

Update: 12/11/2013

*Abdus Salam School of Mathematical Sciences (ASSMS), Lahore*  
**6th World Conference on 21st Century Mathematics 2013.**  
<http://www.sms.edu.pk/wc2013.php>

## Lecture on the *abc* conjecture and some of its consequences

by

*Michel Waldschmidt*

### Abstract

The *abc* Conjecture was proposed in the 80's by J. Oesterlé and D.W. Masser. This simple statement implies a number of results and conjectures in number theory. We state this conjecture and list a few of the many consequences.

This conjecture has gained increasing awareness in August 2012 when Shinichi Mochizuki released a series of four preprints containing a claim to a proof of the *abc* Conjecture using his *Inter-universal Teichmüller Theory*:

<http://www.kurims.kyoto-u.ac.jp/~motizuki/top-english.html>

### 1 The radical of a positive integer, *abc*-triples and *abc*-hits

According to the fundamental theorem of arithmetic, any integer  $n \geq 2$  can be written as a product of prime numbers:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}.$$

The *radical* (also called *kernel* or *core*)  $\text{Rad}(n)$  of  $n$  is the product of the distinct primes dividing  $n$ :

$$\text{Rad}(n) = p_1 p_2 \cdots p_t.$$

An *abc-triple* is a triple  $(a, b, c)$  of three positive coprime integers such that  $a + b = c$  with  $a < b$ . The smallest example of an *abc-triple* is  $(1, 2, 3)$ .

An *abc-hit* is an *abc-triple*  $(a, b, c)$  such that  $\text{Rad}(abc) < c$ . For instance  $(1, 8, 9)$ , is an *abc-hit*, since  $1 + 8 = 9$ ,  $\text{gcd}(1, 8, 9) = 1$  and

$$\text{Rad}(1 \cdot 8 \cdot 9) = \text{Rad}(2^3 \cdot 3^2) = 2 \cdot 3 = 6 < 9.$$

Among  $15 \cdot 10^6$  *abc-triples* with  $c < 10^4$ , there are 120 *abc-hits*<sup>1</sup>.

There are infinitely many *abc-hits*. Indeed, take  $k \geq 1$ ,  $a = 1$ ,  $c = 3^{2^k}$ ,  $b = c - 1$ . By induction on  $k$ , one checks that  $2^{k+2}$  divides  $3^{2^k} - 1$ . Hence

$$\text{Rad}((3^{2^k} - 1) \cdot 3^{2^k}) \leq \frac{3^{2^k} - 1}{2^{k+1}} \cdot 3 < 3^{2^k}.$$

Hence

$$(1, 3^{2^k} - 1, 3^{2^k})$$

is an *abc-hit*. From this argument one deduces (see [71]):

**Lemma 1.** *There exist infinitely many abc-triples  $(a, b, c)$  such that*

$$c > \frac{1}{6 \log 3} R \log R,$$

where  $R = \text{Rad}(abc)$ .

It is not known (but conjectured in [31]) whether there are *abc-triples*  $(a, b, c)$  for which  $c > \text{Rad}(abc)^2$ . The largest known value of  $\lambda$  for which there exists an *abc-triple*  $(a, b, c)$  with  $c > \text{Rad}(abc)^\lambda$  is  $\lambda = 1.62991\dots$ , which is reached by Reyssat's example with

$$a = 2, \quad b = 3^{10} \cdot 109 = 6\,436\,341, \quad c = 23^5 = 6\,436\,343.$$

Indeed one checks

$$2 + 3^{10} \cdot 109 = 23^5, \quad \text{Rad}(2 \cdot 3^{10} \cdot 109 \cdot 23^5) = 2 \cdot 3 \cdot 23 \cdot 109 = 15\,042.$$

When  $(a, b, c)$  is an *abc-triple*, define

$$\lambda(a, b, c) = \frac{\log c}{\log \text{Rad}(abc)}.$$

---

<sup>1</sup>See the tables of [http://rekenmeemetabc.nl/Synthese\\_resultaten](http://rekenmeemetabc.nl/Synthese_resultaten)

In 2013, there are 140 known values of  $\lambda(a, b, c)$  which are  $\geq 1.4$ . Besides Reyssat's example, the largest value of  $\lambda(a, b, c)$  is  $1.625990\dots$ , obtained by Benne de Weger

$$a = 11^2, \quad b = 3^2 \cdot 5^6 \cdot 7^3 = 48\,234\,375, \quad c = 2^{21} \cdot 23 = 48\,234\,496 :$$

$$11^2 + 3^2 \cdot 5^6 \cdot 7^3 = 2^{21} \cdot 23, \quad \text{Rad}(2^{21} \cdot 3^2 \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 23) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 53\,130.$$

According to S. Laishram and T. N. Shorey [39], an explicit version, due to A. Baker [2], of the *abc* Conjecture, namely Conjecture 15 below, yields

**Conjecture 1.** *For any abc-triple  $(a, b, c)$ ,*

$$c < \text{Rad}(abc)^{7/4}.$$

## 2 *abc* Conjecture

Here is the *abc* Conjecture of Œsterlé [58] and Masser [50].

**Conjecture 2.** *Let  $\varepsilon > 0$ . Then the set of abc triples  $(a, b, c)$  for which*

$$c > \text{Rad}(abc)^{1+\varepsilon}$$

*is finite.*

It is easily seen that Conjecture 2 is equivalent to the following statement:

- *For each  $\varepsilon > 0$ , there exists  $\kappa(\varepsilon)$  such that, for any abc triple  $(a, b, c)$ ,*

$$c < \kappa(\varepsilon) \text{Rad}(abc)^{1+\varepsilon}.$$

This may be viewed as a lower bound for  $\text{Rad}(abc)$  in terms of  $c$ .

An unconditional result in the direction of the *abc* Conjecture has been obtained in 1986 by Stewart and Tijdeman [71] using lower bounds for linear combinations of logarithms, in the complex case as well as in the  $p$ -adic case:

$$\log c \leq \kappa R^{15}$$

with an absolute constant  $\kappa$ . This estimate has been refined by Stewart and Yukunrui, who proved in 1991 [72]: for any  $\varepsilon > 0$  and for  $c$  sufficiently large in terms of  $\varepsilon$ ,

$$\log c \leq \kappa(\varepsilon) R^{(2/3)+\varepsilon}.$$

In 2001, they refined in [73] the exponent  $2/3$  to  $1/3$  when they established the best known estimate so far:

**Theorem 1** (Stewart-Yu Kunrui). *There exists an absolute constant  $\kappa$  such that any  $abc$  triple  $(a, b, c)$  satisfies*

$$\log c \leq \kappa R^{1/3} (\log R)^3$$

with  $R = \text{Rad}(abc)$ . In other terms,

$$c \leq e^{\kappa R^{1/3} (\log R)^3}.$$

J. Esterlé and A. Nitaj (see [56]) proved that the  $abc$  Conjecture implies the truth of a previous conjecture by L. Szpiro on the conductor of elliptic curves (see [36] p 227):

**Conjecture 3** (Szpiro's Conjecture). *Given any  $\varepsilon > 0$ , there exists a constant  $C(\varepsilon) > 0$  such that, for every elliptic curve with minimal discriminant  $\Delta$  and conductor  $N$ ,*

$$|\Delta| < C(\varepsilon) N^{6+\varepsilon}.$$

According to [78, 41, 37], the next statement is equivalent to the  $abc$  Conjecture.

