Inequality Aversion and Separability in Social Risk Evaluation*

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Abstract

This paper examines how to satisfy "independence of the utilities of the dead" (Blackorby et al., 1995; Bommier and Zuber, 2008) in the class of "expected equally distributed equivalent" social orderings (Fleurbaey, 2010) and inquires into the possibility to keep some aversion to inequality in this context. It is shown that the social welfare function must either be utilitarian or take a special multiplicative form. The multiplicative form is compatible with any degree of inequality aversion, but only under some constraints on the range of individual utilities.
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1 Introduction

The evaluation of social situations involving risk has been a debated topic ever since Harsanyi (1955) published his theorem on utilitarianism. He interpreted his theorem as vindicating utilitarianism. But an equivalent interpretation is that this is an impossibility theorem for those who would like to give some priority to the worst-off. If one wants to incorporate such priority in the evaluation criterion, one must relax one of Harsanyi's central postulates, social rationality or Pareto.[1]

Based on the observation that the Pareto principle in risky contexts is not as compelling as in riskless contexts, because when individuals take risks, by definition they are not fully informed about the final consequences of their choices, Fleurbaey (2010) has proposed to restrict the application of the Pareto principle to riskless situations and to risky situations that involve no inequalities ex post. With such restrictions, one obtains a class of criteria that compute the expected value of the "equally distributed equivalent" (EDE) utility.[2] Any degree of inequality aversion can be put in the EDE function. In the extreme, the expected value of the lowest utility, or expected maximin, is such a criterion. A leximin refinement, which lexicographically examines the expected value of utility at successive ranks in the distribution, is also characterized in Fleurbaey (2010).

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[1] On the interpretation of the implications of Harsanyi's theorem, see Weymark (1991) and Broome (1991). For a defense of Paretian ("ex ante") criteria that evaluate the distribution of individual expected utilities with some inequality aversion, see, e.g., Diamond (1967) and Epstein and Segal (1992). For a defense of rational ("ex post") criteria that compute the expected value of an inequality averse social welfare function, see, e.g., Adler and Sanchirico (2006) and Fleurbaey (2010).

[2] The equally distributed equivalent (Atkinson, 1970) of a given distribution of utility is the level of utility that, if enjoyed uniformly by all individuals, would yield the same social welfare as the contemplated distribution.

Unlike the standard leximin criterion, which is subgroup separable, the expected leximin criterion is unfortunately highly non-separable across individuals. If Robinson wants to climb a tree, this is fine if he is worse-off than Friday in all states of nature, or better-off in all states of nature. But if he may be better-off or worse-off than Friday depending on whether he falls from the tree or not, his adventure decreases the expected value of the lowest utility. One therefore sees that the evaluation depends on the utility level of Friday, even when Friday is on the other side of the island, totally unconcerned.

A natural question which then arises is whether introducing some requirement of separability imposes serious restrictions on the degree of inequality aversion that can be incorporated in the expected EDE criterion. More generally, the tension between the Pareto principle, inequality aversion, and separability that is highlighted in this literature deserves further scrutiny in the direction of having some separability with perhaps less inequality aversion, and possibly less of the Pareto principle.

Our results show that the outlook of the tension is rather complex, and involves a fourth consideration, namely, the domain of admissible individual utilities. It is possible to introduce a substantial form of separability and retain an arbitrarily high degree of inequality aversion, but provided the utility domain is specific (and narrower, the greater the inequality aversion). For standard utility domains (e.g., the positive real line), the degree of inequality aversion is quite limited.

Interestingly, in all configurations it is shown that the functional form of the social welfare function must take a simple multiplicative form. Finally, we also show that separability may come in conflict with the Pareto principle when it takes a slightly stronger form.

The definition of separability for risky prospects is a delicate issue. Consider the following prospects, described by matrices in which a cell gives the utility of an individual in a particular state of the world (rows are for two individuals, columns for two equiprobable states). An egalitarian with even a weak degree of inequality aversion would like the social ordering to satisfy

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \text{ preferred to } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

because individual expected utilities are the same and less inequality ex post is obtained in the preferred prospect. The second individual faces the same personal prospect in both social prospects. If full separability were applied, one could arbitrarily change this "unconcerned" individual's prospect in order to obtain

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ preferred to } \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

in contradiction with the same egalitarian rationale. This example shows that full separability would make it impossible to be sensitive to the correlation between individual utilities.

One must therefore be cautious and consider only weaker forms of separability. In this paper we restrict the application of separability to unconcerned individuals who bear no risk. Changing their riskless prospect cannot then affect the correlation of their payoffs with the others. In this paper we consider two forms of separability. In the main one studied here, social preference is not altered if one changes the prospect of unconcerned riskfree individuals. In the stronger variant, social preference is not affected if one removes or introduces individuals who are unconcerned and riskfree.

This paper is related to two other papers. In a more specific model with successive generations facing a risk of utility and a risk of extinction, Bommier and Zuber (2008) study a condition of independence with respect to the utility of the past generations (that is called "independence of the utilities of the dead", following Blackorby et al. (1995)). This condition is similar to ours because in their model past generations are unconcerned and bear no risk.

Bommier and Zuber, however, restricted the Pareto principle to situations in which risks are independent across individuals and assumed that the evaluation relied on expected social welfare. Here, in contrast, we restrict Pareto to riskless and to egalitarian situations, and, as far as social rationality is concerned, we only assume that social evaluation satisfies statewise dominance (except in one result). Because of this difference, we obtain different functional forms that do not satisfy their Pareto condition, although the multiplicative structure is similar. Interestingly, however, we actually retrieve their criterion in the discussion of the stronger condition of separability, with a variant of the Pareto principle that applies to situations in which only one individual takes a risk. Finally, another difference between the two papers is that we consider a more general class of utility domains and this opens new possibilities.

The other related paper is Fleurbaey et al. (2010)[3], which also introduces this form of separability with respect to unconcerned and riskfree individuals, but focuses on a different issue, namely, the possibility to let the evaluation of ex post consequences depend on fairness in the distribution of ex ante prospects. This has strong implications on the application of statewise dominance, which then becomes a very weak rationality condition. This other paper also does not assume that the ex ante evaluation of individual prospects, in the application of the Pareto principle, is made in terms of expected utility.

A connection between separability and inequality aversion has also been made in a different setting by Foster and Shneyerov (2000). In the context of inequality

[3]Fleurbaey, M., Gajdos, T., Zuber, S.: Social rationality, separability, and equity under uncertainty. CORE Discussion Paper, no 2010/37 (2010).

measurement, they study the problem of defining within-group inequality either directly or as the difference between total inequality and between-group inequality. For a specific definition of within-group and between-group inequality, they show that this "path independence" property entails subgroup separability and moreover pinpoints a specific member of the generalized entropy class. Formally, our separability property is weaker, and we obtain a class of social orderings which allows for some limited variations of inequality aversion.

The paper is organized as follows. The next section introduces the framework. The axioms and our main result are presented in Section 3. The implications for inequality are examined in Sections 4. Section 5 studies the stronger form of separability. The final section concludes. An appendix, made available on the authors' website, provides further material about the results of Sections 4 and 5.

2 The framework

The framework is the same as in Fleurbaey (2010). The population is finite and fixed, $N = \{1, \ldots, n\}$. The set of states of the world is finite, $S = \{1, \ldots, m\}$, and the evaluator has a fixed probability vector $\pi = (\pi_s)_{s \in S}$, with $\sum_{s \in S} \pi_s = 1$. This probability vector corresponds to the evaluator's best estimate of the likelihood of the various states of the world. We therefore abstract from the problem of aggregating beliefs. Given that what happens in null states can be disregarded, we simply assume that $\pi_s > 0$ for all $s \in S$. Vector inequalities are denoted $\geq$, $>$ and $\gg$ as usual.

The evaluator's problem is to rank prospects $U = (U_i^s)_{i \in N, s \in S} \in \mathbb{R}^{nm}$, where U_i^s describes the utility attained by individual i in state s. Let $X \subseteq \mathbb{R}$ be an interval (not necessarily bounded) and $\mathcal{L} = X^{nm}$ denote the relevant set of prospects over which the evaluation must be made. The social ordering (i.e., a complete, transitive binary relation) over the set $\mathcal{L}$ is denoted R (with strict preference P and indifference I).

