# Nonlinear Methods in the Study of Singular Partial Differential Equations

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This thesis is presented in fulfilment of
the requirements for the degree of
*Doctor of Philosophy*

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Faculty of Science, Engineering and Technology
Victoria University of Technology

December 2004

to my parents with love and appreciation

# Abstract

Nonlinear singular partial differential equations arise naturally when studying models from such areas as Riemannian geometry, applied probability, mathematical physics and biology.

The purpose of this thesis is to develop analytical methods to investigate a large class of nonlinear elliptic PDEs underlying models from physical and biological sciences. These methods advance the knowledge of qualitative properties of the solutions to equations of the form $\Delta u = f(x, u)$ where $\Omega$ is a smooth domain in $\mathbb{R}^N$ (bounded or possibly unbounded) with compact (possibly empty) boundary $\partial\Omega$. A non-negative solution of the above equation subject to the singular boundary condition $u(x) \to \infty$ as $\text{dist}(x, \partial\Omega) \to 0$ (if $\Omega \not\equiv \mathbb{R}^N$), or $u(x) \to \infty$ as $|x| \to \infty$ (if $\Omega = \mathbb{R}^N$) is called a blow-up or *large solution*; in the latter case the solution is called an *entire large* solution.

Issues such as existence, uniqueness and asymptotic behavior of blow-up solutions are the main questions addressed and resolved in this dissertation. The study of similar equations with homogeneous Dirichlet boundary conditions, along with that of ODEs, supplies basic tools for the theory of blow-up. The treatment is based on devices used in Nonlinear Analysis such as the maximum principle and the method of sub and super-solutions, which is one of the main tools for finding solutions to boundary value problems. The existence of blow-up solutions is examined not only for semilinear elliptic equations, but also for systems of elliptic equations in $\mathbb{R}^N$ and for singular mixed boundary value problems. Such a study is motivated by applications in various fields and stimulated by very recent trends in research at the international level.

The influence of the nonlinear term $f(x, u)$ on the uniqueness and asymptotics of the blow-up solution is very delicate and still eludes researchers, despite a very extensive literature on the subject. This challenge is met in a general setting capable of modelling competition near the boundary (that is, $0 \cdot \infty$ near $\partial\Omega$), which is very suitable to applications in population dynamics. As a special feature, we develop innovative methods linking, for the first time, the topic of blow-up in PDEs with regular variation theory (or Karamata's theory) arising in applied probability. This interplay between PDEs and probability theory plays a crucial role in proving the uniqueness of the blow-up solution in a setting that removes previous restrictions imposed in the literature. Moreover, we unveil the intricate pattern of the blow-up solution near the boundary by establishing the *two-term* asymptotic expansion of the solution and its variation speed (in terms of Karamata's theory).

The study of singular phenomena is significant because computer modelling is usually inefficient in the presence of singularities or fast oscillation of functions. Using the asymptotic methods developed by this thesis one can find the appropriate functions modelling the singular phenomenon. The research outcomes prove to be of significance through their potential applications in population dynamics, Riemannian geometry and mathematical physics.

# Declaration

I, Florica-Corina Cîrstea, declare that the PhD thesis entitled "Nonlinear Methods in the Study of Singular Partial Differential Equations" is no more than 100,000 words in length, exclusive of tables, appendices, references and footnotes. This thesis contains no material that has been submitted previously, in whole or in part, for the award of any other academic degree or diploma. Except where otherwise indicated, this thesis is my own work.

I also hereby declare that this thesis is written in accordance with the University's Policy with respect to the Use of Project Reports and Higher Degree Theses.

Signature Date

# Acknowledgements

The research of this thesis has been undertaken with an International Postgraduate Research Scholarship awarded by Victoria University (VU) with the support of the Australian Government through DETYA. I gratefully acknowledge the support of these institutions. This dissertation would not have been possible without their funding.

My warmest thanks go to my supervisors for their dedication, support and wise advice: principal supervisor, Professor Sever S. Dragomir, for his wonderful guidance, insightful comments and commitment to a high standard; Associate Professor Neil Barnett for his support and input I hold valuable; and Associate Professor Pietro Cerone for his helpful insights, continual interest and dedication. To Pietro, as a Head of School, who succeeded in creating a great atmosphere that stimulated me to greater accomplishment, I am so grateful.