**Conjecture 4** (Generalized Szpiro's Conjecture). *Given any  $\varepsilon > 0$  and  $M > 0$ , there exists a constant  $C(\varepsilon, M) > 0$  such that, for all integers  $x$  and  $y$  such that the number  $D = 4x^3 - 27y^2$  is not 0 and such that the greatest prime factor of  $x$  and  $y$  is bounded by  $M$ ,*

$$\max\{|x|^3, y^2, |D|\} < C(\varepsilon, M) \text{Rad}(D)^{6+\varepsilon}.$$

In view of Conjecture 3, it is natural to introduce another exponent related with the  $abc$  Conjecture. When  $(a, b, c)$  is an  $abc$  triple, define

$$\varrho(a, b, c) = \frac{\log abc}{\log \text{Rad}(abc)}.$$

From the  $abc$  Conjecture it follows that for any  $\varepsilon > 0$ , there are only finitely many  $abc$ -triples  $(a, b, c)$  such that  $\varrho(a, b, c) > 3 + \varepsilon$ .

Here are the two largest known values for  $\varrho(a, b, c)$ , both found by A. Nitaj [56]

$a + b = c$	$\varrho(a, b, c)$
$13 \cdot 19^6 + 2^{30} \cdot 5 = 3^{13} \cdot 11^2 \cdot 31$	4.41901...
$2^5 \cdot 11^2 \cdot 19^9 + 5^{15} \cdot 37^2 \cdot 47 = 3^7 \cdot 7^{11} \cdot 743$	4.26801...

In 2013, there are 47 known  $abc$ -triples  $(a, b, c)$  satisfying  $\rho(a, b, c) > 4$ .

In 2006, the Mathematics Department of Leiden University in the Netherlands, together with the Dutch Kennislink Science Institute, launched the ABC@Home project, a grid computing system which aims to discover additional  $abc$ -triples. Although no finite set of examples or counterexamples can resolve the  $abc$  conjecture, it is hoped that patterns in the triples discovered by this project will lead to insights about the conjecture and about number theory more generally. ABC@Home is an educational and non-profit distributed computing project finding  $abc$ -triples related to the  $abc$  conjecture.

Surveys on the  $abc$  Conjecture have been written by S. Lang [41, 42]; (see also §7 p. 194–200 of [43]), by A. Nitaj [57] and W.M. Schmidt [66] Epilogue p. 205.

The *Congruence  $abc$  Conjecture* is discussed in [55], §5.5 and 5.6.

Generalizations of the  $abc$  Conjecture to more than three numbers, namely to  $a_1 + \dots + a_n = 0$ , have been investigated by J. Browkin and J. Brzeziński [14] in 1994 and by Hu, Pei-Chu and Yang, Chung-Chun in 2002 [35].

## 3 Consequences

### 3.1 Fermat's Last Theorem

Assume  $x, y, z, n$  are positive integers satisfying  $x^n + y^n = z^n$ ,  $\gcd(x, y, z) = 1$  and  $x < y$ . Then  $(x^n, y^n, z^n)$  is an  $abc$ -triple with

$$\text{Rad}(x^n y^n z^n) \leq xyz < z^3.$$

If the explicit  $abc$  Conjecture  $c < \text{Rad}(abc)^2$  of [31] is true, then one deduces  $z^n < z^6$ , hence  $n \leq 5$ .

### 3.2 Perfect powers

Define a *perfect power* as a positive integer of the form  $a^b$  where  $a$  and  $b$  are positive integers and  $b \geq 2$ . The sequence of perfect powers starts with

1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128,  
144, 169, 196, 216, 225, 243, 256, 289, 324, 343, 361, 400,  
441, 484, 512, 529, 576, 625, 676, 729, 784, 841, 900, 961,  
1000, 1024, 1089, 1024, 1089, 1156, 1225, 1296, 1331, 1369,  
1444, 1521, 1600, 1681, 1728, 1764, ...

The reference of this sequence in Sloane's Encyclopaedia of Integer Sequences is <http://oeis.org/A001597>.

From the *abc* Conjecture 2, one easily deduces the following Conjecture due to Subbayya Sivasankaranarayana Pillai [60] (see also [61, 62])

**Conjecture 5** (Pillai). *In the sequence of perfect powers, the difference between two consecutive terms tends to infinity.*

Pillai's Conjecture 5 can also be stated in an equivalent way as follows:

- Let  $k$  be a positive integer. The equation

$$x^p - y^q = k,$$

where the unknowns  $x$ ,  $y$ ,  $p$  and  $q$  take integer values, all  $\geq 2$ , has only finitely many solutions  $(x, y, p, q)$ .

For  $k = 1$ , Mihăilescu's solution of Catalan's Conjecture states that the only solution to Catalan's equation [62, 6]

$$x^p - y^q = 1$$

is  $3^2 - 2^3 = 1$ . It is a remarkable fact that there is no value of  $k \geq 2$  for which one knows that Pillai's equation  $x^p - y^q = k$  has only finitely many solutions.

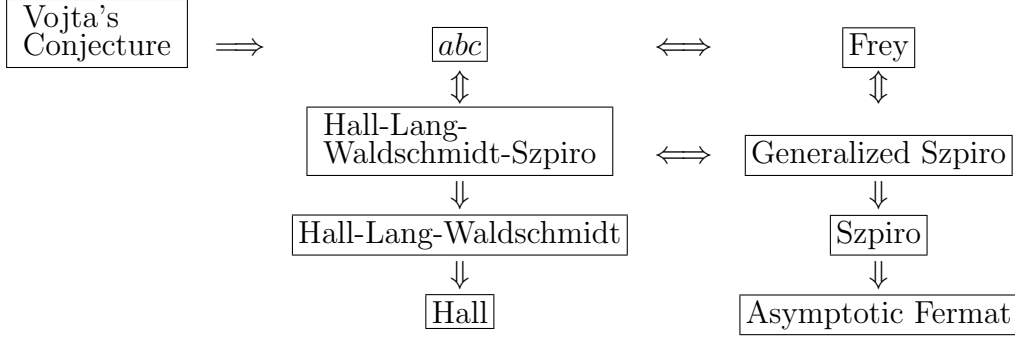
The *abc* Conjecture implies the following stronger version of Pillai's Conjecture (see the introduction of Chapters X and XI of [40]):

**Conjecture 6** (Lang-Waldschmidt). *Let  $\varepsilon > 0$ . There exists a constant  $c(\varepsilon) > 0$  with the following property. If  $x^p \neq y^q$ , then*

$$|x^p - y^q| \geq c(\varepsilon) \max\{x^p, y^q\}^{\kappa - \varepsilon} \quad \text{with} \quad \kappa = 1 - \frac{1}{p} - \frac{1}{q}.$$

The motivation of this Conjecture in [40] is the quest for a strong (essentially optimal) lower bound for linear combinations of logarithms of algebraic numbers.

