1. Numerical Model

We begin with the primitive hydrodynamic equations of motion in two dimensions, assuming hydrostatic equilibrium:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\mathbf{v} \cdot \nabla \rho - \rho \nabla \cdot \mathbf{v} && (1)\\
\frac{\partial \mathbf{v}}{\partial t} &= -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{\rho} \nabla p && (2)\\
\frac{\partial}{\partial t}\left(\frac{e}{\rho}\right) &= -\mathbf{v} \cdot \nabla \left(\frac{e}{\rho}\right) - \frac{p}{\rho} \nabla \cdot \mathbf{v}. && (3)
\end{aligned}
$$

We use the equation of state for an ideal gas: $p = (\gamma - 1)e = \rho C_V T$, where R is the *specific* gas constant and the adiabatic constant $\gamma = 1.4$ for a diatomic gas. (We actually use a value of $\gamma = 1.389$, appropriate for molecular hydrogen at 900 K.) Setting the specific heat capacity to $C_V = R/(\gamma - 1)$, we can substitute

$$T = \frac{1}{C_V} \frac{e}{\rho} \qquad (4)$$

into equation (3) to obtain the hydrodynamic equations in terms of the dynamical variables $T, \mathbf{v}$, and ρ:

$$
\begin{aligned}
\frac{\partial T}{\partial t} &= -\mathbf{v} \cdot \nabla T - (\gamma - 1) T \nabla \cdot \mathbf{v} && (5)\\
\frac{\partial \mathbf{v}}{\partial t} &= -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{RT}{\rho} \nabla \rho - R \nabla T && (6)\\
\frac{\partial \rho}{\partial t} &= -\mathbf{v} \cdot \nabla \rho - \rho \nabla \cdot \mathbf{v}. && (7)
\end{aligned}
$$

In order to obtain a model appropriate to planetary atmospheric flows, we add a thermal forcing term to (5) and a Coriolis term to (6), obtaining the equations of hydrodynamic motion on an irradiated, rotating sphere:

$$
\begin{aligned}
\frac{\partial T}{\partial t} &= -\mathbf{v} \cdot \nabla T - (\gamma - 1) T \nabla \cdot \mathbf{v} + f_{rad} && (8)\\
\frac{\partial \mathbf{v}}{\partial t} &= -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{RT}{\rho} \nabla \rho - R \nabla T - 2\Omega_{rot} \sin\theta (\hat{n} \times \mathbf{v}) && (9)\\
\frac{\partial \rho}{\partial t} &= -\mathbf{v} \cdot \nabla \rho - \rho \nabla \cdot \mathbf{v}, && (10)
\end{aligned}
$$

where f_{rad} gives the thermal forcing, Ω_{rot} is the angular frequency of the planet's rotation, θ is the latitude, and $\hat{n}$ is a unit vector normal to the planet's surface.

We calculate f_{rad} using a one-layer, two-frequency radiative transfer scheme. We take the mean opacity for incident stellar radiation to be k_1, while the mean opacity for outgoing long-wave radiation is k_2. In this greatly simplified model, a thin layer of the atmosphere with thickness dz and at a pressure depth p will absorb a flux equal to

$$dF = \left(\frac{k_1 F_*(p)}{\cos\alpha} + k_2 \sigma T_{night}^4 - k_2 \sigma T^4 \right) \rho \; dz, \tag{11}$$

where $F_*(p)$ is the incident stellar flux at a pressure depth p, α is the zenith angle of the star, T_{night} is the temperature of the layer being modeled in the absence of solar heating., and $\sigma = 5.67 \cdot 10^{-8}$ W m^{-2} K^{-4} is the Stefan-Boltzman constant. T_{night} is the result of the combination of the intrinsic planetary flux due to tidal heating and the heating supplied by the atmosphere at high optical depth, and is calculated using a one-dimensional radiative transfer code. Energy conservation requires that $dF = \rho C_p f_{rad} dz$, so the thermal forcing must be

$$f_{rad} = \left(\frac{1}{C_p} \right) \left(\frac{k_1 F_*(p)}{\cos\alpha} + k_2 \sigma T_{night}^4 - k_2 \sigma T^4 \right). \tag{12}$$

