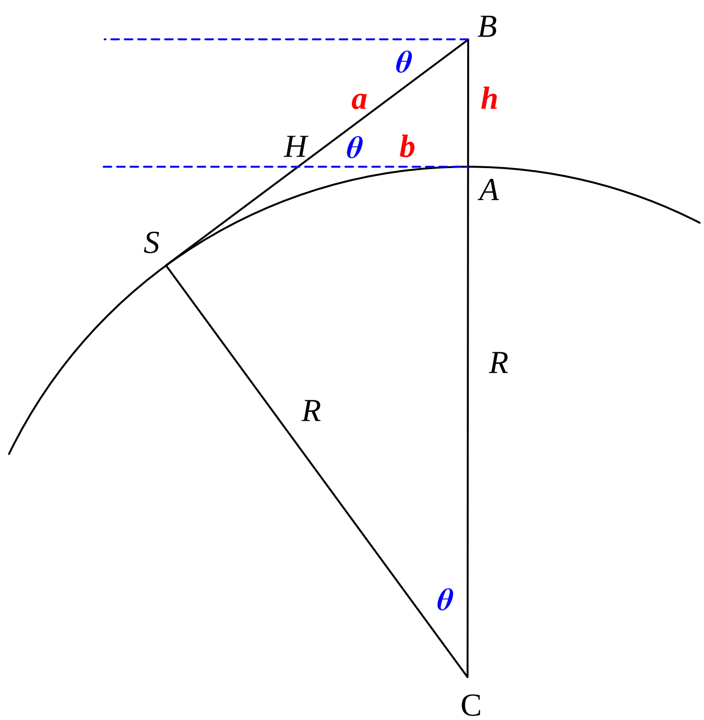


Figure 2: Biruni realized that the figure linking the earth's centre C, the mountaintop B, and the (sea or flat enough) horizon S was a huge right triangle on which the law of sines could be made to yield the earth's radius!

[1] I wish, esteemed reader, that you try all this out by yourself and, should you find any errors, I'd be delighted if you would let me know at the email address provided at the beginning of this article.

3. Biruni's formula

Applying the *law of sines* we have

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{h}{\sin H}\ , \qquad \text{and} \qquad \frac{b}{\sin B}=\frac{b}{\sin(90^\circ-\theta)}\ , \qquad \text{and} \qquad \frac{h}{\sin H}=\frac{h}{\sin\theta}\ .$$

It follows that $\dfrac{h}{\sin\theta}=\dfrac{b}{\sin(90^\circ-\theta)}$, so $b=\dfrac{h\ \sin(90^\circ-\theta)}{\sin\theta}$.

He also saw that the distance b is the same as the distance HS and that, by the Pythagorean theorem, $a=\sqrt{b^2+h^2}$.

Knowing this, he could now deal with the bigger triangle: $BS=a+HS=a+b$.

Applying again the law of sines gives $\dfrac{BS}{\sin C}=\dfrac{R}{\sin B}=\dfrac{R+h}{\sin S}$,

and replacing the convenient equivalents gives $\dfrac{BS}{\sin\theta}=\dfrac{R}{\sin(90^\circ-\theta)}=\dfrac{R+h}{\sin 90^\circ}$.

It follows that $R=\dfrac{(R+h)\sin(90^\circ-\theta)}{\sin 90^\circ}=(R+h)\cos\theta$, so $\cos\theta=\dfrac{R}{R+h}=\dfrac{1}{1+\dfrac{h}{R}}$.

Hence $1=\cos\theta+\dfrac{h\ \cos\theta}{R}$, or $1-\cos\theta=\dfrac{h\ \cos\theta}{R}$,

and this finally gives Biruni's famous equation $\boxed{R=\dfrac{h\ \cos\theta}{1-\cos\theta}}$,

which can be further simplified to $R=\dfrac{h}{\dfrac{1}{\cos\theta}-1}$ or $R=\dfrac{h}{\sec\theta-1}$.

(The *geometric* distance to the horizon can also be deduced as $BS=R\ \tan\theta$.)