P. Vojta, in [78] Chap.V appendix *ABC*, explained connections between various conjectures. Here is a figure from that reference:



In the special case  $p = 3, q = 2$ , Conjecture 6 reads: *If  $x^3 \neq y^2$ , then*

$$|x^3 - y^2| \geq c(\varepsilon) \max\{x^3, y^2\}^{(1/6)-\varepsilon}.$$

In 1971, Marshall Hall Jr [33] proposed a stronger Conjecture without the  $\varepsilon$  (what is called *Hall's Conjecture* in [78] has the  $\varepsilon$ ):

**Conjecture 7** (M. Hall Jr.). *There exists an absolute constant  $c > 0$  such that, if  $x^3 \neq y^2$ , then*

$$|x^3 - y^2| \geq c \max\{x^3, y^2\}^{1/6}.$$

This statement does not follow from the *abc* Conjecture 2. In [33], M. Hall Jr discusses possible values for his constant  $c$  in Conjecture 7. In the other direction, L.V. Danilov [17] (see also [37]) proved that the inequality

$$0 < |x^3 - y^2| < 0.971|x|^{1/2}$$

has infinitely many solutions in integers  $x, y$ . According to F. Beukers and C.L. Stewart [5], this conjecture maybe too optimistic. Indeed they conjecture:

**Conjecture 8** (Beukers–Stewart). *Let  $p, q$  be coprime integers with  $p > q \geq 2$ . Then, for any  $c > 0$ , there exist infinitely many positive integers  $x, y$  such that*

$$0 < |x^p - y^q| < c \max\{x^p, y^q\}^\kappa \quad \text{with} \quad \kappa = 1 - \frac{1}{p} - \frac{1}{q}.$$

### 3.3 Generalized Fermat Equation

Consider the equation (see for instance [77])

$$x^p + y^q = z^r \quad (3.1)$$

where the unknowns  $(x, y, z, p, q, r)$  take their values in the set of tuples of positive integers for which  $x, y, z$  are relatively prime and  $p, q, r$  are  $\geq 2$ .

Define

$$\chi = \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1.$$

If  $\chi \geq 0$ , then  $(p, q, r)$  is a permutation of one of

$$(2, 2, k) \quad (k \geq 2), \quad (2, 3, 3), \quad (2, 3, 4), \\ (2, 3, 5), \quad (2, 4, 4), \quad (2, 3, 6), \quad (3, 3, 3);$$

in each of these cases, all solutions  $(x, y, z)$  are known, often there are infinitely many of them (see [4, 16, 38, 39]).

Assume now  $\chi > 0$ . Then only 10 solutions  $(x, y, z, p, q, r)$  with  $x, y, z$  relatively prime (up to obvious symmetries) to the equation (3.1) are known; by increasing order for  $z^r$ , they are:

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2, \\ 3^5 + 11^4 = 122^2, \quad 33^8 + 1\,549\,034^2 = 15\,613^3, \\ 1\,414^3 + 2\,213\,459^2 = 65^7, \quad 9\,262^3 + 15\,312\,283^2 = 113^7, \\ 17^7 + 76\,271^3 = 21\,063\,928^2, \quad 43^8 + 96\,222^3 = 30\,042\,907^2.$$

Beal's problem, including a 50 000 US\$ prize (see [52]), is:

**Problem 9** (Beal's Problem). *Assume  $\chi < 0$ . Either find another solution to equation (3.1), or prove that there is no further solution.*

A related conjecture, due to R. Tijdeman and D. Zagier [52], is:

**Conjecture 10** (Tijdeman-Zagier). *The equation (3.1) has no solution in positive integers  $(x, y, z, p, q, r)$  with each of  $p, q$  and  $r$  at least 3 and with  $x, y, z$  relatively prime.*

The next conjecture is proposed by H. Darmon and A. Granville [18] :



**Conjecture 11** (Fermat-Catalan Conjecture). *The set of solutions  $(x, y, z, p, q, r)$  with  $\chi < 0$  to the equation (3.1) is finite.*

It is easy to deduce Conjecture 11 from the *abc* Conjecture 2, once one notices that for  $p, q, r$  positive integers, the assumption  $\chi < 0$  implies

$$\chi \leq -\frac{1}{42}.$$

In 1995, H. Darmon and A. Granville [18] proved unconditionally that for fixed  $(p, q, r)$  with  $\chi < 0$ , there are only finitely many  $(x, y, z)$  satisfying equation (3.1).

### 3.4 Wieferich Primes

A *Wieferich prime* is a prime number  $p$  such that  $p^2$  divides  $2^{p-1} - 1$ . Note that the definition in [55], §5.4 is the opposite. The only known Wieferich primes below  $4 \cdot 10^{12}$  are 1093 and 3511.

J.H. Silverman [69] showed that if the *abc* Conjecture 2 is true, given a positive integer  $a > 1$ , there exist infinitely many primes  $p$  such that  $p^2$  does not divide  $a^{p-1} - 1$ . A consequence is that there are infinitely many primes which are not Wieferich primes, a result which is known only if one assumes the *abc* Conjecture. See also [32].

### 3.5 Erdős–Woods Conjecture

There are infinitely many pairs of positive integers  $(x, y)$  with  $x < y$  such that  $x$  and  $y$  have the same radical, and, at the same time,  $x + 1$  and  $y + 1$  have the same radical. Indeed, for  $k \geq 1$ , the pair of numbers  $(x, y)$  with

$$x = 2^k - 2 = 2(2^{k-1} - 1) \quad \text{and} \quad y = (2^k - 1)^2 - 1 = 2^{k+1}(2^{k-1} - 1)$$

satisfy this condition, since

$$x + 1 = 2^k - 1 \quad \text{and} \quad y + 1 = (2^k - 1)^2.$$

There is one further sporadic known example, namely  $(x, y) = (75, 1215)$ , since

$$75 = 3 \cdot 5^2 \quad \text{and} \quad 1215 = 3^5 \cdot 5 \quad \text{with} \quad \text{Rad}(75) = \text{Rad}(1215) = 3 \cdot 5 = 15,$$

while

$$76 = 2^2 \cdot 19 \quad \text{and} \quad 1216 = 2^6 \cdot 19 \quad \text{with} \quad \text{Rad}(76) = \text{Rad}(1216) = 2 \cdot 19 = 38.$$

It is not known whether there are further examples. It is not even known whether there exist two distinct integers  $x, y$  such that

$$\text{Rad}(x) = \text{Rad}(y), \quad \text{Rad}(x+1) = \text{Rad}(y+1) \quad \text{and} \quad \text{Rad}(x+2) = \text{Rad}(y+2).$$

The comparatively weaker assertion below [44, 45, 46, 47] would have interesting consequences in logic:

**Conjecture 12** (Erdős–Woods Conjecture). *There exists an absolute constant  $k$  such that, if  $x$  and  $y$  are positive integers satisfying*

$$\text{Rad}(x+i) = \text{Rad}(y+i)$$

for  $i = 0, 1, \dots, k-1$ , then  $x = y$ .