Let U_i denote $(U_i^s)_{s\in S}$ and U^s denote $(U_i^s)_{i\in N}$. Let $[U^s]$ denote the riskless prospect in which vector U^s occurs in all states of the world. Two subsets of $\mathcal{L}$ must be singled out: $\mathcal{L}^c$ will denote the subset of riskless prospects (i.e., $U^s = U^t$ for all $s, t \in S$); $\mathcal{L}^e$ will denote the subset of egalitarian prospects (i.e., $U_i = U_j$ for all $i, j \in N$). For two prospects $U, \widetilde{U} \in \mathcal{L}$ and a subset $Q \subset N$, let $(U_Q, \widetilde{U}_{N\backslash Q})$ denote the prospect V such that $V_i = U_i$ for all $i \in Q$ and $V_i = \widetilde{U}_i$ for all $i \in N\backslash Q$.

The utility numbers U_i^s are assumed to be fully measurable and interpersonally comparable. They may measure any subjective or objective notion of advantage that the evaluator considers relevant for social evaluation. It is assumed that, for one-person evaluations, the evaluator considers that the expected value $EU_i = \sum_{s\in S} \pi_s U_i^s$ correctly measures agent i's ex-ante interests. We also assume that for every $s \in S$, the vector U^s fully describes the relevant features of the final situation occurring in state s. Thus, the social preferences over final situations need not be state dependent. This means that U^s is deemed better than V^s in state s if and only if $[U^s] R [V^s]$. In other words, there is no need to introduce preferences over final consequences as they are equivalent to the social ordering R restricted to riskless prospects. This is a convenient and innocuous simplification.

3 Multiplicative and Additive Criteria

We now introduce some requirements that one may wish to impose on the social ordering R. First, as explained above, there are two Pareto conditions, one for riskless situations, the other for situations in which full equality prevails in all states of the world.

Axiom 1 (Strong Pareto for no risk) *For all $U, V \in \mathcal{L}^c$, if $U_i \geq V_i$ for all*

$i \in N$, then URV. If furthermore $U_j > V_j$ for some $j \in N$, then UPV.

Axiom 2 (Weak Pareto for equal risk) *For all $U, V \in \mathcal{L}^e$, if $EU_i > EV_i$ for all $i \in N$, then UPV.*

Social rationality is expressed here by statewise dominance. This is a compelling requirement. Violating it would mean that one would sometimes prefer a prospect that is bound to generate worse consequences than another.[4]

Axiom 3 (Weak dominance) *For all $U, V \in \mathcal{L}$, if $[U^s]R[V^s]$ for all $s \in S$, then URV.*

The last key requirement is an independence condition, which says that the social ranking of two prospects is independent of the level of utility of individuals who bear no risk and have the same utility in the two prospects.

Axiom 4 (Independence of the utilities of the sure) *For all $U, V \in \mathcal{L}$ and $\widetilde{U}, \widetilde{V} \in \mathcal{L}^c$, and for all $Q \subset N$,*

$$\left(U_Q, \widetilde{U}_{N\setminus Q}\right) R \left(V_Q, \widetilde{U}_{N\setminus Q}\right) \Longleftrightarrow \left(U_Q, \widetilde{V}_{N\setminus Q}\right) R \left(V_Q, \widetilde{V}_{N\setminus Q}\right).$$

The restriction to individuals who bear no risk is important, as explained in the introduction. If, in our model, individuals are successive generations and we interpret the index $i = 1, \ldots, n$ as the birth date of a generation, one may want to apply independence of the utilities of the sure to the first generations up to any particular date. This is what Blackorby et al. (1995) and Bommier and

[4]Certain apparent violations of dominance seem rational (Grant, 1995). If a parent would rather flip a coin to allocate a sweet between two children than give it to one child without flipping a coin, this seems to violate dominance because the final distribution of sweets is the same anyway. But this behavior is compatible with dominance if, as is natural, one incorporates the fairness of the procedure in the description of the final consequences.

Zuber (2008) have called "Independence of the utilities of the dead".[5] One may object that Independence of the utility of the sure is stronger because it applies to any subpopulation, whereas with Independence of the utility of the dead the unconcerned are always the past generations. Under Anonymity, however, the two axioms are equivalent.[6]

Finally, we will make use of two basic axioms of anonymity and continuity.

Axiom 5 (Anonymity) *For all $U, V \in \mathcal{L}$, if there exists a bijection $\rho : N \to N$ such that $U_i = V_{\rho(i)}$ for all $i \in N$, then UIV.*

Axiom 6 (Continuity) *For all $U, V \in \mathcal{L}$, if $(U(k))_{k\in\mathbb{N}} \in \mathcal{L}^{\mathbb{N}}$ is such that $U(k) \to U$ and $U(k)RV$ for all $k \in \mathbb{N}$, then URV; if $VRU(k)$ for all $k \in \mathbb{N}$, then VRU.*

We are now able to state our main result.

Proposition 1 *The social ordering R satisfies the six axioms if and only if one of the following two statements holds:*

1. *For all $U, V \in \mathcal{L}$*

$$URV \iff \sum_{s\in S} \pi_s \frac{1}{n} \sum_{i\in N} U_i^s \geq \sum_{s\in S} \pi_s \frac{1}{n} \sum_{i\in N} V_i^s. \tag{1}$$

[5] Independence of the utilities of the dead is an important principles in intergenerational ethics. In the certainty case, it corresponds to Postulate 3b of Koopmans (1960) as discussed by Asheim Mitra and Tungodden (2012) who defend recursive social welfare objectives defined by this Postulate and a stationarity condition. These principles however apply to situations with an infinite number of generations. This infinite population case raises specific issues as discussed by Lauwers (2012). We do not address these issues in the present paper.

[6] The reader can in fact check that in the proof of Proposition 1, only Independence of the utilities of the dead is actually used.

2. *There exist $\alpha, \beta \in \mathbb{R}$ satisfying $\alpha x + \beta > 0$ for all $x \in X$ such that, for all $U, V \in \mathcal{L}$,*

$$URV \iff \frac{1}{\alpha}\sum_{s\in S}\pi_s \prod_{i\in N}(\alpha U_i^s+\beta)^{\frac{1}{n}} \geq \frac{1}{\alpha}\sum_{s\in S}\pi_s \prod_{i\in N}(\alpha V_i^s+\beta)^{\frac{1}{n}}. \tag{2}$$

Proof. If the social ordering R satisfies (1) or (2), then it clearly satisfies the axioms.

Now assume that the social ordering R satisfies the axioms. Let $\mathbf{1}_n$ denote the n-vector $(1,\ldots,1)$. By Strong Pareto for no risk, for every $U^s \in X^n$, there exists $a, b \in X$ such that $[b\mathbf{1}_n]\, R\, [U^s]\, R\, [a\mathbf{1}_n]$. By Continuity, there exists $x \in X$ such that $[x\mathbf{1}_n]\, I\, [U^s]$. By Strong Pareto for no risk, it is unique. This value of x defines the EDE function $e(U^s)$. By Anonymity, e is symmetric. By Strong Pareto for no risk, it is increasing in each argument. By definition, it satisfies $e(x,\ldots,x) = x$ for all $x \in X$.

By Weak dominance, for all $U \in \mathcal{L}$, $UI\left(e\left(U^1\right),\ldots,e\left(U^m\right)\right)$. The quantity $\sum_{s\in S}\pi_s e(U^s)$ belongs to X because X is an interval. By Continuity and Weak Pareto for equal risk, one must have

$$\left(e\left(U^1\right),\ldots,e\left(U^m\right)\right) I \left[\left(\sum_{s\in S}\pi_s e(U^s)\right)\mathbf{1}_n\right].$$

Therefore, by transitivity and Strong Pareto for no risk, for all $U, V \in \mathcal{L}$, $URV \iff \sum_{s\in S}\pi_s e(U^s) \geq \sum_{s\in S}\pi_s e(V^s)$.