4. Exactly how exact?

Armed with his newly found formula and with the data he had taken at Nandana (652.055 cubits for the height, and 0º34′ for the dip), he stated that the earth's radius was 12,851,369.845 cubits (6335.725 km).[1] Had he had a calculator to hand, he would have got 13,331,728.352 cubits (6572.542 km) instead but, in his time, calculations were performed in a rather mental way. Instead of using cos(0º34′), he used sin(90º – 0º34′), that is, sin(89º26′), which is pretty much the same, but for the fact that it was

[1] Or, as Biruni puts it, 12,851,369 cubits and 50′42″.

then also fashionable to express numbers in the old Babylonian way and, the value he gives in his *Masudic Canon* for this sine is 0º59′59″49‴2⁗28′′′′′, which is slightly inaccurate. The correct number (given by our modern computers and expressed in the same historical fashion) would have been 0º59′59″49‴26⁗9′′′′′; hence the disagreement. He then went on to find the earth's girth, which he stated to be 80,780,039 cubits and 1′33″ (clearly, the result of rounding π to 22/7).

5. Biruni's Peak

Let me readdress now the issue of the mountain. I have already mentioned how Mercier (1994:183) pinpointed it as likely to be one standing about 1.4 km south-southwest of the fort and 478 m above sea level. I haven't found such a place. Instead, I found a mountain 780 m southwest of the fort (32º43′14.60″ N, 73º13′26.00″ E) that stands 533 m above sea level. Because Biruni said 'west,' I also checked another candidate 3.8 km west of the fort (32º43′46.08″ N, 73º11′20″ E) standing 677 m above sea level. From them, the plain to the south looks strikingly flat. It is hard to imagine a flatter thing, or one better suited for Biruni's purpose. Yet, of course, it cannot be as flat as the sea.

Slowly and somewhat unevenly, it slopes away to the south getting higher and then lower than at the mountain's foot. In Figure 3 you can see how the plain evolves as seen from the 533 m peak: it first sinks to about 202 m above sea level, then rises again to about 211 or even 212 m, and then resumes sloping away for as far as the eye can see. All this is important because, if you take the mountain's height and subtract from it the higher of the plain's levels, you find how much the mountain stands above these levels, that is, about 321 m or 652 cubits, which is Biruni's figure exactly. But it wouldn't be wise to measure a mountain from this far away. If you come closer and allow Biruni to take his measurements from the area between 2½ and 3½ km south of our peak and gently sloping from about 212 to 211 m above sea level, you get again 652 cubits exactly. The choice of this area has the advantage not only of being close enough to allow reliable measurements, but also, and above all, of being on a level with the apparent horizon, which is seen from it as the plain's upper levels. Seemingly, this was Biruni's logical way of finding the 'plain's [otherwise relative] level.'

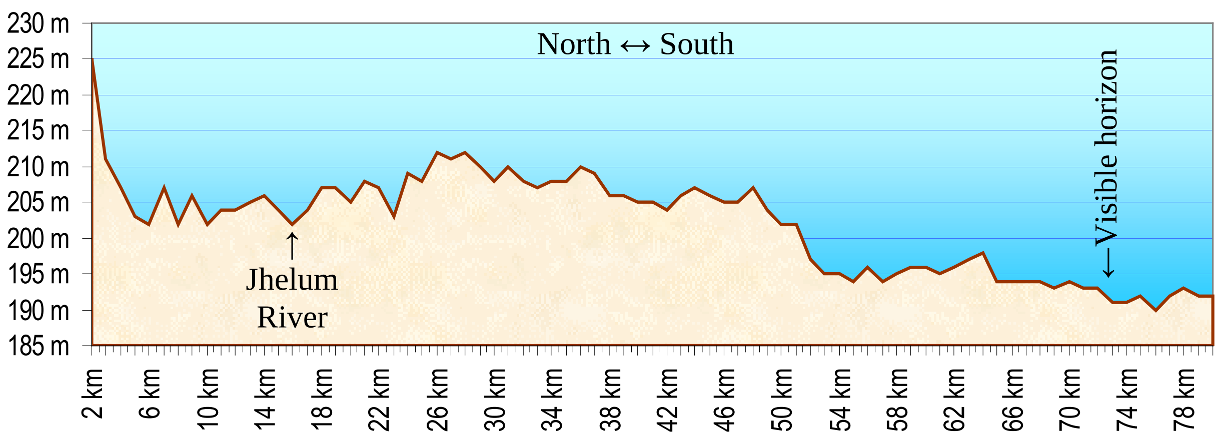


Figure 3: The plain is not perfectly flat. It slopes unevenly away to the south. For illustrative purposes, heights and distances are not to scale nor is the earth's curvature shown. Between about 26 and 28 km south of the 533 m high peak, the plain rises to about 211 or 212 m above the sea. Relative to this area, our peak stands 321½ m (or 652 cubits) high.