A special thank you to my collaborators (in alphabetical order) for interesting discussions and suggestions: Neil Barnett (VU), Sever Dragomir (VU), Yihong Du (University of New England), Constantin Niculescu and Vicenţiu Rădulescu (University of Craiova).

My great appreciation and enormous thanks go to the Australian research community for sustained encouragement and support, particularly: Professor Neil S. Trudinger for his warm invitation, hospitality and support to visit the Centre for Mathematics and its Applications at the Australian National University and give two presentations in April 2003 and January 2004; Professor Norman Dancer for his kind support and invitation to give a talk at the "Mini-conference on Mathematical Problems arising in Mathematical Biology" organized at the University of Sydney in July 2003; Associate Professor Yihong Du

for his cordial support and hospitality during my invited visits to the University of New England in December 2003 and November 2004. His interesting comments and suggestions on a draft of my thesis are much appreciated.

I gratefully acknowledge that an interesting research question formulated by Prof. Dancer (*vis-à-vis* my talk at the Trilateral Workshop organized by Prof. Trudinger in February 2003) stimulated me to write §5.3 of Chapter 5.

To the Conference Committee for the School of Computer Science and Mathematics, who provided funding of travel to conferences, I sincerely appreciate your support.

To the Postgraduate Research Unit for organizing high-quality workshops that contributed to my personal and professional development.

To the Scholarship Coordinator, Lesley Birch, who has promptly helped me to resolve many questions regarding the IPRS Scholarship, I deeply value your dedication.

To Dr. Nicoleta Dragomir, whose friendship over the years of my PhD has provided constant emotional strength and challenge to strive to achieve, I cherish our relationship.

To Shirley Herrewyn and Janet Grady, whose assistance with various administrative tasks has given me the peace of mind to create, thank you for your commitment.

To Ponnusamy Rajendran, Cameron Giles, Mark Mojic and Dahn Ho, whose support to obtain and maintain whatever electronic equipment or computer software I needed has helped me bring this thesis into existence.

To Ewa Sztendur, George Hanna, Mladen Georgievski, Leon Gor, and Gabriele Sorrentino, my dear friends, whose lively conversations have been the source of great joy, I deeply value your friendship.

To my teachers throughout my educational years, who have shaped much of my skills and strategies, thank you all for passing on your knowledge and passion.

Most of all, to my parents (Gabi and Ştefan Cîrstea) and my brother (Marian), whose unconditional love is invaluable to me. I deeply admire your courage and strength. Thank you for your encouragement to pursue my dreams even when this meant to be so far from you.

# Papers published during the author's candidature

From the material in this thesis there are, at the time of submission, a number of papers that have been published in refereed publications.

• The second part of Chapter 2 (namely, §2.2) has been published in *J. Math. Pures Appl.*, **81** (2002), 827-846. V. Rădulescu is a co-author. The results rely on a previous paper (with the same co-author) published in *Nonlinear Anal., T.M.A.*, **48** (2002), 521-534. For completeness, the findings of the latter article are presented in the first part of Chapter 2.

• The work from Chapter 3, sections 3.1 and 3.2, has been published in *Commun. Contemp. Math.*, **4** (2002), 559-586 and *Houston J. Math.*, **29** (2003), 821-829. V. Rădulescu is a co-author.

• A part of Chapter 4 is based on results published in *Proceedings of the Fourth International Conference on Modelling and Simulation*, 2002, Victoria University of Technology, Melbourne, pp. 364–368 (no co-authors).

• Other parts of Chapter 4 have been published in *C. R. Math. Acad. Sci. Paris*, volumes **335** (2002), 447-452 and **336** (2003), 231-236, jointly with V. Rădulescu.

• The work from §3.3 of Chapter 3 and §4.3 of Chapter 4 has been published in *C. R. Math. Acad. Sci. Paris*, **339** (2004), 119-124, co-authored with V. Rădulescu.

• A part of the work presented in Chapter 5 has been published in *C. R. Math. Acad. Sci. Paris*, **339** (2004), 689–694 (no co-authors).

◊ A paper based on the results from §5.2 of Chapter 5 has been accepted for publication in *Proceedings of the London Mathematical Society*. Y. Du is a co-author.

◊ Some work from Chapter 4 (co-authored with V. Rădulescu) has been accepted for publication in *Asymptotic Analysis*.

◊ A manuscript encompassing some results in §5.3 of Chapter 5 and treating also a complementary case will be submitted for publication in the early half of 2005 (no co-authors).