The remainder of the proof is closely related to a similar result by Keeney and Raiffa in the case of multidimensional risks (Keeney and Raiffa, 1976, Th. 6.1, p. 289). Let u^* be an arbitrary number in X. Let $\hat{e}$ be the function defined as $\hat{e} \equiv e - u^*$, which implies $\hat{e}(u^*,\ldots,u^*) = 0$. By definition, the function $\hat{e}$ is symmetric, and for all $U, V \in \mathcal{L}$, $URV \iff \sum_{s\in S}\pi_s\hat{e}(U^s) \geq \sum_{s\in S}\pi_s\hat{e}(V^s)$.

Independence of the utilities of the sure tells us that, for all $i \in \{1, \ldots, n-1\}$, for all $U \in \mathcal{L}^c$ and all $V, \widetilde{V} \in \mathcal{L}$:

$$\sum_{s \in S} \pi_s \hat{e}(U_1^s, \ldots, U_i^s, V_{i+1}^s, \ldots, V_n^s) \geq \sum_{s \in S} \pi_s \hat{e}(U_1^s, \ldots, U_i^s, \widetilde{V}_{i+1}^s, \ldots, \widetilde{V}_n^s)$$
$$\iff \sum_{s \in S} \pi_s \hat{e}(u^*, \ldots, u^*, V_{i+1}^s, \ldots, V_n^s) \geq \sum_{s \in S} \pi_s \hat{e}(u^*, \ldots, u^*, \widetilde{V}_{i+1}^s, \ldots, \widetilde{V}_n^s).$$

Because vNM utility functions are unique up to an increasing affine transformation, there must exist two functions f_i and g_i such that:

$$\begin{aligned} \hat{e}(U_1^s, \ldots, U_i^s, U_{i+1}^s, \ldots, U_n^s) &= f_i(U_1^s, \ldots, U_i^s) + \\ &\quad g_i(U_1^s, \ldots, U_i^s) \hat{e}(u^*, \ldots, u^*, U_{i+1}^s, \ldots, U_n^s), \end{aligned} \tag{3}$$

where $g_i(U_1^s, \ldots, U_i^s) > 0$ for all $(U_1^s, \ldots, U_i^s) \in X^i$.

Define $a_1 \equiv f_1$, $b_1 \equiv g_1$, and, for all $i \in \{2, \ldots, n-1\}$, $a_i(U_i^s) = f_i(u^*, \ldots, u^*, U_i^s)$ and $b_i(U_i^s) = g_i(u^*, \ldots, u^*, U_i^s)$, and $a_n(U_n^s) = \hat{e}(u^*, \ldots, u^*, U_n^s)$. Equation (3) implies that, for all $i \in \{1, \ldots, n-1\}$:[7]

$$\begin{aligned} \hat{e}(u^*, \ldots, u^*, U_i^s, U_{i+1}^s, \ldots, U_n^s) &= a_i(U_i^s) + \\ &\quad b_i(U_i^s) \hat{e}(u^*, \ldots, u^*, U_{i+1}^s, \ldots, U_n^s). \end{aligned} \tag{4}$$

Repeated applications of Equation (4) yield:

$$\begin{aligned} \hat{e}(U_1^s, \ldots, U_n^s) &= a_1(U_1^s) + b_1(U_1^s) \left(a_2(U_2^s) + b_2(U_2^s) \left(\ldots \right) \right) \\ &= a_1(U_1^s) + \sum_{i=2}^{n} a_i(U_i^s) \prod_{j=1}^{i-1} b_j(U_j^s). \end{aligned}$$

Using the normalization condition $\hat{e}(u^*, \ldots, u^*) = 0$ in Equation (4), we also obtain that $a_i(U_i^s) = \hat{e}(u^*, \ldots, u^*, U_i^s, u^*, \ldots, u^*)$ for all $i \in \{1, \ldots, n-1\}$ (the same is also true for a_n by definition). Therefore, by symmetry of $\hat{e}$, all the functions a_i are the same (increasing) function ϕ, such that $\phi(u^*) = 0$.

[7] In the case $i = 1$, the equation is $\hat{e}(U_s^1, U_s^2, \ldots, U_s^n) = a_1(U_s^1) + b_1(U_s^1) \hat{e}(u^*, U_s^2, \ldots, U_s^n)$.

The symmetry of the function $\hat{e}$ also implies that, for all $i \in \{1, \dots, n-1\}$:

$$\hat{e}(U_1^s, \dots, U_i^s, U_{i+1}^s, \dots, U_n^s) = \hat{e}(U_1^s, \dots, U_{i+1}^s, U_i^s, \dots, U_n^s).$$

Using Equation (4) applied to $(u^*, \dots, u^*, U_i^s, U_{i+1}^s, u^*, \dots, u^*)$ and $a_i \equiv \phi$, this yields, for all $(U_i^s, U_{i+1}^s) \in X^2$:

$$\phi(U_i^s) + b_i(U_i^s)\phi(U_{i+1}^s) = \phi(U_{i+1}^s) + b_i(U_{i+1}^s)\phi(U_i^s). \tag{5}$$

If $U_i^s = u^*$, we obtain $b_i(U_i^s) = 1$. If U_i^s and U_{i+1}^s are both different from u^*, we obtain:

$$\frac{1 - b_i(U_i^s)}{\phi(U_i^s)} = \frac{1 - b_i(U_{i+1}^s)}{\phi(U_{i+1}^s)}.$$

Therefore there exists a constant $k_i = (b_i(U_i^s) - 1)/\phi(U_i^s)$ for all U_i^s, or equivalently, $b_i(U_i^s) = 1 + k_i\phi(U_i^s)$. Note that we need $b_i(x) > 0$ for all $x \in X$ and therefore $1 + k_i\phi(x) > 0$ for all $x \in X$.

Symmetry also implies that:

$$\hat{e}(u^*, \dots, u^*, U_i^s, U_{i+1}^s, u^*, \dots, u^*) = \hat{e}(U_i^s, U_{i+1}^s, u^*, \dots, u^*),$$

so that $\phi(U_i^s) + (1 + k_i\phi(U_i^s))\,\phi(U_{i+1}^s) = \phi(U_i^s) + (1 + k_1\phi(U_i^s))\,\phi(U_{i+1}^s)$ and therefore k_i is equal to a given constant k for all $i \in \{1, \dots, n-1\}$. In the end, we obtain that:

$$\hat{e}(U_1^s, \dots, U_n^s) = \phi(U_1^s) + \sum_{i=2}^{n} \phi(U_i^s) \prod_{j=1}^{i-1} \big(1 + k\phi(U_j^s)\big) \tag{6}$$

There are two cases.

Case 1: $k = 0$. In this case, (6) implies that $\hat{e}(U_1^s, \dots, U_n^s) = \sum_{i \in N} \phi(U_i^s)$, so that $e(U_1^s, \dots, U_n^s) = u^* + \sum_{i \in N} \phi(U_i^s)$. Note that the condition $1 + k\phi(x) > 0$ is always satisfied in that case. The condition $e(x, \dots, x) = x$ implies $\phi(x) = (x - u^*)/n$, which yields (1).

Case 2: $k \neq 0$. In this case, (6) can be rewritten

$$\begin{aligned} 1 + k\hat{e}(U_1^s, \dots, U_n^s) &= 1 + k\phi(U_1^s) + \sum_{i=2}^{n} k\phi(U_i^s) \prod_{j=1}^{i-1} \left(1 + k\phi(U_j^s)\right) \\ &= \prod_{i=1}^{n} \left(1 + k\phi(U_i^s)\right), \end{aligned}$$

so that

$$\hat{e}(U_1^s, \dots, U_n^s) = \frac{1}{k} \left(\prod_{i=1}^{n} \left(1 + k\phi(U_i^s)\right) - 1 \right),$$

and $e(U_1^s, \dots, U_n^s) = u^* + \hat{e}(U_1^s, \dots, U_n^s)$. The condition $e(x, \dots, x) = x$ implies $\phi(x) = 1/k \left((kx + 1 - ku^*)^{1/n} - 1 \right)$, so that

$$e(U_1^s, \dots, U_n^s) = \frac{1}{\alpha} \prod_{i \in N} (\alpha U_i^s + \beta)^{\frac{1}{n}} + u^*.$$

where $\alpha = k$ and $\beta = 1 - ku^*$. The condition $1 + k\phi(x) > 0$ for all $x \in X$ implies that we must have $\alpha x + \beta > 0$ for all $x \in X$. This yields (2). ■

The first possibility highlighted in this result is unappealing to an egalitarian because it features standard utilitarianism. The second possibility makes it possible to introduce inequality aversion, but this partly depends on the value of the parameters α, β. We study this issue in the next section.