M. Langevin [44, 46, 47] (cf. [37]) proved that this Conjecture follows from the *abc* Conjecture 2. See also [40] and [3] for connections with conjectures 6 and 7.

### 3.6 Waring’s Problem

In 1770, a few months before J.L. Lagrange solved a conjecture of Bachet (1621) and Fermat (1640) by proving that every positive integer is the sum of at most four squares of integers, E. Waring wrote (see [81]) :

- “*Omnis integer numerus vel est cubus, vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compositus, est etiam quadrato-quadratus vel e duobus, tribus, 4, usque ad novemdecim compositus, & sic deinceps*”<sup>2</sup>

Waring’s function  $g$  is defined as follows: For any integer  $k \geq 2$ ,  $g(k)$  is the least positive integer  $s$  such that any positive integer  $N$  can be written  $x_1^k + \dots + x_s^k$ .

For each integer  $k \geq 2$ , define  $I(k) = 2^k + \lceil (3/2)^k \rceil - 2$ . It is easy to show that  $g(k) \geq I(k)$  (this result is due to J. A. Euler, son of Leonhard

---

<sup>2</sup> “Every integer is a cube or the sum of two, three, . . . nine cubes; every integer is also the square of a square, or the sum of up to nineteen such; and so forth. Similar laws may be affirmed for the correspondingly defined numbers of quantities of any like degree.”

Euler). Indeed, write the Euclidean division of  $3^k$  by  $2^k$ , with quotient  $q$  and remainder  $r$ :

$$3^k = 2^k q + r \quad \text{with} \quad 0 < r < 2^k, \quad q = \left\lceil \left(\frac{3}{2}\right)^k \right\rceil$$

and consider the integer

$$N = 2^k q - 1 = (q - 1)2^k + (2^k - 1)1^k.$$

Since  $N < 3^k$ , writing  $N$  as a sum of  $k$ -th powers can involve no term  $3^k$ , and since  $N < 2^k q$ , it involves at most  $(q - 1)$  terms  $2^k$ , all others being  $1^k$ ; hence it requires a total number of at least  $(q - 1) + (2^k - 1) = I(k)$  terms.

The next conjecture [54] is due to C.A. Bretschneider (1853).

**Conjecture 13** (Ideal Waring's Theorem). *For any  $k \geq 2$ ,  $g(k) = I(k)$ .*

A slight improvement of the upper bound  $r < 2^k$  for the remainder  $r = 3^k - 2^k q$  would suffice for proving Conjecture 13. Indeed, L.E. Dickson and S.S. Pillai (see for instance [34], Chap. XXI or [54] p. 226, Chap. IV) proved independently in 1936 that if  $r = 3^k - 2^k q$  satisfies

$$r \leq 2^k - q - 2, \tag{3.2}$$

then  $g(k) = I(k)$ . The condition (3.2) is satisfied for  $4 \leq k \leq 471\,600\,000$ .

According to K. Mahler, the upper bound (3.2) is valid for all sufficiently large  $k$ . Hence the ideal Waring's Theorem

$$g(k) = I(k)$$

holds also for all sufficiently large  $k$ . However, Mahler's proof uses a  $p$ -adic Diophantine argument related to the Thue–Siegel–Roth Theorem which does not yield effective results.

S. David (see [60]) noticed that the estimate (3.2) for sufficiently large  $k$  follows from the *abc* Conjecture 2. S. Laishram checked that the ideal Waring's Theorem  $g(k) = I(k)$  follows from the explicit *abc* Conjecture 15.

### 3.7 A problem of P. Erdős solved by C.L. Stewart

Let us denote by  $P(m)$  the greatest prime factor of an integer  $m \geq 2$ . In 1965, P. Erdős conjectured

$$\frac{P(2^n - 1)}{n} \rightarrow \infty \quad \text{when } n \rightarrow \infty.$$

In 2002, R. Murty and S. Wong [53] proved that this is a consequence of the *abc* Conjecture 2.

In 2012, C.L. Stewart [70] proved Erdős's Conjecture (in a wider context of Lucas and Lehmer sequences) in the stronger form:

$$P(2^n - 1) > n \exp(\log n / 104 \log \log n).$$

## 4 Stronger than *abc*: best possible estimate?

Let  $\delta > 0$ . In 1986, C.L. Stewart and R. Tijdeman [71] proved that there are infinitely many *abc*-triples  $(a, b, c)$  for which

$$c > R \exp \left( (4 - \delta) \frac{(\log R)^{1/2}}{\log \log R} \right).$$

This is much better than the lower bound  $c > R \log R$  obtained in Lemma 1. The coefficient  $4 - \delta$  has been improved by M. van Frankenhuijsen [21] into 6.068 in 2000. In the same paper, M. van Frankenhuijsen suggested that there may exist two positive absolute constants  $\kappa_1$  and  $\kappa_2$  such that, for any *abc*-triples  $(a, b, c)$ ,

$$c < R \exp \left( \kappa_1 \left( \frac{\log R}{\log \log R} \right)^{1/2} \right),$$

while for infinitely many *abc*-triples  $(a, b, c)$ ,

$$c > R \exp \left( \kappa_2 \left( \frac{\log R}{\log \log R} \right)^{1/2} \right).$$

O. Robert, C.L. Stewart and G. Tenenbaum suggest in [63] the following more precise limit for the *abc* Conjecture, which would yield these statements with  $\kappa_1 = 4\sqrt{3} + \varepsilon$  for  $c$  sufficiently large in terms of  $\varepsilon$  and  $\kappa_2 = 4\sqrt{3} - \varepsilon$  for any  $\varepsilon > 0$ .

**Conjecture 14** (Robert-Stewart-Tenenbaum). *There exist positive constants  $\kappa_1, \kappa_2, \kappa_3$  such that, for any  $abc$ -triple  $(a, b, c)$  with  $R = \text{Rad}(abc)$ ,*

$$c < \kappa_1 R \exp \left( 4\sqrt{3} \left( \frac{\log R}{\log \log R} \right)^{1/2} \left( 1 + \frac{\log \log \log R}{2 \log \log R} + \frac{\kappa_2}{\log \log R} \right) \right)$$

*and there exist infinitely many  $abc$ -triples  $(a, b, c)$  for which*

$$c > R \exp \left( 4\sqrt{3} \left( \frac{\log R}{\log \log R} \right)^{1/2} \left( 1 + \frac{\log \log \log R}{2 \log \log R} + \frac{\kappa_3}{\log \log R} \right) \right).$$

The only heuristic argument used in [63] is that, whenever  $a$  and  $b$  are relatively prime positive integers, the radicals of  $a$ ,  $b$  and  $a+b$  are statistically independent. The estimates from (14) are based on the work [64] on the number of positive integers  $N(x, y)$  bounded by  $x$  whose radical is at most  $y$ .