We may approximate $F_*(p)$ by ignoring the variation in k_1 with temperature and pressure; this gives

$$F_*(p) = F_*(0) e^{-\frac{k_1 p}{g \cos\alpha}}. \tag{13}$$

But $F_*(0)$ is simply the flux at the top of the atmosphere, which is given by

$$F_*(0) = (1 - A) \left(\frac{L_*}{4\pi a^2} \right) \cos\alpha, \tag{14}$$

where A is the Bond Albedo, L_* is the stellar luminosity, and a is the distance between the star and the planet. Then the thermal forcing is

$$f_{rad} = \left(\frac{1}{C_p} \right) \left(k_1 (1 - A) \left(\frac{L_*}{4\pi a^2} \right) e^{-\frac{k_1 p}{g \cos\alpha}} + k_2 \sigma T_{night}^4 - k_2 \sigma T^4 \right). \tag{15}$$

With a little bit of algebraic manipulation, this can be placed in a more transparent form:

$$f_{rad} = \beta \left(T_{eq}^4 - T^4 \right), \tag{16}$$

where $\beta = \sigma k_2 / C_p$ and the equilibrium temperature T_{eq} follows

$$T_{eq}^4 = \left(\frac{k_1}{k_2} \right) T_{ss}^4 x^{\sec\alpha} + T_{night}^4, \tag{17}$$

with $\sigma T_{ss}^4 = (1 - A) L_* / (4\pi a^2)$ and $x = \exp(-k_1 p / g)$. The forcing that results from this scheme strongly depends on the choice of k_1, k_2, and p. Motivated by many-layer radiative models of HD 209458 b, we choose $k_1 = 2 \cdot 10^{-4}$ m^2 kg^{-1}, $k_2 = 4 \cdot 10^{-4}$ m^2 kg^{-1}, and $p = 250(g/10 \text{ m s}^{-2})$ mbar, where g is the acceleration due to gravity at the planet's surface.

While we believe this model represents an improvement over earlier work, it is nevertheless unable to account for a number of possibly important effects. Two-dimensional hydrodynamics obviously cannot account for vertical atmospheric flows. Since the infrared photosphere is well inside the radiative zone for the planets under consideration, it is likely that convective effects are negligible. However, the inability of the model to account for vertical expansion under heating could cause some error.

In our radiative model, we ignore the variation of opacities with temperature and pressure, which is known to be significant. We furthermore do not consider the variation in opacity with wavelength within the infrared regime; this implies that, for example, the 24 micron photosphere and the 3 micron photosphere occur at the same pressure level. Both of these simplifications are likely to cause discrepancy between our predictions and observations.

Despite these shortcomings, this model represents a significant improvement over earlier simulations in a number of areas. Shallow water models fail to model accurately the behavior of gaseous flows on even a qualitative level; temperature waves redistribute heat far more efficiently than is realistic. Since our code employs fully compressible hydrodynamics, we are able to simulate heat transfer within atmospheric flows with far greater realism than is possible using incompressible shallow-water dynamics.

Furthermore, our radiative scheme provides a more realistic model of the thermal forcing than the Newtonian relaxation employed in earlier hydrodynamic simulations. While Newtonian relaxation is a reasonable approximation when the temperature is close to the equilibrium temperature, it becomes dramatically less accurate as the temperature perturbations grow larger. In the forcing regimes expected on extrasolar planets, the Newtonian approximation can overestimate the actual rate of temperature change by as much as a factor of 3. Our radiative model therefore allows us to treat the stellar heating more accurately than previous simulations; this is particularly important in cases where the thermal forcing varies significantly with time, either due to non-synchronous rotation or to a highly eccentric orbit.