In addition, other papers published or submitted for publication during the author's candidature are listed below.

• Barnett, N. S., F.-C. Cîrstea and S. S. Dragomir, Some inequalities for the integral mean of Hölder continuous functions defined on disks in a plane. *Inequality Theory and Applications, Vol. 2 (Chinju/Masan, 2001)*, 7–18, *Nova Sci. Publ., Hauppauge, NY*, 2003.

• Cîrstea, F.-C. and C. Niculescu, Existence and non-existence results for some degenerate quasilinear problems with indefinite nonlinearities. *Differential Equations and Applications*, 63–80, *Nova Sci. Publ., Hauppauge, NY*, 2004.

• Cîrstea, F.-C., M. Ghergu and V. Rădulescu, Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane-Emden-Fowler type, *J. Math. Pures Appl.*, to appear.

• Cîrstea, F.-C. and V. Rădulescu, Boundary blow-up in nonlinear elliptic equations of Bieberbach–Rademacher type, submitted for publication to *Transactions of the American Mathematical Society*.

• Cîrstea, F.-C. and S. S. Dragomir, Representation of multivariate functions via the potential theory, to be submitted.

**Note.** My research to date has been aligned to the very prestigious and internationally recognized school of Prof. H. Brezis. The research questions investigated in Chapters 3 and 4, in particular the uniqueness issue in the competing case, have been formulated in 2001 by Prof. Brezis to Prof. V. Rădulescu.

Though other research topics have been pursued under the principal supervision of Prof. Dragomir, as shown above, I became more and more involved in the topic of blow-up solutions. This is because I came up with an original and highly innovative approach that uses regular variation theory from applied probability to resolve the challenging question of uniqueness and asymptotic behavior.

Thus, on my own initiative, I studied thoroughly the book of Seneta (1976), as well as the relevant parts from Resnick (1987) and Bingham et al. (1987). Since the research questions on blow-up solutions have been initially brought to me by Prof. Rădulescu, I sent him for a while the corresponding drafts of my research progress and received feedback on them via email.

I would like to point out that the results are my own contribution towards obtaining my PhD. The final version of the published papers may slightly differ from my original drafts in the form of presentation and bibliography section. I consider my overall contribution to the published papers to be no less than 75%.

# Presentations given during the author's candidature

During my candidature I have given a number of presentations to prestigious Conferences and Workshops, as well as to Seminars at the Australian National University, University of Sydney and the University of New England. On these occasions I have presented my research findings enclosed in this dissertation as follows:

• **International Conference on Mathematical Inequalities and their Applications, I**, 6–8 December 2004, Victoria University.

"Representation of multivariate functions via the potential theory"

• **Invited Lecture in the Seminar Series** of the *School of Mathematics, Statistics and Computer Sciences*, University of New England, Armidale, 17 November 2004.

"Existence of singular solutions for elliptic equations with mixed boundary conditions" (results from Chapter 3)

• **The 48th Annual Meeting of the Australian Mathematical Society**, RMIT University, Melbourne, Australia, September 28 - October 1, 2004; `http://www.ma.rmit.edu.au/austms04/`

"Singular solutions for nonlinear elliptic equations with mixed boundary conditions" (results included in Chapter 5)

• **Mathematical Sciences Institute (MSI) Colloquium**, organized by Dr. James McCoy, Australian National University, Canberra, 29 January 2004.

"Nonlinear elliptic equations with singular boundary conditions" (results from Chapter 4)

• *AMSI National Research Symposium* **Non-linear Partial Differential Equations and their Applications**, organized by Dr. Yihong Du, Min-Chun Hong and Chris Radford, University of New England, Armidale, 8-12 December 2003.

"On the positive solutions for some semilinear elliptic equations/systems with singular boundary conditions " (results from Chapter 2)

• **Post ICIAM Mini-conference on Mathematical Problems Arising in Mathematical Biology**, organized by *Professor Norman Dancer*, University of Sydney, 15 July 2003.

"Boundary blow-up solutions of nonlinear elliptic equations" (results from Chapters 3 and 4)

• **Invited talk in the Seminar Series on PDE's**, organized by *Professor Neil Trudinger*, Australian National University, Canberra, 7 April 2003.

"Asymptotics for the blow-up boundary solution of the logistic equation with absorption" (results included in Chapter 4)

• **Trilateral Workshop on Analysis and Applications**, organized by *Professor Neil Trudinger*, 3-7 February 2003, Murramarang Resort, South Durras, NSW, Australia.