## 5 Explicit $abc$ Conjecture

In 1996, A. Baker [1] suggested the following statement. Let  $(a, b, c)$  be an  $abc$ -triple and let  $\varepsilon > 0$ . Then

$$c \leq \kappa (\varepsilon^{-\omega} R)^{1+\varepsilon}$$

where  $\kappa$  is an absolute constant,  $R = \text{Rad}(abc)$  and  $\omega = \omega(abc)$  is the number of distinct prime factors of  $abc$ .

A. Granville noticed that the minimum of the function on the right hand side over  $\varepsilon > 0$  occurs essentially with  $\varepsilon = \omega / \log R$ . This incited Baker [2] to propose a slightly sharper form of his previous conjecture, namely :

$$c \leq \kappa R \frac{(\log R)^\omega}{\omega!}.$$

He made some computational experiments in order to guess an admissible value for his absolute constant  $\kappa$ , and he ended up with the following precise statement:

**Conjecture 15** (Explicit  $abc$  Conjecture). *Let  $(a, b, c)$  be an  $abc$ -triple. Then*

$$c \leq \frac{6}{5} R \frac{(\log R)^\omega}{\omega!},$$

*with  $R = \text{Rad}(abc)$  and  $\omega = \omega(abc)$ .*

P. Philippon in 1999 [59] (Appendix) pointed out how sharp lower bounds for linear forms in logarithms, involving several metrics, would imply the *abc* Conjecture. Effective and explicit versions of the *abc* Conjecture have plenty of consequences [10, 8, 13, 39, 65]. Here is a very few set of examples.

The *Nagell–Ljunggren equation* is the equation

$$y^q = \frac{x^n - 1}{x - 1}$$

where the unknowns  $x, y, n, q$  take their values in the set of tuples of positive integers satisfying  $x > 1, y > 1, n > 2$  and  $q > 1$ . This equation means that in basis  $x$ , all the digits of the perfect power  $y^q$  are 1 (this is a so-called *repunit*).

According to [39], if the explicit *abc* Conjecture 15 of Baker is true, then the only solutions are

$$11^2 = \frac{3^5 - 1}{3 - 1}, \quad 20^2 = \frac{7^4 - 1}{7 - 1}, \quad 7^3 = \frac{18^3 - 1}{18 - 1}.$$

Further consequences of the explicit *abc* Conjecture 15 are discussed in [39], in particular on the *Goormaghtigh's Conjecture*, which states that the only numbers with at least three digits and with all digits equal to 1 in two different bases are 31 (in bases 2 and 5) and 8191 (in bases 2 and 90):

$$\frac{5^3 - 1}{5 - 1} = \frac{2^5 - 1}{2 - 1} = 31 \quad \text{and} \quad \frac{90^3 - 1}{90 - 1} = \frac{2^{13} - 1}{2 - 1} = 8191.$$

In other terms, the Goormaghtigh's Conjecture asserts that if  $(x, y, m, n)$  is a tuple of positive integers satisfying  $x > y > 1, n > 2, m > 2$  and

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1},$$

then  $(x, y, m, n)$  is either  $(5, 2, 3, 5)$  or  $(90, 2, 3, 13)$ . Surveys on such questions have been written by T.N. Shorey [67, 68].

## 6 *abc* for number fields

In 1991, N. Elkies [19] deduced Faltings's Theorem on the finiteness of the set of rational points on an algebraic curve of genus  $\geq 2$  (previously *Mordell's*

conjecture) from a generalization he proposed of the *abc* Conjecture to number fields. See also [31].

In 1994, E. Bombieri [7] deduced from a generalization of the *abc* Conjecture to number fields a refinement of the Thue–Siegel–Roth Theorem on the rational approximation of algebraic numbers

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{q^{2+\varepsilon}},$$

where he replaces  $\varepsilon$  by

$$\kappa(\log q)^{-1/2}(\log \log q)^{-1},$$

with  $\kappa$  depending only on the algebraic number  $\alpha$ .

A. Granville and H.M. Stark [30] proved that the uniform *abc* Conjecture for number fields implies a lower bound for the class number of an imaginary quadratic number field; K. Mahler had shown that this implies that the associated *L*–function has no Siegel zero. See also [31].

Further work on the *abc* Conjecture for number fields ( see [8]) are due to M. van Frankenhuijsen [20, 21, 22, 24], N. Broberg [9], J. Browkin [11, 12], A. Granville and H.M. Stark [30], K. Győry, D.W. Masser [51], A. Surroca [75, 76], P.C. Hu and C.C Yang [36] § 5.6 and [37].

## 7 Further consequences of the *abc* Conjecture

Further consequences of the *abc* Conjecture 2 are quoted in [56], including:

- Erdős’s Conjecture on consecutive powerful numbers. The *abc* Conjecture 2 implies that the set of triples of consecutive powerful integers (namely integers of the form  $a^2b^3$ ) is finite. R. Mollin and G. Walsh conjecture that there is no such triple.
- Dressler’s Conjecture: between two positive integers having the same prime factors, there is always a prime.
- Squarefree and powerfree values of polynomials [15, 29].
- Lang’s conjectures: lower bounds for heights, number of integral points on elliptic curves [25, 26, 27].
- Bounds for the order of the Tate–Shafarevich group [28].
- Vojta’s Conjecture for curves [78, 79, 80].
- Greenberg’s Conjecture on Iwasawa invariants  $\lambda$  and  $\mu$  in cyclotomic extensions.

- Exponents of class groups of quadratic fields.
- Fundamental units in quadratic and biquadratic fields.

## 8 *abc* and meromorphic function fields

There is a rich theory related with Nevanlinna value distribution theory. See for instance P. Vojta [78, 79, 80], Machiel van Frankenhuijsen [22, 23], Hu, Pei–Chu and Yang, Chung-Chun [35, 36, 37]. Notice in particular that Vojta’s Conjecture on algebraic points of bounded degree on a smooth complete variety over a global field of characteristic zero implies the *abc* Conjecture 2.

## 9 *ABC* Theorem for polynomials

We end this lecture with a proof of an analog of the *abc* conjecture for polynomials – see for instance [31, 43].

Let  $K$  be an algebraically closed field. The *radical* of a monic polynomial

$$P(X) = \prod_{i=1}^n (X - \alpha_i)^{a_i} \in K[X],$$

with  $\alpha_i$  pairwise distinct, is defined as

$$\text{Rad}(P)(X) = \prod_{i=1}^n (X - \alpha_i) \in K[X].$$

The following result is due to W.W. Stothers [74] and R. Mason [48, 49]. It can also be deduced from earlier results by A. Hurwitz.