4 Transfer principle and inequality aversion

Inequality aversion, or equivalently, priority for the worse-off, may be captured by requiring the social ordering to satisfy the Pigou-Dalton transfer principle. If i's prospect strictly dominates j's prospect in every state of the world, making a transfer of utility from i to j in every state (without reversing their relative positions) improves the social prospect.[8]

[8] For a comparison of various multidimensional versions of the Pigou-Dalton principle, see Diez et al. (2007).

Axiom 7 (Multidimensional transfer principle) *For all $U, V \in \mathcal{L}$, if there exist $i, j \in N$ and $\delta \in \mathbb{R}^m_{++}$ such that*

$$U_i = V_i - \delta \gg V_j + \delta = U_j,$$

and for all $k \in N \setminus \{i, j\}$, $U_k = V_k$, then $U P V$.

Proposition 2 *The social ordering R satisfies the same axioms as in Proposition 1, with Anonymity replaced by Multidimensional transfer principle, if and only if one of the three following statements holds true:*

1. *There exists a scalar $\varepsilon \in \mathbb{R}_{++}$ satisfying $\varepsilon x + 1 > 0$ for all $x \in X$ and such that for all $U, V \in \mathcal{L}$,*

$$U R V \Longleftrightarrow \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon U_i^s + 1)^{\frac{1}{n}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon V_i^s + 1)^{\frac{1}{n}}. \qquad (7)$$

2. *There exists a scalar $\varepsilon \in \mathbb{R}_{++}$ satisfying $\varepsilon x - 1 > 0$ for all $x \in X$ and such that for all $U, V \in \mathcal{L}$,*

$$U R V \Longleftrightarrow \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon U_i^s - 1)^{\frac{1}{n}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon V_i^s - 1)^{\frac{1}{n}}. \qquad (8)$$

3. *$X \subset \mathbb{R}_{++}$ and for all $U, V \in \mathcal{L}$,*

$$U R V \Longleftrightarrow \sum_{s \in S} \pi_s \prod_{i \in N} (U_i^s)^{\frac{1}{n}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (V_i^s)^{\frac{1}{n}}. \qquad (9)$$

Proof. One can easily check that the proposed social welfare functions satisfy all the axioms. For the Multidimensional transfer principle, this follows from the fact that the transfer δ^s improves the distribution in every $s \in S$.

By Lemma 1 proven below, R satisfies Anonymity. Proposition 1 is therefore valid under the current list of axioms, so that either (1) or (2) holds.

Consider (1) first. When U and V are defined as in the Multidimensional transfer principle, it is clear that $\sum_{s\in S}\pi_s\frac{1}{n}\sum_{i\in N}U_i^s=\sum_{s\in S}\pi_s\frac{1}{n}\sum_{i\in N}V_i^s$, therefore the axiom cannot be satisfied.

For the case (2), if $\beta\neq 0$ we can rewrite:

$$\begin{aligned} URV &\iff \frac{1}{\alpha}\sum_{s\in S}\pi_s\prod_{i\in N}(\alpha U_i^s+\beta)^{\frac{1}{n}}\geq \frac{1}{\alpha}\sum_{s\in S}\pi_s\prod_{i\in N}(\alpha V_i^s+\beta)^{\frac{1}{n}} \\ &\iff sign(\alpha)\sum_{s\in S}\pi_s\prod_{i\in N}\left(sign(\alpha)\varepsilon U_i^s+sign(\beta)\right)^{\frac{1}{n}}\geq \\ &\qquad\qquad sign(\alpha)\sum_{s\in S}\pi_s\prod_{i\in N}\left(sign(\alpha)\varepsilon V_i^s+sign(\beta)\right)^{\frac{1}{n}}, \end{aligned}$$

where $\varepsilon=|\alpha|\,/\,|\beta|$. There are four subcases, depending on $sign(\alpha)$ and $sign(\beta)$.

Now, considering $U,V\in\mathcal{L}^c$, we obtain that $URV\iff\sum_{i\in N}\phi(U_i^s)\geq\sum_{i\in N}\phi(V_i^s)$, where $\phi(x)=sign(\alpha)\ln\left(sign(\alpha)\varepsilon x+sign(\beta)\right)$. On $\mathcal{L}^c$, Multidimensional transfer principle implies the usual Pigou-Dalton transfer principle, which is satisfied if and only if ϕ is a strictly concave function. This is the case here only when $sign(\alpha)>0$, which leaves us with the two possibilities (7) and (8), depending on the sign of β.

If $\beta=0$, one then has

$$URV\iff sign(\alpha)\sum_{s\in S}\pi_s\prod_{i\in N}(sign(\alpha)U_i^s)^{\frac{1}{n}}\geq sign(\alpha)\sum_{s\in S}\pi_s\prod_{i\in N}(sign(\alpha)V_i^s)^{\frac{1}{n}},$$

and here again the Multidimensional transfer principle implies $sign(\alpha)>0$, which yields (9). ■

Lemma 1 *If the social ordering R satisfies Strong Pareto for no risk, Continuity, Weak Dominance, Independence of the utilities of the sure, and Multidimensional transfer principle, then it satisfies Anonymity.*

Proof. In virtue of the Debreu-Gorman theorem, by Strong Pareto for no risk, Independence of the utility of the sure, and Continuity, there exist continuous increasing functions $(\varphi_i)_{i\in N}$ such that for all $U,V \in \mathcal{L}^c$,

$$URV \Longleftrightarrow \sum_{i\in N} \varphi_i(U_i^s) \geq \sum_{i\in N} \varphi_i(V_i^s),$$

where any s can be taken.

By Lemma 2 stated below, Multidimensional transfer principle implies that the functions $(\varphi_i)_{i\in N}$ are identical up to a constant. As the constants play no role in the ordering, there is no loss of generality in taking the functions $(\varphi_i)_{i\in N}$ to be identical. This means that Anonymity is satisfied over $\mathcal{L}^c$. By Weak Dominance, Anonymity is then satisfied over the whole set $\mathcal{L}$. ■

Lemma 2 *If an ordering over X^n, where $X \subset \mathbb{R}$ is an interval, is represented by $\sum_{i=1}^n \varphi_i(x_i)$, where each φ_i is a continuous function, and satisfies the Pigou-Dalton transfer principle (for all i,j, $x < x'$, $0 < \delta \leq (x'-x)/2$, $\varphi_i(x+\delta) + \varphi_j(x'-\delta) > \varphi_i(x) + \varphi_j(x')$), then it satisfies anonymity (for all i, j, x, x', $\varphi_i(x) + \varphi_j(x') = \varphi_i(x') + \varphi_j(x)$).*

Proof. Suppose that anonymity is not satisfied. There exist u, v such that $\varphi_i(v) + \varphi_j(u) < \varphi_i(u) + \varphi_j(v)$.

Without loss of generality let us assume $u < v$ and let $\Delta = v - u$. Let $k \in \mathbb{N}$. The Pigou-Dalton transfer principle imposes that for every $t = 0, ..., k-2$,

$$\varphi_i\left(u + \frac{t+1}{k}\Delta\right) + \varphi_j\left(u + \frac{t+1}{k}\Delta\right) > \varphi_i\left(u + \frac{t}{k}\Delta\right) + \varphi_j\left(u + \frac{t+2}{k}\Delta\right).$$

Summing over t, one obtains

$$\varphi_i\left(u + \frac{k-1}{k}\Delta\right) + \varphi_j\left(u + \frac{1}{k}\Delta\right) > \varphi_i(u) + \varphi_j(u+\Delta).$$

Taking the limit when $k \to \infty$, and invoking the continuity of the functions, one obtains $\varphi_i(u+\Delta) + \varphi_j(u) \geq \varphi_i(u) + \varphi_j(u+\Delta)$,i.e., $\varphi_i(v) + \varphi_j(u) \geq \varphi_i(u) + \varphi_j(v)$, a contradiction. ■

Looking at the proof, it is worth noting that the result would not be changed if we used a weaker axiom making only simple Pigou-Dalton transfers in riskless situations. The stronger axiom has been introduced here because it is worth checking that it can be satisfied in this context. We also show in Appendix 1 that our seven axioms are independent (in the sense that each one is required to get the result).