Measuring the dip of the horizon from the Salt Range tops is easier said than done. As Rizvi (1979:619) reports, it takes a good deal of patience to wait for the right atmospheric conditions to get a clear view of the horizon. It is best to try after rain has cleared away the haze and dust from the air. It is possible that Biruni may have lived in rainier times for, after all, there must be some reason why the name Nandana means *Paradise* in Sanskrit. Even so, the dip he reports, 0º34′, though strikingly accurate, is not given to the seconds of a degree and, therefore, does not allow us to use mathematics this time to find out whether he averaged it out of several measurements or not. Perhaps the prevailing poor visibility forced him to be satisfied with fewer measurements, or perhaps he had some other reason for the figure he gave.

I found by a formula that the *apparent* dip θ of the horizon (that is, taking into account ray-bending at mean air conditions)[1] from this mountain is about 0º32′, which is almost the same as Biruni reports. Care must be taken to provide the formula with the right mountain height h for this problem, which is neither 533 m above sea level (we'll consider this one later for sport) nor Biruni's 321½ m above the plain's top grounds, but 340 m above the farthest visible point on the horizon (which is about 72½ km from our peak and about 193 m above sea level). 0º32′ is close to Biruni's figure, but not quite. At first sight, a difference of just a couple of minutes seems quite negligible, but it is hardly so, and this is one of the drawbacks of Biruni's method: that it is so dependent on taking the dip angle with such hair splitting accuracy, that just a minute of a degree results in a difference of hundreds of kilometres for the computed earth-radius, not to mention the difficulty in taking such angle with any reliable accuracy at all![2] Let us pay more attention to Biruni's own words (*Canon* 5.7, ed. 1954-6 2:530; *Tahdid*, Ali's tr., p. 188):

> 'I changed to another way owing to having found in a region in India a mountain peak facing toward a wide flat plain whose flatness served as the smooth surface of the sea. Then on its peak I gauged the intersection of heaven and earth [the horizon] in the prospect, and I found it by an instrument to incline from the East-West line [southern astr. hor.] a little less than ⅓¼ of a degree, and I took it as 0º34′. I derived the height of the mountain taking the summit in two places, and I found it to be $652\frac{1}{20}$ cubits.'

So he took the dip as 'a little less than ⅓¼ of a degree' and rounded it down to 0º34′. Yet, a *little* less than ⅓¼ of a degree should round up to 0º35′. Why 0º34′ and not 0º35′? There is just one minute of difference, but one minute that makes all the difference: 0º34′ got him closer to Mamun's value than 0º35′ or 0º33′. Certainly, Mamun's value was serving him as a reference all the time, and the difference between 32′, 33′, 34′, or 35′ is so tiny that he might indeed have believed in earnest that 34′ must be the right angle his instrument was reading. Had he chosen a different dip, he would have gone wrong by hundreds of kilometres. It is only by chance, therefore, that the compensating features of the lucky scenario chosen led him to a figure so close to the truth! Even his own measurements do lead to a value slightly worse than Mamun's, should the cosine mistake alone be corrected! (Oh, by the way, note that the little ⅓¼ detail may lead us to suspect that his instrument was accurate to 5 minutes of a degree; I leave the guesswork to you!)

Scientifically speaking, Biruni's method leads to results infected by ray bending when applied to other scenarios. For example, as mentioned above, in the completely imaginary case that our 533 m mountain looked south onto the sea (which, remember, it does not), the apparent (refracted) dip of the horizon would be about 0º40′22″. With these data, Biruni's formula gives an earth-radius (for Nandana's latitude) of 7734 km, which is too large by about 6/5 due to ray bending. His celebrated method, nevertheless, bears the immortal originality of saving a lot of walking through deserts and, provided we take accurate measurements and compensate for ray bending, it certainly does correctly yield the earth's radius! (see Figure 4 below.)

[1] Mean air conditions are taken to be 15ºC of temperature, 1013.25 mb of pressure, 6.49ºC/km of lapse rate, 0% of relative humidity, and 450 ppm of CO_2 content.
[2] The main difficulties arise from the ever-changing air conditions and from the limitations of human eyesight.