**Theorem 2** (*ABC* Theorem). *Let  $A, B, C$  be three relatively prime polynomials in  $K[X]$  with  $A + B = C$  and let  $R = \text{Rad}(ABC)$ . Then*

$$\max\{\deg(A), \deg(B), \deg(C)\} < \deg(R).$$

This result can be compared with the *abc* Conjecture 2, where the degree of a polynomial replaces the logarithm of a positive integer.

The proof uses the remark that the radical is related with a gcd: for  $P \in K[X]$  a monic polynomial, we have

$$\text{Rad}(P) = \frac{P}{\gcd(P, P')}. \tag{9.1}$$



Indeed, the common zeroes of

$$P(X) = \prod_{i=1}^n (X - \alpha_i)^{a_i} \in K[X]$$

and  $P'$  are the  $\alpha_i$  with  $a_i \geq 2$ . They are zeroes of  $P'$  with multiplicity  $a_i - 1$ . Hence (9.1) follows.

Now suppose  $A + B = C$  with  $A, B, C$  relatively prime. Notice that

$$\text{Rad}(ABC) = \text{Rad}(A)\text{Rad}(B)\text{Rad}(C).$$

We may suppose  $A, B, C$  to be monic and, say,  $\deg(A) \leq \deg(B) \leq \deg(C)$ . Write

$$A + B = C, \quad A' + B' = C'.$$

Then the three determinants

$$\begin{vmatrix} A & B \\ A' & B' \end{vmatrix} = AB' - A'B, \quad \begin{vmatrix} A & C \\ A' & C' \end{vmatrix} = AC' - A'C, \quad \begin{vmatrix} C & B \\ C' & B' \end{vmatrix} = CB' - C'B$$

take the same value; in particular

$$AB' - A'B = AC' - A'C.$$

Recall  $\gcd(A, B, C) = 1$ . Since  $\gcd(C, C')$  divides  $AC' - A'C = AB' - A'B$ , it divides also

$$\frac{AB' - A'B}{\gcd(A, A') \gcd(B, B')}$$

which, according to (9.1), is a polynomial of degree strictly less than

$$\deg(\text{Rad}(A)) + \deg(\text{Rad}(B)) = \deg(\text{Rad}(AB)).$$

Hence

$$\deg(\gcd(C, C')) < \deg(\text{Rad}(AB)).$$

Using (9.1) again, we deduce

$$\deg(C) = \deg(\text{Rad}(C)) + \deg(\gcd(C, C')),$$

hence

$$\deg(C) < \deg(\text{Rad}(C)) + \deg(\text{Rad}(AB)) = \deg(\text{Rad}(ABC)).$$

## References

- [1] A. BAKER – «Logarithmic forms and the *abc*-conjecture», in *Number theory (Eger, 1996)*, de Gruyter, Berlin, 1998, p. 37–44.
- [2] — , «Experiments on the *abc*-conjecture», *Publ. Math. Debrecen* **65** (2004), no. 3-4, p. 253–260.
- [3] R. BALASUBRAMANIAN, T. N. SHOREY & M. WALDSCHMIDT – «On the maximal length of two sequences of consecutive integers with the same prime divisors», *Acta Math. Hungar.* **54** (1989), no. 3-4, p. 225–236.
- [4] F. BEUKERS – «The Diophantine equation  $Ax^p + By^q = Cz^r$ », *Duke Math. J.* **91** (1998), no. 1, p. 61–88.
- [5] F. BEUKERS & C. L. STEWART – «Neighboring powers», *J. Number Theory* **130** (2010), no. 3, p. 660–679, Addendum: *J. Number Theory* **130** (2010), no. 7, p. 1571.
- [6] Y. F. BILU – «Catalan’s conjecture (after Mihăilescu)», *Astérisque* (2004), no. 294, p. vii, 1–26, Séminaire Bourbaki - Volume 2002/2003 - Exposés 909-923.
- [7] E. BOMBIERI – «Roth’s theorem and the *abc*-conjecture», 1994, preprint ETH Zürich.
- [8] E. BOMBIERI & W. GUBLER – *Heights in diophantine geometry*, Cambridge: Cambridge University Press, 2006.
- [9] N. BROBERG – «Some examples related to the *abc*-conjecture for algebraic number fields», *Math. Comp.* **69** (2000), no. 232, p. 1707–1710.
- [10] J. BROWKIN – «A consequence of an effective form of the *abc*-conjecture», *Colloq. Math.* **82** (1999), no. 1, p. 79–84.
- [11] — , «The *abc*-conjecture», in *Number theory*, Trends Math., Birkhäuser, Basel, 2000, p. 75–105.
- [12] — , «The *abc*-conjecture for algebraic numbers», *Acta Math. Sin. (Engl. Ser.)* **22** (2006), no. 1, p. 211–222.

- [13] —, «A weak effective *abc*-conjecture», *Funct. Approx. Comment. Math.* **39** (2008), no. part 1, p. 103–111.
- [14] J. BROWKIN & J. BRZEZIŃSKI – «Some remarks on the *abc*-conjecture», *Math. Comp.* **62** (1994), no. 206, p. 931–939.
- [15] J. BROWKIN, M. FILASETA, G. GREAVES & A. SCHINZEL – «Squarefree values of polynomials and the *abc*-conjecture», in *Sieve methods, exponential sums, and their applications in number theory (Cardiff, 1995)*, London Math. Soc. Lecture Note Ser., vol. 237, Cambridge Univ. Press, Cambridge, 1997, p. 65–85.
- [16] H. COHEN – *Number theory. Vol. II. Analytic and modern tools*, Graduate Texts in Mathematics, vol. 240, Springer, New York, 2007.
- [17] L. V. DANILOV – «Letter to the editors: “The Diophantine equation  $x^3 - y^2 = k$  and a conjecture of M. Hall” [Mat. Zametki **32** (1982), no. 3, 273–275; MR0677595 (84c:10014)]», *Mat. Zametki* **36** (1984), no. 3, p. 457–458.
- [18] H. DARMON & A. GRANVILLE – «On the equations  $z^m = F(x, y)$  and  $Ax^p + By^q = Cz^r$ », *Bull. London Math. Soc.* **27** (1995), no. 6, p. 513–543.
- [19] N. D. ELKIES – «*ABC* implies Mordell», *Internat. Math. Res. Notices* (1991), no. 7, p. 99–109.
- [20] M. VAN FRANKENHUIJSEN – «The *ABC* conjecture implies Roth’s theorem and Mordell’s conjecture», *Mat. Contemp.* **16** (1999), p. 45–72, 15th School of Algebra (Portuguese) (Canela, 1998).
- [21] —, «A lower bound in the *abc* conjecture», *J. Number Theory* **82** (2000), no. 1, p. 91–95.
- [22] —, «The *ABC* conjecture implies Vojta’s height inequality for curves», *J. Number Theory* **95** (2002), no. 2, p. 289–302.
- [23] —, «*ABC* implies the radicalized Vojta height inequality for curves», *J. Number Theory* **127** (2007), no. 2, p. 292–300.
- [24] —, «About the *ABC* conjecture and an alternative», in *Number theory, analysis and geometry*, Springer, New York, 2012, p. 169–180.