Social welfare functions satisfying the transfer principle are said to be inequality averse. It remains to study how much inequality aversion is compatible with formulae (7) and (8). To that effect we will compare the inequality aversion of the contemplated orderings with that of benchmark orderings. It is enough to focus on riskless prospects, and we can therefore rely on standard concepts of unidimensional inequality measurement. We have the following standard method to compare inequality aversion:

Definition 1 *A social ordering R is more inequality averse than a social ordering $\tilde{R}$ if, for all $U \in \mathcal{L}^c$ and $V \in \mathcal{L}^c \cap \mathcal{L}^e$, $URV \Longrightarrow U\tilde{R}V$.*

In the case of social orderings represented for riskless prospects by symmetric additive social welfare functions $\sum_{i \in N} \phi(U_i^s)$, there are standard results indicating that the more concave the function ϕ, the more inequality averse the social ordering. When $\sum_{i \in N} \phi(U_i^s)$ takes the classical isoelastic form $\frac{1}{1-\alpha} \sum_{i \in N} (U_i^s)^{1-\alpha}$, it is convenient to measure its degree of inequality aversion by α.

Clearly, all the social welfare functions in the families (7) and (8) are more inequality averse than the social ordering represented by the utilitarian social welfare function (1), which has a degree of inequality aversion equal to 0.

One can also compare them with the social ordering represented by (9), which is for sure prospects ordinally equivalent to $\sum_{i \in N} \ln U_i^s$ and has a degree of inequality aversion equal to 1. We obtain the following results:

Proposition 3

1. *Social welfare functions from family (7):*

   (a) *Are more inequality averse the larger* ε.

   (b) *Become ordinally equivalent to (1) when* $\varepsilon \to 0$ *and to (9) when* $\varepsilon \to +\infty$.[9]

2. *Social welfare functions from family (8):*

   (a) *Are less inequality averse the larger* ε.

   (b) *Become ordinally equivalent to (9) when* $\varepsilon \to +\infty$.

   (c) *Are more inequality averse than* $\frac{1}{1-\alpha}\sum_{i\in N}(U_i^s)^{1-\alpha}$, *for a given* $\alpha > 1$, *if* $0 < \varepsilon x - 1 < -1/(1-\alpha)$ *for all* $x \in X$.

Proof. 1.a. As indicated above, a social ordering represented by $\sum_{i\in N}\phi(U_i^s)$ is more inequality averse than a social ordering represented by $\sum_{i\in N}\tilde{\phi}(U_i^s)$ if and only if there exists a concave function ψ such that $\phi = \psi \circ \tilde{\phi}$. Let $\varphi_\varepsilon(x) = \ln(\varepsilon x + 1)$. On riskless prospects, (7) is ordinally equivalent to $\sum_{i\in N}\varphi_\varepsilon(U_i^s)$. One has $\varphi_\varepsilon(x) = \psi_{\varepsilon,\varepsilon'} \circ \varphi_{\varepsilon'}(x)$, where the function $\psi_{\varepsilon,\varepsilon'}(z) = \ln\left((\varepsilon/\varepsilon')\exp(z) + 1 - \varepsilon/\varepsilon'\right)$ is strictly concave if $\varepsilon > \varepsilon'$. Then the social ordering on riskless prospects represented by $\sum_{i\in N}\varphi_\varepsilon(U_i^s)$ is more inequality averse than the social ordering represented by $\sum_{i\in N}\varphi_{\varepsilon'}(U_i^s)$.

1.b. When $\varepsilon \to 0$, $(\varepsilon x + 1)^{1/n} \approx 1 + \varepsilon x/n$. Therefore the function $\sum_{s\in S}\pi_s \prod_{i\in N}(\varepsilon U_i^s + 1)^{\frac{1}{n}}$ becomes ordinally equivalent to $\sum_{s\in S}\pi_s\sum_{i\in N}U_i^s$.

The function $\sum_{s\in S}\pi_s\prod_{i\in N}(\varepsilon U_i^s + 1)^{\frac{1}{n}}$ is ordinally equivalent to $\sum_{s\in S}\pi_s\prod_{i\in N}(U_i^s + 1/\varepsilon)^{\frac{1}{n}}$, which tends to $\sum_{s\in S}\pi_s\prod_{i\in N}(U_i^s)^{\frac{1}{n}}$ when $\varepsilon \to +\infty$.

[9] It is permissible to let $\varepsilon \to +\infty$ only if $\inf X \geq 0$.

2.a. Let $\chi_\varepsilon(x) = \ln(\varepsilon x - 1)$. With a similar argument as for point 1, one shows that the social ordering represented by $\sum_{i \in N} \chi_\varepsilon(U_i^s)$ is more inequality averse than the social ordering represented by $\sum_{i \in N} \chi_{\varepsilon'}(U_i^s)$ if $\varepsilon < \varepsilon'$.

2.b. When $\varepsilon \to +\infty$, the argument is similar as for (7).

2.c. One has $\chi_\varepsilon(x) = \ln(\varepsilon\,[(1-\alpha)\,z]^{\frac{1}{1-\alpha}} - 1)$ whenever $z = \frac{1}{1-\alpha}x^{1-\alpha}$. The function $\psi_{\varepsilon,\alpha}(z) = \ln(\varepsilon\,[(1-\alpha)\,z]^{\frac{1}{1-\alpha}} - 1)$ is strictly concave if $\varepsilon\,[(1-\alpha)\,z]^{\frac{1}{1-\alpha}} - 1 < -1/\,(1-\alpha)$ for all z. ■

The families (7) and (8) seemingly cover a wide range of attitudes towards inequality. However the social welfare function represented by (7) is well-defined on (subsets of) the interval $(-1/\epsilon, +\infty)$ while the social welfare function represented by (8) is well-defined on (subsets of) the interval $(1/\epsilon, +\infty)$. So the form of the set X will constrain possible degree of inequality aversions. A noteworthy configuration is the following:

Corollary 1 *If $X = \mathbb{R}_{++}$, $(0,a]$ or $(0,a)$ (where $a \in \mathbb{R}_{++}$), $\sum_{i \in N} \ln U_i^s$ is the most inequality averse social ordering satisfying the seven axioms.*

To obtain a greater inequality aversion, a further restriction of the domain is required:

Corollary 2 *A social ordering satisfying the seven axioms is more inequality averse than $\frac{1}{1-\alpha}\sum_{i \in N}(U_i^s)^{1-\alpha}$, for a given $\alpha > 1$, if and only if $X \subset \left(\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\frac{\alpha}{\alpha-1}\right)$ and it belongs to family (8).*

Therefore, although in theory any positive degree of inequality aversion can be surpassed by social orderings satisfying the seven axioms, this may require a calibration of individual utilities which squeezes them into a tiny interval. More precisely, the greater the degree of inequality aversion one wishes to put into social evaluation, the more difficult it may be to measure utilities in a reasonable range. Whether the evaluator is free to rescale utility numbers before applying a formula like (8) is a delicate issue that depends on what utility is supposed to measure.

5 Separability versus Pareto

Another problematic consideration is that, even though the *utility* of the past generations can be ignored in the application of the social orderings highlighted in Proposition 2, the *number* of individuals in society, and therefore in the past generations, still plays a role in the computation. One might want to have independence not just of the utility of the sure, but of the existence of the sure.

If one combines independence of the existence of the sure with Weak Pareto for equal risk, one obtains the following stronger version of Weak Pareto for equal risk, that applies to the subgroup of concerned individuals independently of its size.

Axiom 8 (Weak Pareto for subgroup equal risk) *For all* $U, V \in \mathcal{L}^e$ *and* $\widetilde{U} \in \mathcal{L}^c$, *and for all* $Q \subset N$, *if* $EU_i \geq EV_i$ *for all* $i \in Q$, *then* $\left(U_Q, \widetilde{U}_{N\setminus Q}\right) P \left(V_Q, \widetilde{U}_{N\setminus Q}\right)$.

As shown in Fleurbaey (2010), this axiom brings us back into the grip of Harsanyi's utilitarianism. In the context of the EDE criteria studied in this paper, it seems that we cannot allow more separability than permitted by independence of the utility of the sure.