"Uniqueness and asymptotics for the blow-up boundary solution of the logistic equation with absorption" (results from Chapters 3 and 4); `http://wwwmaths.anu.edu.au/events/trilateral/`

• **The 4th International Conference on Modelling and Simulation**, 11-13 November 2002, Victoria University, Melbourne.

"On the uniqueness of solutions with boundary blow-up for a class of logistic equations" (results included in Chapter 4)

• **The 38th Applied Mathematics Conference**, ANZIAM 2002, Canberra, Australia, 2-6 February 2002.

"Blow-up boundary solutions of semilinear elliptic problems" (results included in §2.1 of Chapter 2)

• **Seminar to the RGMIA**, Victoria University, Melbourne.

◇ 17 April 2002: "Solutions with boundary blow-up for logistic equations" (results from Chapter 3)

◇ 19 November 2001: "Existence and uniqueness of positive solutions to a semilinear elliptic problem in $\mathbf{R}^N$"

◇ 13 August 2001: "Blow-up boundary solutions of semilinear elliptic problems" (results from Chapter 2)

# Contents

# Notation

$\mathbb{R}^N$: Euclidian $N$-space, $N \geq 2$, with points $x = (x_1, \ldots, x_N)$, $x_i \in \mathbb{R}$ (real numbers); $|x| = \left(\sum_{i=1}^N x_i^2\right)^{1/2}$.

$\partial\Omega$: boundary of $\Omega$; $\overline{\Omega}$ = closure of $\Omega = \Omega \cup \partial\Omega$.

$\omega \subset\subset \Omega$: $\omega$ is an open set *strongly included* in $\Omega$, i.e., $\overline{\omega}$ is compact and $\overline{\omega} \subset \Omega$.

$\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$=gradient of $u$.

$\beta = (\beta_1, \ldots, \beta_N)$, $\beta_i$ = integer $\geq 0$; we define

$$D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \cdots \partial x_N^{\beta_N}}, \quad \text{where } |\beta| = \sum_{i=1}^N \beta_i \text{ is a multi-index.}$$

$\Delta u = \sum_{i=1}^N \partial^2 u/\partial x_i^2$= Laplacian of $u$.

$C(\Omega)$ ($C(\overline{\Omega})$): the set of continuous functions on $\Omega$ ($\overline{\Omega}$).

$C^k(\Omega)$: the set of functions having all derivatives of order $\leq k$ continuous in $\Omega$ ($k$ = integer $\geq 0$); $C^\infty(\Omega) = \cap_{k \geq 0} C^k(\Omega)$.

$C^k(\overline{\Omega})$: the set of functions in $C^k(\Omega)$ all of whose derivatives of order $\leq k$ have continuous extensions to $\overline{\Omega}$; $C^\infty(\overline{\Omega}) = \cap_{k \geq 0} C^k(\overline{\Omega})$.

$\operatorname{supp} u$: the support of $u$, the closure of the set on which $u \neq 0$.

$C_c(\Omega)$: the set of continuous functions with compact support in $\Omega$.

$C_c^k(\Omega) = C^k(\Omega) \cap C_c(\Omega)$, where $k$ = integer or $k = \infty$.

$C^{0,\mu}(\overline{\Omega}) = \left\{ u \in C(\Omega) : \sup_{x,y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\mu} \right\}$ with $0 < \mu < 1$.

$C^{k,\mu}(\overline{\Omega}) = \{u \in C^k(\Omega) : \ D^j u \in C^{0,\mu}(\overline{\Omega}) \ \forall j, \ |j| \leq k\}$.

$L^p(\Omega) = \left\{ u \text{ is measurable on } \Omega : \int_\Omega |u(x)|^p \, dx < \infty \right\}$, $1 \leq p < \infty$.

$L^\infty(\Omega) = \{u \text{ is measurable on } \Omega : \ |u(x)| \leq C \text{ a.e. in } \Omega, \text{ for some } C > 0\}$.

$W^{1,p}$, $W_0^{1,p}$, $H^1 = W^{1,2}$, $H_0^1 = W_0^{1,2}$, $H^m = W^{m,2}$ Sobolev spaces.