- [25] G. FREY – «Links between elliptic curves and solutions of  $A - B = C$ », *J. Indian Math. Soc. (N.S.)* **51** (1987), p. 117–145 (1988).
- [26] — , «Links between solutions of  $A - B = C$  and elliptic curves», in *Number theory (Ulm, 1987)*, Lecture Notes in Math., vol. 1380, Springer, New York, 1989, p. 31–62.
- [27] D. GOLDFELD – «Modular forms, elliptic curves and the  $ABC$ -conjecture», in *A panorama of number theory or the view from Baker’s garden (Zürich, 1999)*, Cambridge Univ. Press, Cambridge, 2002, p. 128–147.
- [28] D. GOLDFELD & L. SZPIRO – «Bounds for the order of the Tate-Shafarevich group», *Compositio Math.* **97** (1995), no. 1-2, p. 71–87, Special issue in honour of Frans Oort.
- [29] A. GRANVILLE – « $abc$  allows us to count squarefrees.», *Int. Math. Res. Not.* **1998** (1998), no. 19, p. 991–1009.
- [30] A. GRANVILLE & H. STARK – « $ABC$  implies no “Siegel zeros” for  $L$ -functions of characters with negative discriminant.», *Invent. Math.* **139** (2000), no. 3, p. 509–523.
- [31] A. GRANVILLE & T. J. TUCKER – «It’s as easy as  $abc$ .», *Notices Am. Math. Soc.* **49** (2002), no. 10, p. 1224–1231,  
<http://www.ams.org/notices/200210/fea-granville.pdf>.
- [32] H. GRAVES & M. RAM MURTY – «The  $abc$  conjecture and non-Wieferich primes in arithmetic progressions», *J. Number Theory* **133** (2013), no. 6, p. 1809–1813.
- [33] M. HALL, JR. – «The Diophantine equation  $x^3 - y^2 = k$ », in *Computers in number theory (Proc. Sci. Res. Council Atlas Sympos. No. 2, Oxford, 1969)*, Academic Press, London, 1971, p. 173–198.
- [34] G. H. HARDY & E. M. WRIGHT – *An introduction to the theory of numbers*, sixth éd., Oxford University Press, Oxford, 2008, Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.

- [35] P.-C. HU & C.-C. YANG – «A generalized  $abc$ -conjecture over function fields», *J. Number Theory* **94** (2002), no. 2, p. 286–298.
- [36] — , *Distribution theory of algebraic numbers*, de Gruyter Expositions in Mathematics, vol. 45, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.
- [37] — , «Roth’s theorem and  $abc$ -conjecture», *J. Shandong Univ. Nat. Sci.* **44** (2009), no. 8, p. 1–12,  
[http://lxbwk.njournal.sdu.edu.cn/EN/volumn/volumn\\_91.shtml](http://lxbwk.njournal.sdu.edu.cn/EN/volumn/volumn_91.shtml).
- [38] A. KRAUS – «On the equation  $x^p + y^q = z^r$ : a survey», *Ramanujan J.* **3** (1999), no. 3, p. 315–333.
- [39] S. LAISHRAM & T. N. SHOREY – «Baker’s explicit  $abc$ -conjecture and applications», *Acta Arith.* **155** (2012), no. 4, p. 419–429.
- [40] S. LANG – *Elliptic curves: Diophantine analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 231, Springer-Verlag, Berlin, 1978.
- [41] — , «Old and new conjectured Diophantine inequalities», *Bull. Amer. Math. Soc. (N.S.)* **23** (1990), no. 1, p. 37–75.
- [42] — , «Die  $abc$ -Vermutung», *Elem. Math.* **48** (1993), no. 3, p. 89–99.
- [43] — , *Algebra*, third éd., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.
- [44] M. LANGEVIN – «Cas d’égalité pour le théorème de Mason et applications de la conjecture ( $abc$ )», *C. R. Acad. Sci. Paris Sér. I Math.* **317** (1993), no. 5, p. 441–444.
- [45] — , «Sur quelques conséquences de la conjecture ( $abc$ ) en arithmétique et en logique», *Rocky Mountain J. Math.* **26** (1996), no. 3, p. 1031–1042, Symposium on Diophantine Problems (Boulder, CO, 1994).
- [46] — , «Imbrications entre le théorème de Mason, la descente de Belyi et les différentes formes de la conjecture ( $abc$ )», *J. Théor. Nombres Bordeaux* **11** (1999), no. 1, p. 91–109, Les XXèmes Journées Arithmétiques (Limoges, 1997).

- [47] — , «Liens entre le théorème de Mason et la conjecture  $(abc)$ », in *Number theory (Ottawa, ON, 1996)*, CRM Proc. Lecture Notes, vol. 19, Amer. Math. Soc., Providence, RI, 1999, p. 187–213.
- [48] R. C. MASON – *Diophantine equations over function fields*, London Mathematical Society Lecture Note Series, vol. 96, Cambridge University Press, Cambridge, 1984.
- [49] — , «The study of Diophantine equations over function fields», in *New advances in transcendence theory (Durham, 1986)*, Cambridge Univ. Press, Cambridge, 1988, p. 229–247.
- [50] D. W. MASSER – «Open problems», in *Proceedings of the Symposium on Analytic Number Theory*, London: Imperial College, 1985, Edited by Chen, W. W. L.
- [51] — , «On  $abc$  and discriminants», *Proc. Amer. Math. Soc.* **130** (2002), no. 11, p. 3141–3150 (electronic).
- [52] R. MAULDIN – «A generalization of Fermat’s Last Theorem: The Beal conjecture and prize problem.», *Notices Am. Math. Soc.* **44** (1997), no. 11, p. 1436–1437.
- [53] R. MURTY & S. WONG – «The  $ABC$  conjecture and prime divisors of the Lucas and Lehmer sequences», in *Number theory for the millennium, III (Urbana, IL, 2000)*, A K Peters, Natick, MA, 2002, p. 43–54.
- [54] W. NARKIEWICZ – *Classical problems in number theory*, Monografie Matematyczne [Mathematical Monographs], vol. 62, Państwowe Wydawnictwo Naukowe (PWN), Warsaw, 1986.
- [55] M. B. NATHANSON – *Elementary methods in number theory*, Graduate Texts in Mathematics, vol. 195, Springer-Verlag, New York, 2000.
- [56] A. NITAJ – «The  $abc$  conjecture home page», <http://www.math.unicaen.fr/~nitaj/abc.html>.
- [57] — , «La conjecture  $abc$ », *Enseign. Math. (2)* **42** (1996), no. 1-2, p. 3–24.
- [58] J. OESTERLÉ – «Nouvelles approches du “théorème” de Fermat», *Astérisque* (1988), no. 161-162, p. Exp. No. 694, 4, 165–186 (1989), Séminaire Bourbaki, Vol. 1987/88.