But this may become possible if Weak Pareto for equal risk is abandoned or modified. Consider the following weakening of Weak Pareto for subgroup equal risk (and of Weak Pareto for equal risk), where the group of concerned individuals may be restricted to a subset of possible subgroups of N.

Axiom 9 (Weak Pareto for restricted subgroup risk) *There exists $\mathcal{Q} \subset 2^N \setminus \emptyset$ such that for all $U, V \in \mathcal{L}^e$ and $\widetilde{U} \in \mathcal{L}^c$, and for all $Q \in \mathcal{Q}$, if $EU_i > EV_i$ for all $i \in Q$, then $\left(U_Q, \widetilde{U}_{N\setminus Q}\right) P \left(V_Q, \widetilde{U}_{N\setminus Q}\right)$.*

This axiom encompasses cases of particular interest. When $N \in \mathcal{Q}$ it implies Weak Pareto for equal risk. When $\mathcal{Q}$ contains all singleton sets, the axiom covers situations in which one individual takes risks that do not affect the other members of the society. One could argue that choices for such individual risks should be respected.

The point of introducing this axiom is to make it possible to study what sets $\mathcal{Q}$ are compatible with combining this axiom with other axioms. In this way one can analyze the extent of separability that is permitted by the approach. To do so, we need to strengthen our rationality requirements to remain within the scope of expected utility theory (see Appendix 2 for details). We therefore make the following assumption, which implies both Continuity and Weak Dominance.

Axiom 10 (Expected utility hypothesis) *For all $U, V \in \mathcal{L}$, there exists a continuous function F unique up to positive affine transformations such that*

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s F(U^s) \geq \sum_{s\in S} \pi_s F(V^s).$$

Using this axiom, we obtain the following characterization result.

Proposition 4 *The social ordering R satisfies Strong Pareto for no risk, Independence of the utilities of the sure, Multidimensional transfer principle, Weak Pareto for restricted subgroup risk, and Expected utility hypothesis if and only if one of the three following statements holds true:*

1. *There exists $q \in \{1, \ldots, n\}$ and $\varepsilon \in \mathbb{R}_{++}$ satisfying $\varepsilon x + 1 > 0$ for all $x \in X$ and such that for all $U, V \in \mathcal{L}$,*

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon U_i^s + 1)^{\frac{1}{q}} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (\varepsilon V_i^s + 1)^{\frac{1}{q}}. \tag{10}$$

2. There exists $q \in \{1, \ldots, n\}$ and $\varepsilon \in \mathbb{R}_{++}$ satisfying $\varepsilon x - 1 > 0$ for all $x \in X$ and such that for all $U, V \in \mathcal{L}$,

$$URV \iff \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon U_i^s - 1)^{\frac{1}{q}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon V_i^s - 1)^{\frac{1}{q}}. \tag{11}$$

3. $X \subset \mathbb{R}_{++}$ and there exists $q \in \{1, \ldots, n\}$ such that for all $U, V \in \mathcal{L}$,

$$URV \iff \sum_{s \in S} \pi_s \prod_{i \in N} (U_i^s)^{\frac{1}{q}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (V_i^s)^{\frac{1}{q}}. \tag{12}$$

Proof. By Expected utility hypothesis, $URV \iff \sum_{s \in S} \pi_s F(U^s) \geq \sum_{s \in S} \pi_s F(V^s)$, and F is continuous. By the same argument as in the proof of Lemma 1, Strong Pareto for no risk, Independence of the utilities of the sure, and Multi-dimensional transfer principle imply that the function F must be increasing and symmetric.

Thus we obtain an equivalence similar to (3) in the proof of Proposition 1. Furthermore, F can be normalized so that $F(u^*, \ldots, u^*) = 0$ like function $\hat{e}$ in the proof of Proposition 1. Using Independence of the utilities of the sure, we can proceed as in the proof of Proposition 1 to obtain that:

$$F(U_1^s, \ldots, U_n^s) = \phi(U_1^s) + \sum_{i=2}^{n} \phi(U_i^s) \prod_{j=1}^{i-1} \left(1 + k\phi(U_j^s)\right) \tag{13}$$

where ϕ is a continuous and increasing function such that $1 + k_i\phi(x) > 0$ for all $x \in X$.

The fact that F is symmetric means that R satisfies Anonymity. By Anonymity, if $Q \in \mathcal{Q}$, so does every subgroup of size $|Q|$. Let q be a particular size that is admitted in $\mathcal{Q}$. There are two cases.

Case 1: $k = 0$. In this case, Weak Pareto for restricted subgroup risk, applied to subgroups of size q, implies that

$$\sum_{s \in S} \pi_s q\phi(U_i^s) \geq \sum_{s \in S} \pi_s q\phi(V_i^s) \iff \sum_{s \in S} \pi_s U_i^s \geq \sum_{s \in S} \pi_s V_i^s.$$

As VNM functions are unique up to an increasing affine transform, there must exist $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$ such that $q\phi(x) = \alpha x + \beta$. Therefore

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s U^s \geq \sum_{s\in S} \pi_s V^s \tag{14}$$

Case 2: $k \neq 0$. In this case, Equation (16) can be rewritten:

$$F(U_1^s, \ldots, U_n^s) = \frac{1}{k}\left(\prod_{i=1}^{n} (1 + k\phi(U_i^s)) - 1\right).$$

Hence Weak Pareto for restricted subgroup risk implies that:

$$\sum_{s\in S} \pi_s \frac{1}{k}\left(1 + k\phi(U_i^s)\right)^q \geq \sum_{s\in S} \pi_s \frac{1}{k}\left(1 + k\phi(U_i^s)\right)^q \Longleftrightarrow \sum_{s\in S} \pi_s U_i^s \geq \sum_{s\in S} \pi_s V_i^s.$$

Applying the same reasoning as above there must exist $\alpha \in \mathbb{R}_{++}$ and $\beta \in \mathbb{R}$ such that $\frac{1}{k}\left(1 + k\phi(x)\right)^q = \alpha x + \beta$.

When $\beta = 0$, it is necessary that $kx > 0$ in order to have $1 + k\phi(x) > 0$. Then $1 + k\phi(x) = (k\alpha)^{1/q} x^{1/q}$ if $k > 0$, and $(-k\alpha)^{1/q}(-x)^{1/q}$ if $k < 0$. Therefore, either $X \subset \mathbb{R}_{++}$ and

$$URV \Longleftrightarrow \sum_{s\in S} \pi_s \prod_{i\in N} (U_i^s)^{1/q} \geq \sum_{s\in S} \pi_s \prod_{i\in N} (V_i^s)^{1/q} \tag{15}$$

or $X \subset \mathbb{R}_{--}$ and

$$URV \Longleftrightarrow -\sum_{s\in S} \pi_s \prod_{i\in N} (-U_i^s)^{1/q} \geq -\sum_{s\in S} \pi_s \prod_{i\in N} (-V_i^s)^{1/q}.$$

The latter case is excluded by Multidimensional transfer principle.

When $\beta \neq 0$, it is necessary that $k(\alpha x + \beta) > 0$. One has

$$\begin{aligned} 1 + k\phi(x) &= \left[k(\alpha x + \beta)\right]^{1/q} \\ &= |\beta k|^{1/q}\left[\left(\frac{sign(k)\alpha}{|\beta|} x + sign(\beta k)\right)\right]^{1/q}. \end{aligned}$$

This gives us four possibilities, depending on $sign(k)$ and $sign(\beta k)$. As in the previous paragraph, we need $k > 0$ to satisfy the transfer axiom, which leaves us with the two possibilities (10) and (11). ■

The criteria highlighted in Proposition 4 are closely related to the classes of criteria (7), (8) and (9). Indeed, as far as the analysis of inequality aversion is concerned, they induce the same results as in Proposition 3.

As far as Weak Pareto for restricted subgroup risk is concerned, the analysis is clear and rather negative. Only one group size can be admitted.