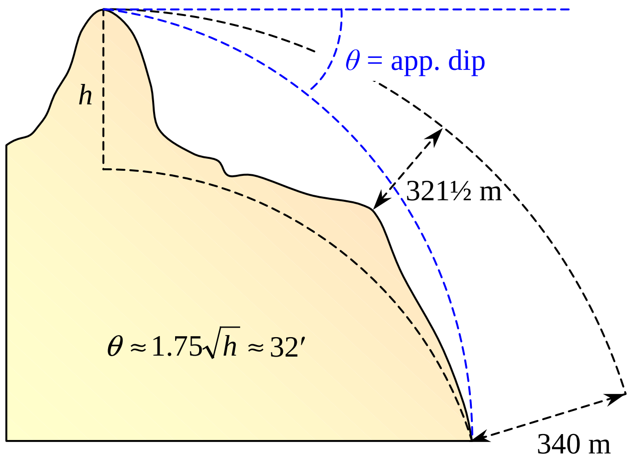


Figure 4: Biruni's method leads to results infected by ray bending. Yet, provided you allow for this and take precise measurements, it certainly yields the earth's radius![1]

To be fair to Biruni, it must be said that he was not just lucky. Clearly, he picked his landscape most carefully after testing his method on several places, noting the disparity of the results, and finally picking the one that best matched that of Mamun's teams (see Figure 5). It is to his credit that he openly regarded his own method as a curious complement to their work, accepting their result as more reliable, because, as he said (*Canon* 5.7, tr. Mercier 1994:183), 'Their instrument was more refined, and they took greater pains in its accomplishment.'

Figure 5: View of Biruni's happy choice of landscape.

[1] Note added 2012: See an interactive illustration of B's method at http://www.geogebratube.org/student/m7159

6. A final note

When I started this work, I first suspected another peak west of Nandana of being the one Biruni used, perhaps because of the resemblance between its 677 m height and Biruni's 652-cubit figure. (What a meaningless connection!) After drawing the profile of the landscape south of this, I was inspired to subtract the mountain's height from the highest point on the profile. The result was close to Biruni's mountain's height, but not quite: this was not the peak! Then I remembered another peak near Nandana Fort and, at once, I knew this was the one I was seeking. After drawing the new profile and doing the same intuitive subtraction, it was most gratifying to see a perfect match with Biruni's number! All that remained was to make sense of it all.

The reader is invited to suggest new ideas, easier wordings for the difficult concepts expressed in this paper, and new mathematical approaches to reveal Biruni's actual instrument and readings, or to contest any part of this paper. I'm especially interested in trying to find the two actual points from which he measured the mountain. Though the most likely candidates are given in the text, I have included here a fuller list (see Table 1), which of course can be expanded by the caring reader:

Table 1. Points from which Biruni may have measured the mountain's height.

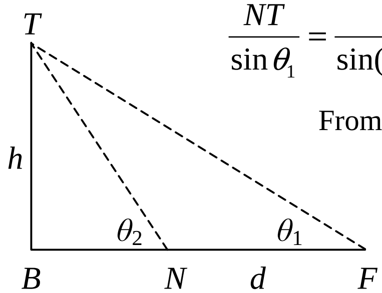
Likelihood	Instrument's accuracy	Dist. between the 2 points in metres	Dist. between the 2 points in cubits	1st angle	2nd angle	Derived height in cubits
Likely	½°	896.767	1819	5½°	7½°	652^{cb} 3′17″10‴
Likely	¼°	677.012	1373¼	5¼°	6½°	652^{cb} 3′17″54‴
Possible	¼°	2070.230	4199¼	4°	7¼°	652^{cb} 3′18″05‴
Possible	⅓°	972.360	1972⅓	4¾°	6⅓°	652^{cb} 3′17″50‴
Possible	⅓°	2099.256	4258⅛	4°	7⅓°	652^{cb} 3′18″00‴
Possible	5′	2100.920	4261½	4°10′	7°55′	652^{cb} 3′17″47‴
Possible	5′	547.887	1111⅓	4°40′	5°25′	652^{cb} 3′18″36‴
Possible	5′	71.916	145⅞	4°35′	4°40′	652^{cb} 3′18″22‴
Unlikely	1′	753.304	1528	4°5′	4°54′	652^{cb} 3′18″20‴
Unlikely	1′	1153.127	2339	4°24′	6°4′	652^{cb} 3′17″32‴
Unlikely	1′	497.314	1008¾	4°	4°29′	652^{cb} 3′17″57‴
Unlikely	1°	2774.029	5626½⅓	4°	10°	652^{cb} 3′18″02‴
Unlikely	½°	3916.885	7945	3½°	13½°	652^{cb} 3′17″52‴
Unlikely	¼°	1867.484	3788	4½°	8¼°	652^{cb} 3′18″27‴
Unlikely	¼°	1681.623	3411	6½°	15¾°	652^{cb} 3′17″33‴
Unlikely	⅓°	3942.192	7996⅓	2¾°	6⅔°	652^{cb} 3′17″56‴
Unlikely	⅓°	2154.287	4369¾	3¾°	6⅔°	652^{cb} 3′18″07‴
Unlikely	⅓°	1542.535	3128⅞	5¼°	9⅓°	652^{cb} 3′17″57‴
Unlikely	⅓°	865.523	1755⅝	6½°	9⅓°	652^{cb} 3′17″59‴