- [59] P. PHILIPPON – «Quelques remarques sur des questions d’approximation diophantienne», *Bull. Austral. Math. Soc.* **59** (1999), no. 2, p. 323–334, Addendum, id., vol. **61** (2000), no. 1, p. 167–169.
- [60] S. S. PILLAI – *Collected works of S. Sivasankaranarayana Pillai*, Ramanujan Mathematical Society Collected Works Series, vol. 1, Ramanujan Mathematical Society, Mysore, 2010, Edited by R. Balasubramanian and R. Thangadurai.
- [61] P. RIBENBOIM – *13 lectures on Fermat’s last theorem*, Springer-Verlag, New York, 1979.
- [62] — , *Catalan’s conjecture, are 8 and 9 the only consecutive powers?*, Academic Press Inc., Boston, MA, 1994.
- [63] O. ROBERT, C. L. STEWART & G. TENENBAUM – «A refinement of the *abc* conjecture», 2013, to appear.  
[http://www.iecn.u-nancy.fr/~tenenb/PUBLIC/Prepublications\\_et\\_publications/](http://www.iecn.u-nancy.fr/~tenenb/PUBLIC/Prepublications_et_publications/).
- [64] O. ROBERT & G. TENENBAUM – «Sur la répartition du noyau d’un entier», 2013, *Indag. Math.*, to appear.  
[http://www.iecn.u-nancy.fr/~tenenb/PUBLIC/Prepublications\\_et\\_publications/](http://www.iecn.u-nancy.fr/~tenenb/PUBLIC/Prepublications_et_publications/).
- [65] N. SARADHA – «Application of the explicit *abc*-conjecture to two Diophantine equations», *Acta Arith.* **151** (2012), no. 4, p. 401–419.
- [66] W. M. SCHMIDT – *Diophantine approximations and Diophantine equations*, Lecture Notes in Mathematics, vol. 1467, Springer-Verlag, Berlin, 1991.
- [67] T. N. SHOREY – «Exponential Diophantine equations involving products of consecutive integers and related equations», in *Number theory*, Trends Math., Birkhäuser, Basel, 2000, p. 463–495.
- [68] — , «An equation of Goormaghtigh and Diophantine approximations», in *Currents trends in number theory (Allahabad, 2000)*, Hindustan Book Agency, New Delhi, 2002, p. 185–197.
- [69] J. H. SILVERMAN – «Wieferich’s criterion and the *abc*-conjecture», *J. Number Theory* **30** (1988), no. 2, p. 226–237.

- [70] C. L. STEWART – «On divisors of Lucas and Lehmer numbers», *Acta Mathematica* (2013), to appear.
- [71] C. L. STEWART & R. TIJDEMAN – «On the Oesterlé-Masser conjecture», *Monatsh. Math.* **102** (1986), no. 3, p. 251–257.
- [72] C. L. STEWART & K. YU – «On the *abc* conjecture», *Math. Ann.* **291** (1991), no. 2, p. 225–230.
- [73] — , «On the *abc* conjecture. II», *Duke Math. J.* **108** (2001), no. 1, p. 169–181.
- [74] W. W. STOTHERS – «Polynomial identities and Hauptmoduln», *Quart. J. Math. Oxford Ser. (2)* **32** (1981), no. 127, p. 349–370.
- [75] A. SURROCA – «Siegel’s theorem and the *abc* conjecture», *Riv. Mat. Univ. Parma (7)* **3\*** (2004), p. 323–332.
- [76] — , «Sur l’effectivité du théorème de Siegel et la conjecture *abc*», *J. Number Theory* **124** (2007), no. 2, p. 267–290.
- [77] R. TIJDEMAN – «Exponential Diophantine equations 1986–1996», in *Number theory (Eger, 1996)*, de Gruyter, Berlin, 1998, p. 523–539.
- [78] P. VOJTA – *Diophantine approximations and value distribution theory*, Lecture Notes in Mathematics, vol. 1239, Springer-Verlag, Berlin, 1987.
- [79] — , «A more general *abc* conjecture», *Internat. Math. Res. Notices* (1998), no. 21, p. 1103–1116.
- [80] — , «On the *ABC* conjecture and Diophantine approximation by rational points», *Amer. J. Math.* **122** (2000), no. 4, p. 843–872, Correction: *Amer. J. Math.* **123** (2001), no. 2, 383–384.
- [81] T. D. WOOLEY – «Waring’s problem, the declining exchange rate between small powers, and the story of 13,792», University of Bristol, 19/11/2007  
<http://www.maths.bris.ac.uk/~matdw/bristol%20b16.pdf>.

### Further resources:



- Additional information about the *abc* conjecture is available at  
<http://www.astro.virginia.edu/~eww6n/math/abcConjecture.html>.
- ABC@Home, a project led by Hendrik W. Lenstra Jr., B. de Smit and W. J. Palenstijn  
<http://www.abcahome.com/>
- Ivars Peterson. — The Amazing ABC Conjecture.  
[http://www.sciencenews.org/sn\\_arc97/12\\_6\\_97/mathland.htm](http://www.sciencenews.org/sn_arc97/12_6_97/mathland.htm)
- Pierre Colmez. —  $a + b = c$ ? Images des Mathématiques, CNRS, 2012.  
<http://images.math.cnrs.fr/a-b-c.html>
- Bart de Smit / ABC triples.  
<http://www.math.leidenuniv.nl/~desmit/abc/>
- Reken mee met *abc*  
[http://rekenmeemetabc.nl/Synthese\\_resultaten](http://rekenmeemetabc.nl/Synthese_resultaten)

*Reken mee met abc is een project dat gericht is op scholieren en andere belangstellenden. Op deze website vind je allerlei interessante artikelen, wedstrijden en informatie voor een praktische opdracht of profielwerkstuk voor het vak wiskunde. Daarnaast kun je je computer laten meerekenen aan een groot rekenproject gebaseerd op een algoritme om abc-drietallen te vinden.*

*Reken mee met abc is a project aimed at students and other interested parties. On this website you can find all sorts of interesting articles, contests and information for a practical assignment or workpiece profile for mathematics. In addition, you can take your computer to a large project based on an algorithm to abc-triples.*

Michel WALDSCHMIDT  
 Université Pierre et Marie Curie-Paris 6  
 Institut de Mathématiques de Jussieu IMJ UMR 7586  
 Théorie des Nombres Case Courrier 247  
 4 Place Jussieu  
 F-75252 Paris Cedex 05 France  
 e-mail: [miw@math.jussieu.fr](mailto:miw@math.jussieu.fr)  
 URL: <http://www.math.jussieu.fr/~miw/>