In the case $q = 1$, Pareto for q group risk collapses to a property of Pareto for individual risk: the risk preferences of the individual are respected if all other individuals are indifferent and bear no risk. In this case, the criteria in (10) exactly correspond to the multiplicative social welfare functions satisfying "risk equity" in Bommier and Zuber (2008). It is worth noting that their multiplicative social welfare functions satisfying "catastrophe avoidance",[10] which would correspond to case 3 in the proof ($\varepsilon < 0$, $\varepsilon x + 1 > 0$ for all $x \in X$), are ruled out by the transfer principle. They did not find the social welfare functions displayed in (11) or (12) because they assumed that $X \subset \mathbb{R}_+$ and $0 \in X$, which excludes these two cases.

Let us now come back to the idea of independence of the *existence* of the (unconcerned) sure. The criteria listed in Proposition 4 all satisfy this property fully, with no restriction on the size of the concerned or the unconcerned subgroups. They are therefore strongly separable. Their limitation is on the side of the Pareto principle, as they satisfy Pareto only when the concerned subgroup

[10]Risk equity and catastrophe avoidance are two principles introduced in Keeney (1980). The former is the principle that, when individuals face independent risks of a specific damage (accident), inequalities in their probabilities of damage are undesirable. The latter principle seeks to minimize the risk of having a large number of fatalities. Keeney showed that the two principles are antinomic, because the best way to avoid a catastrophe is to concentrate the risk on a few (sacrificed) individuals. In an intergenerational setting with uncertain existence of future generations, Bommier and Zuber (2008) show that risk equity (resp., catastrophe avoidance) induces a low (resp., high) social discount rate.

has a particular size. In particular, they fail Pareto for equal risk unless $q = n$.

6 Conclusion

In this paper, we have shown that social rationality (embodied in Weak Dominance) and a reasonable dose of the Pareto principle (Pareto for no risk, Pareto for equal risk) can be reconciled with inequality aversion and some independence with respect to unconcerned individuals bearing no risk. In particular, the Nash product has been singled out as the social ordering giving the most priority to the worst-off in the relevant case where the utility possibility set is the positive real line.

In the context of the evaluation of social situations involving risks, this already constitutes some progress. Indeed, in view of the results involving Independence of the utilities of the sure (or "the dead") in Bommier and Zuber (2008), or the results involving Pareto for subgroup equal risk in Fleurbaey (2010), one might have feared that the degree of inequality aversion would be severely constrained. Our results open a wider range of possibilities.

Truly enough, the tension between social rationality, Pareto, inequality aversion, and separability remains substantial, as epitomized in Proposition 4.

To conclude, we briefly mention two solutions to this tension. One is to restrict the set of possible prospects $\mathcal{L}$ and not simply the utility possibility set X. The ex post generalized Gini criteria introduced in Fleurbaey (2010), for instance, can satisfy the full Pareto principle and strong separability properties (including Independence of the existence of the dead), in an intertemporal setting, if the successive generations' utility is always increasing in all possible worlds.

Another solution is to reject separability in the context of risk even more than considered in this paper.[11] The Robinson-Friday parable of the introduction can be turned into an argument against separability. In absence of risk, it is natural to focus on the worst-off among the concerned individuals only, implying that the separable leximin criterion is more appealing than the non-separable maximin. In the presence of risk, in contrast, an unconcerned and risk-free individual who is the worst-off either in all states or in no state can similarly be neglected, but an individual who is the worst-off in some states only cannot similarly be neglected, because his utility level affects the probability distribution of the lowest utility. We leave the study of the weaker separability conditions that this argument might suggest for future research.

References

Adler M.D., Sanchirico, C.W. (2006). "Inequality and uncertainty: Theory and legal applications", *University of Pennsylvania Law Review*, 155, 279–377.

Asheim, G.B., Mitra, T., Tungodden, B. (2012). "Sustainable recursive social welfare functions", *Economic Theory*, 49, 267–292

Atkinson, A.B. (1970). "On the measurement of inequality", *Journal of Economic Theory*, 2, 244–263.

Blackorby, C., Bossert, W., Donaldson, D. (1995). "Intertemporal population ethics: critical-level utilitarian principles", *Econometrica*, 63, 1303–1320.

Bommier, A., Zuber, S. (2008). "Can preferences for catastrophe avoidance reconcile social discounting with intergenerational equity?", *Social Choice and Welfare*, 31, 415–434.

Broome, J. (1991). *Weighing Goods. Equality, Uncertainty and Time*, Oxford: Blackwell.

Diamond, P.A. (1967). "Cardinal welfare, individualistic ethics, and interpersonal comparison of utility: comment'", *Journal of Political Economy*, 75, 765–766.

[11] We thank the referee for suggesting this conclusion.

Diez, H., Lasso de la Vega, M.C., de Sarachu, A., Urrutia, A.M. (2007). "A Consistent Multidimensional Generalization of the Pigou-Dalton Transfer Principle: An Analysis," *The B.E. Journal of Theoretical Economics*, Vol. 7: Iss. 1 (Topics), Article 45.

Dhillon, A., Mertens, J.-F. (1999). "Relative utilitarianism", *Econometrica*, 67, 471–498.

Epstein, L.G., Segal, U. (1992). "Quadratic social welfare functions", *Journal of Political Economy*, 100, 691–712.

Fleurbaey, M. (2010). "Assessing risky social situations", *Journal of Political Economy*, 118, 649–680.

Fleurbaey, M., T. Gajdos, S. Zuber (2010), "Social rationality, separability, and equity under uncertainty", mimeo.

Fleurbaey, M., Trannoy, A. (2003). "The impossibility of a Paretian egalitarian", *Social Choice and Welfare*, 21, 243–263.

Foster, J.E., Shneyerov A.A. (2000), "Path independent inequality measures", *Journal of Economic Theory* 91: 199–222.

Gajdos, T., Kandil, F. (2008). "The ignorant observer", *Social Choice and Welfare*, 31, 193–232.

Grant, S. (1995). "Subjective probability without monotonicity: Or how Machina's Mom may also be probabilistically sophisticated", *Econometrica*, 63, 159–189.

Harsanyi, J. (1955). "Cardinal welfare, individualistic ethics and interpersonal comparisons of utility", *Journal of Political Economy*, 63, 309–321.

Keeney, R.L. (1980). "Equity and public risk", *Operations Research*, 28, 527–534.

Keeney, R.L., Raiffa, H. (1976). *Decision with Multiple Objectives*, 1999 Edition, Cambridge: Cambridge University Press.

Karni, E. (1998). "Impartiality: Definition and Representation", *Econometrica*, 66, 1405–1415.

Koopmans, T.C. (1960). "Stationary ordinal utility and impatience", *Econometrica*, 28, 287–309.

Lauwers, L. (2012). "Intergenerational equity, efficiency and constructibility", *Economic Theory*, 49, 227–242.

Weymark, J. A. (1991). "A reconsideration of the Harsany-Sen debate on Utilitarianism", in Elster, J., Roemer, J.E. (eds.), *Interpersonal Comparisons of Well-Being*, Cambridge and Paris: Cambridge University Press and Maison des Sciences de l'Homme, 255–320.

A.1 Appendix 1: Independence of the axioms of Proposition 2

In this Appendix, we show the independence of the axioms in Proposition 2 by exhibiting criteria satisfying all but one of the axioms.

- Criterion satisfying all the axioms but Axiom 1 (Strong Pareto for no risk):

$$URV \Longleftrightarrow \sum_{s\in S}\pi_s\left[\prod_{i\in N}(U_i^s)^{\frac{1}{n}}-\frac{1}{2n}\sum_{i\in N}U_i^s\right]\geq\sum_{s\in S}\pi_s\left[\prod_{i\in N}(V_i^s)^{\frac{1}{n}}-\frac{1}{2n}\sum_{i\in N}V_i^s\right].$$

- Criterion satisfying all the axioms but Axiom 2 (Pareto for equal risk):

$$URV \Longleftrightarrow \sum_{s\in S}\pi_s\prod_{i\in N}U_i^s\geq\sum_{s\in S}\pi_s\prod_{i\in N}V_i^s.$$

- Criterion satisfying all the axioms but Axiom 3 (Weak dominance):

$$URV \Longleftrightarrow \sum_{s\in S}\pi_s\prod_{i\in N}(\alpha U_i^s+(1-\alpha)EU_i)^{\frac{1}{n}}\geq\sum_{s\in S}\pi_s\prod_{i\in N}(\alpha V_i^s+(1-\alpha)EV_i)^{\frac{1}{n}}$$

  with $0<\alpha<1$.