# Chapter 1

# Introduction

**"Nothing happens unless first a dream." (Carl Sandburg)**

## 1.1 Background

A great number of processes in the applied sciences are modelled by nonlinear partial differential equations (PDEs). Many interesting phenomena arise due to the nonlinearity of the problems. The desire to describe and predict such phenomena continues to produce a profound impact on research objectives in both pure and applied mathematics. This has strongly influenced the development of the modern theory of partial differential equations (Gilbarg and Trudinger (1983)), as well as that of the calculus of variations, nonlinear functional analysis (Brezis (1983)) and numerical analysis (see for example Brezis and Browder (1998)).

A very remarkable property of nonlinear problems is the possibility of the eventual occurrence of singularities. For instance, they can arise through the boundary conditions (blow-up on the boundary) or through the singularities contained in the coefficients/nonlinearities of the problem. A broad spectrum of nonlinear problems underlying models from the physical and biological sciences leads to the study of nonlinear PDEs with singular boundary conditions.

The study of singular PDEs, and specifically boundary blow-up problems, has attracted considerable attention starting with the pioneering work of Bieberbach (1916). This interest has been reignited in the last decades from the need to give rigorous answers to important questions of the nonlinear world.

Several models of stochastic control problems involving constraints on the state of the system can be converted to nonlinear second-order elliptic equations with singular boundary conditions via a dynamic programming approach (Lasry and Lions (1989)). Singular boundary conditions may be encountered and be of fundamental use to more general quasilinear elliptic equations such as: the Hamilton-Jacobi-Bellman equations, first-order Hamilton-Jacobi equations, and Monge-Ampère equations (see, e.g., Brezis (1984), Castillo and Albornoz (2003), Crandall and Lions (1987), Trudinger (1986), Urbas (1998, 1999)).

Existence, uniqueness and rate explosion on the boundary for a class of quasilinear elliptic equations are given by Bandle and Giarrusso (1996), Diaz et al. (1996). Gradient bounds and existence were obtained in Lasry and Lions (1989). Local gradient estimates are also provided by Gilbarg and Trudinger (1983), Lasry and Lions (1989), Castillo and Albornoz (2003).

There have been a number of studies that give a rigorous mathematical treatment of the dynamics of some population models (Du and Huang (1999)) including the predator-prey model (Dancer and Du (2002)) and the competition model (Du (2002$a$,$b$)). These works have argued that the qualitative properties of the solutions for singular elliptic PDEs play a crucial role in the understanding of the dynamic behavior for various population models (see also Du (2003)).

In this dissertation, the main concern lies in the theory of blow-up for various classes of semilinear elliptic equations which allow for competition near the boundary. Innovative methods are advanced to settle very challenging questions that have not yet been answered despite vigorous investigation. These methods, which have not been previously exploited in the literature, will contribute to the understanding of the blow-up phenomenon in a general setting. The objectives are principally determined by the latest developments and/or by the theoretical and practical motivation of the equations dealt with.

Singular boundary conditions for semilinear elliptic equations of the form

$$\Delta u = f(x, u) \quad \text{in } \Omega, \tag{1.1}$$

where $\Omega$ is a domain in $\mathbb{R}^N$, arise naturally when studying models from Riemannian geometry (Bieberbach (1916), Loewner and Nirenberg (1974)), mathematical physics (Rademacher (1943)), applied probability (le Gall (1994), Dynkin (1991)), and population dynamics (Du and Huang (1999), García-Melián et al. (2001)). Suitable choices of the nonlinear term meet the need of various applications: the equilibrium of a charged gas in a container (Rademacher (1943)), PDEs invariant under conformal or projective transformations (Loewner and Nirenberg (1974)), or related questions to the classical Thomas-Fermi equations (Robinson (1971)).

A non-negative solution of (1.1), subject to the singular boundary condition $u(x) \to \infty$ as $\mathrm{dist}(x, \partial\Omega) \to 0$ if $\Omega \not\equiv \mathbb{R}^N$ is a bounded/unbounded domain with compact boundary $\partial\Omega$, or $u(x) \to \infty$ as $|x| \to \infty$ if $\Omega = \mathbb{R}^N$, is called a *blow-up* or *large solution*; in the latter case the solution is referred to as an *entire large* solution. We adhere to the definition of a large solution which includes the non-negativity property of the solution as it appears, for instance, in Bandle and Marcus (1992*a*,*b*), Marcus (1992). If the set of positive solutions $P$ of (1.1) is not empty, then a solution $U$ will be called a *maximal solution* if it dominates every function in $P$ (see Bandle and Marcus (1992*b*)).