Note that to stay within the area of about 3 ± ½ km south of our peak and gently sloping from about 212 to 211 m above the sea, the angles must be between 5¼° and 7⅓°. If it is any bigger than 7⅓° you'll get closer to the mountain, but you will also be higher than 212 m. From this position you won't be able to compute the height of Biruni's mountain unless you lower the ground you are standing on by arguing that some sediment has piled up that wasn't there a thousand years ago.[1]

[1] Note added 2012: See an interactive illustration of this point at http://www.geogebratube.org/student/m9062

Appendix

Today, we measure mountain heights with *theodolites*, *graphometers*, or *laser range finders* instead of astrolabes, and use a slightly simpler formula derived where else but from our beloved *law of sines*, which applied to the two triangles in Figure 6, allows us to know that



$$\frac{NT}{\sin\theta_1} = \frac{d}{\sin(\theta_2-\theta_1)} = \frac{FT}{\sin\theta_2} \quad \text{and that} \quad \frac{NT}{\sin 90^\circ} = \frac{h}{\sin\theta_2} = \frac{NB}{\sin(90^\circ-\theta_2)} \ .$$

From these relations we deduce that $h = NT\,\sin\theta_2$ and, therefore,

$$h = \frac{d\,\sin\theta_1 \sin\theta_2}{\sin(\theta_2-\theta_1)}$$, which works like the one Biruni used.

Figure 6: Finding the height of a mountain.

If you have a restless mind and wish to know more about these matters, perhaps you may be intrigued by the following problem: How do you find the height, not of a hill, but of a building on top of a hill? Again the *sine law* comes to our rescue! Have a look at Figure 7.

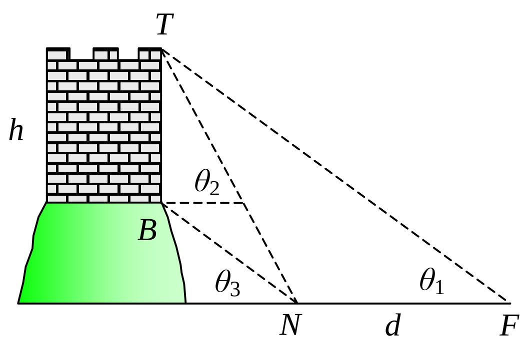


Figure 7: To calculate the height *h* of a building on a hilltop, we need to measure the distance *d* from the two points *N*, *F* at ground level from which we are to take the angles θ to the top *T* and base *B* of the building.

Knowing that $$\frac{NT}{\sin\theta_1} = \frac{d}{\sin(\theta_2-\theta_1)} = \frac{FT}{\sin(180^\circ-\theta_2)}$$

and that $$\frac{NT}{\sin(90^\circ+\theta_3)} = \frac{h}{\sin(\theta_2-\theta_3)} = \frac{NB}{\sin(90^\circ-\theta_2)} \ ,$$

we can deduce that $$NT = \frac{d\,\sin\theta_1}{\sin(\theta_2-\theta_1)} \quad \text{and that} \quad NT = \frac{h\,\sin(90^\circ+\theta_3)}{\sin(\theta_2-\theta_3)} \ .$$

The connection is then plain to see: $$h = \frac{d\,\sin\theta_1 \sin(\theta_2-\theta_3)}{\sin(\theta_2-\theta_1)\cos\theta_3} \ .$$