- Criterion satisfying all the axioms but Axiom 4 (Independence of the utilities of the sure):

$$URV \Longleftrightarrow \sum_{s\in S}\pi_s\left(\tfrac{1}{n}\sum_{i\in N}(U_i^s)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}\geq\sum_{s\in S}\pi_s\left(\tfrac{1}{n}\sum_{i\in N}(V_i^s)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}$$

  with $\varepsilon>0$.

- Criterion satisfying all the axioms but Axiom 6 (Continuity):

  URV iff there exists $\varepsilon>\max\{0,-1/\inf X\}$ such that either

$$\sum_{s\in S}\pi_s\prod_{i\in N}(1+\varepsilon U_i^s)^{\frac{1}{n}}>\sum_{s\in S}\pi_s\prod_{i\in N}(1+\varepsilon V_i^s)^{\frac{1}{n}}$$

or

$$\sum_{s\in S}\pi_s\prod_{i\in N}(1+\varepsilon U_i^s)^{\frac{1}{n}} = \sum_{s\in S}\pi_s\prod_{i\in N}(1+\varepsilon V_i^s)^{\frac{1}{n}}$$
$$\text{and } M(U) < M(V),$$

where

$$M(U) = |\{(i,s) \mid U_i^s < \chi\}|,$$

for $\inf X < \chi < \sup X$.

- Criterion satisfying all the axioms but Axiom 7 (Mulitidimensional transfer principle):

$$URV \Longleftrightarrow \sum_{s\in S}\pi_s\prod_{i\in N}(1-\varepsilon U_i^s)^{\frac{1}{n}} \geq \sum_{s\in S}\pi_s\prod_{i\in N}(1-\varepsilon V_i^s)^{\frac{1}{n}}$$

with $\varepsilon > 0$ and $\varepsilon < (\sup X)^{-1}$ if $\sup X > 0$.

A.2 Appendix 2: Proposition 4 without the expected utility hypothesis

In this Appendix, we discuss what would happen if we used Continuity and Weak Dominance instead of the Expected utility hypothesis in Proposition 4. It is shown that the result would still hold on a subdomain (including sure prospects). But on part of the domain, social preferences may not be expected utilities.

Proposition 5 *If the social ordering R satisfies Strong Pareto for no risk, Weak dominance, Independence of the utilities of the sure, Continuity, Multidimensional transfer principle, and Weak Pareto for restricted subgroup risk, then there exists a subset $\bar{\mathcal{L}} \subset \mathcal{L}$ such that $\mathcal{L}^c \subset \bar{\mathcal{L}}$ and one of the three following statements holds true:*

1. *There exists* $q \in \{1, \ldots, n\}$ *and* $\varepsilon \in \mathbb{R}_{++}$ *satisfying* $\varepsilon x + 1 > 0$ *for all* $x \in X$ *and such that for all* $U, V \in \bar{\mathcal{L}}$,

$$URV \Longleftrightarrow \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon U_i^s + 1)^{\frac{1}{q}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon V_i^s + 1)^{\frac{1}{q}}. \tag{16}$$

2. *There exists* $q \in \{1, \ldots, n\}$ *and* $\varepsilon \in \mathbb{R}_{++}$ *satisfying* $\varepsilon x - 1 > 0$ *for all* $x \in X$ *and such that for all* $U, V \in \bar{\mathcal{L}}$,

$$URV \Longleftrightarrow \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon U_i^s - 1)^{\frac{1}{q}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (\varepsilon V_i^s - 1)^{\frac{1}{q}}. \tag{17}$$

3. $X \subset \mathbb{R}_{++}$ *and there exists* $q \in \{1, \ldots, n\}$ *such that for all* $U, V \in \bar{\mathcal{L}}$,

$$URV \Longleftrightarrow \sum_{s \in S} \pi_s \prod_{i \in N} (U_i^s)^{\frac{1}{q}} \geq \sum_{s \in S} \pi_s \prod_{i \in N} (V_i^s)^{\frac{1}{q}}. \tag{18}$$

Proof. By Lemma 1, R satisfies Anonymity. Let q be a size admitted in $\mathcal{Q}$ (i.e., in Weak Pareto for restricted subgroup risk).

Fix $x_0 \in X$. Let $Y(x_0) \subset X^n$ denote the subset such that for all $U^s \in Y(x_0)$, there exists $e_q(U^s) \in X$ such that the n-vector $\overline{U}^s(U^s)$ defined by $\overline{U}_1^s(U^s) = ... = \overline{U}_q^s(U^s) = e_q(U^s)$ and $\overline{U}_{q+1}^s(U^s) = ... = \overline{U}_n^s(U^s) = x_0$ satisfies $[U^s]\, I \left[\overline{U}^s(U^s)\right]$. By Strong Pareto for no risk and Continuity, $Y(x_0) \neq \varnothing$ for every $x_0 \in X$. By Weak Dominance, for all $U \in Y(x_0)^m$,

$$U I \left(\overline{U}^1(U^1), ..., \overline{U}^m(U^m)\right).$$

By Weak Pareto for restricted subgroup risk, Anonymity, and Continuity, for all $U, V \in Y(x_0)^m$,

$$URV \Longleftrightarrow \sum_{s \in S} \pi_s e_q(U^s) \geq \sum_{s \in S} \pi_s e_q(V^s).$$

By Strong Pareto for no risk and Anonymity, the function e_q must be increasing and symmetric. We can proceed as in the proof of Proposition 4 to obtain the same results, but limited to $U, V \in Y(x_0)^m$.

Now one can change x_0. Taking a sufficiently fine grid, the corresponding $Y(x_0)^m$ overlap and the same criterion must therefore hold on the union $\bar{\mathcal{L}}$ of the sets $Y(x_0)^m$. Note that by Strong Pareto for no risk and Continuity, one has $X^n = \bigcup_{x_0 \in X} Y(x_0)$. Therefore $\mathcal{L}^c \subset \bar{\mathcal{L}} = \bigcup_{x_0 \in X} Y(x_0)^m$, so that the criteria obtained in Proposition 4 are valid over sure prospects. ■

It is important to note that in Proposition 5, one does not have $\mathcal{L} \subset \bar{\mathcal{L}}$ in general. In order to show that the result may not hold over $\mathcal{L}$ we exhibit a counter-example for $n = m = 2$. Extending it to other values of n, m is straightforward.

Let $X = [a, b] \subset \mathbb{R}_{++}$ and $q = 1$. The ordering R is defined as follows:

$$URV \Longleftrightarrow W(U) \geq W(U)$$

for

$$\begin{aligned} W(U) &= \sum_{s \in S} \pi_s U_1^s U_2^s \text{ if } \frac{a}{b} \leq \frac{U_1^2 U_2^2}{U_1^1 U_2^1} \leq \frac{b}{a}, \\ &= \left(\pi_1 \frac{b}{a} + \pi_2 \right) U_1^2 U_2^2 \text{ if } \frac{U_1^2 U_2^2}{U_1^1 U_2^1} < \frac{a}{b}, \\ &= \left(\pi_1 + \frac{b}{a} \pi_2 \right) U_1^1 U_2^1 \text{ if } \frac{U_1^2 U_2^2}{U_1^1 U_2^1} > \frac{b}{a}. \end{aligned}$$

Observe that when $\frac{U_1^2 U_2^2}{U_1^1 U_2^1} < \frac{a}{b}$, for instance, it is impossible to find $x_0, x_1^1, x_1^2 \in X$ satisfying

$$x_0 x_1^1 = U_1^1 U_2^1 \text{ and } x_0 x_1^2 = U_1^2 U_2^2,$$

because this implies

$$\frac{x_1^2}{x_1^1} = \frac{U_1^2 U_2^2}{U_1^1 U_2^1} < \frac{a}{b},$$

which is impossible for $x_1^1, x_1^2 \in [a, b]$. For this ordering, the set $\bigcup_{x_0 \in X} Y(x_0)^m$ is defined by the condition $\frac{a}{b} \leq \frac{U_1^2 U_2^2}{U_1^1 U_2^1} \leq \frac{b}{a}$.

The ordering defined here satisfies all the axioms of the proposition but does not coincide with any of the criteria listed in the proposition (except on $\bigcup_{x_0 \in X} Y(x_0)^m$).