You must be careful, though, for the angles must be taken from the ground, not from eye level and, if you want to save yourself discomfort, you might like to use the help of a mirror like this: build a giant protractor like the one in Figure 8, and tilt the mirror until the image of the mountaintop reflected on it is made to touch the ground at your feet directly under the plummet line of sight. The tilt of the mirror μ gives the angle θ from your feet to the mountaintop by the following relation

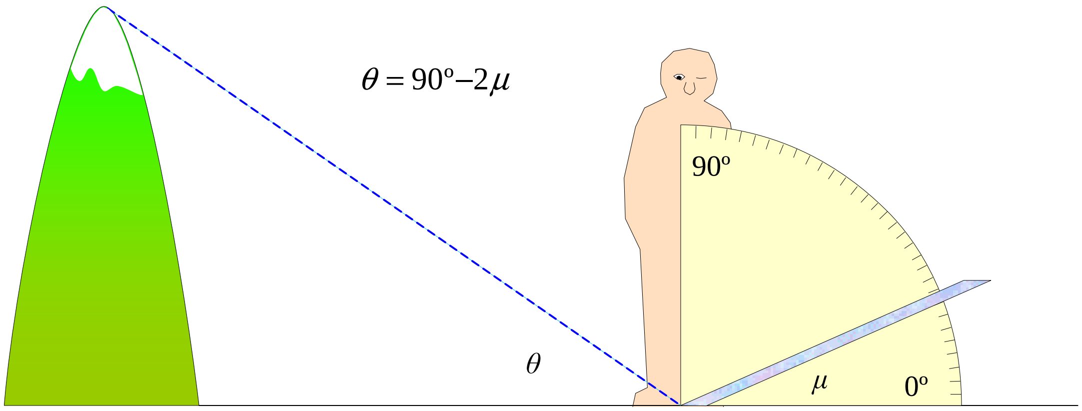


Figure 8: Angles must be taken from the ground, for which purpose you might like to tilt a mirror until the reflected mountaintop touches the ground at your feet under the plummet line of sight. Only bear in mind that, because mirrors are whimsical little things, the angle θ to the mountaintop is 90º minus twice the tilt μ of the mirror.

Alternatively, you may not need to build a protractor at all. Just lay the mirror flat on the ground with the help of a spirit level. Then place a stick upright between the mirror and the mountain, in such a way that the reflected tops of the mountain and the stick are seen to be in line with a spot drawn on the mirror. The angle θ from this spot to the mountaintop is the arctangent of the division of the height h of the stick by the distance d between the spot and the base of the stick (see Figure 9).

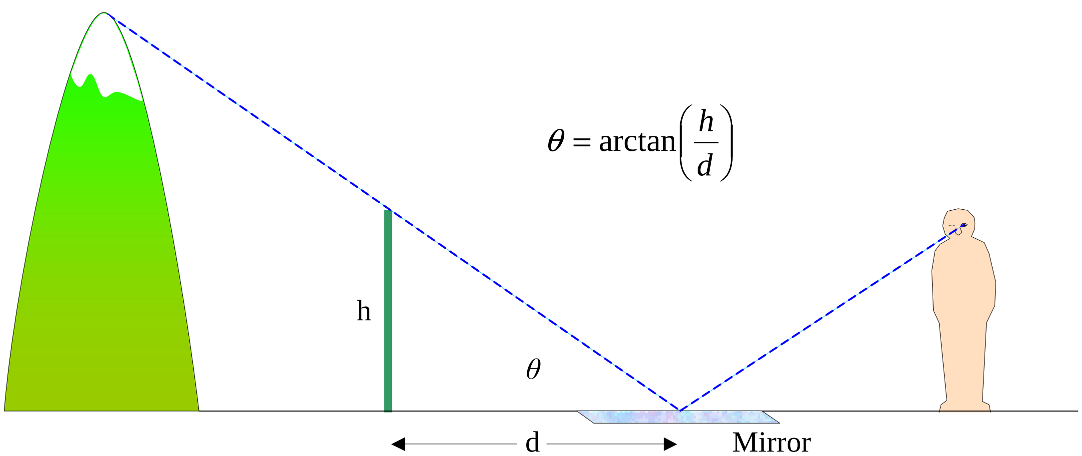


Figure 9: An alternative method of finding the angle θ from a point on the ground to a hilltop makes use of a mirror on the ground and a stick lined up with the reflected hilltop.[1] This method was first described in Euclid's *Optics* (Proposition 19).

[1] The mirror formulas on this page are derived from the all purpose sine law.

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