The most relevant literature regarding the blow-up theory for nonlinear elliptic equations (semilinear, in particular) is reviewed in what follows. When $\Omega$ has a compact boundary, the focus falls mostly on the case that $\partial\Omega$ is smooth (at least $C^2$); it will only be specified if less boundary regularity is involved.

The review refers to the topic of large solutions for scalar equations (§1.2) as well as for systems of equations (§1.3), while §1.2 is organized around the main themes of subsequent research on the subject. One may wish to examine, in as general a framework as possible, three basic questions (McKenna et al. (1997)):

1. Does a large solution exist?

2. Is such a solution unique?

3. How does blow-up occur at the boundary?

Another area of study refers to the existence of multiple blow-up solutions (initiated by McKenna et al. (1997) for the $p$-Laplacian). There is relatively little information in the literature on the multiplicity of blow-up solutions (whose definition, in this context, does not require non-negative solutions). On the grounds that this issue goes beyond the scope of this thesis, the literature related to it is omitted.

## 1.2 Large Solutions for Scalar Equations

### 1.2.1 Existence

The question of existence of large solutions to equations of the form $\Delta u = f(u)$ in a bounded domain $\Omega$ was first addressed by Keller (1957) and Osserman (1957), who arrived at the same conclusion independently. As a main finding, they proved that, subject to some regularity assumptions and restrictions on the growth of $f(u)$, the positive solutions, $u$, are uniformly bounded from above on compact subsets of $\Omega$. When $f$ is locally Lipschitz continuous and non-decreasing on $[0, \infty)$ with $f(0) \geq 0$, then they provide a necessary and sufficient condition for the existence of large solutions, namely

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\, ds. \tag{1.2}$$

Classical examples of nonlinearities satisfying (1.2) are: $f(u) = e^u$ and $f(u) = u^p$ $(p > 1)$. There exists an intensive study of their corresponding blow-up models:

(a) The exponential model

$$\Delta u = e^u \quad \text{in } \Omega \subset \mathbb{R}^N \ (N \geq 2) \tag{1.3}$$

originally analyzed by Bieberbach (1916) $(N = 2)$ and Rademacher (1943) $(N = 3)$, but considered later in papers such as Lazer and McKenna (1993, 1994) and Bandle (2003).

(b) The power model

$$\Delta u = u^p \quad \text{in } \Omega \subset \mathbb{R}^N, \text{ where } p > 1. \tag{1.4}$$

The special case $p = (N+2)/(N-2)$ (where $N > 2$) arising in Riemannian geometry has been studied by Loewner and Nirenberg (1974); their results have been extended by Bandle and Marcus (1992*a*) for *any* $p > 1$.

Matero (1996) extends previous results of Bandle and Essén (1994) on the existence/uniqueness of large solutions for (1.3) and (1.4) to the case of domains with a uniform interior and exterior cone condition. By constructing suitable barriers in a cone, Matero obtains uniform a priori lower and upper bounds for the growth of the solutions and their gradient near $\partial\Omega$. As an application, a boundary blow-up solution in a two-dimensional domain with fractal boundary, called the von Koch snowflake domain, is constructed.

The understanding of models like (1.3) and (1.4) permitted a later development of the blow-up theory for nonlinear elliptic equations (Bandle et al. (1994), Bandle et al. (1997), Bandle and Porru (1994), Du and Huang (1999), Marcus (1992), Marcus and Véron (1997), Ratto et al. (1994)).

Current research focuses on developing new methods to investigate the qualitative properties of the blow-up solutions. The simple replication of previous ideas does not work when dealing with a more general setting.

The qualitative study of the large solutions to equations of the form

$$\Delta u = f(x,u) \quad \text{in } \Omega \subset \mathbb{R}^N, \tag{1.5}$$

brings significant new challenges. This study necessitates a careful analysis of the terms involved in the nonlinearity of $f(x,u)$. The researchers have investigated problems which combine an absorption term $f(u)$ with a weight function $b(x)$.

Cheng and Ni (1992) demonstrated that the equation

$$\Delta u = b(x)u^p \quad \text{in } \Omega \subset \mathbb{R}^N, \quad p > 1, \tag{1.6}$$

considered in a bounded domain $\Omega$, has a large solution assuming that the smooth function $b \geq 0$ is *positive* on $\partial\Omega$. This result was extended by Marcus (1992) to nonlinearities of the form $b(x)f(u)$ with $b$ as before and $f$ satisfying the Keller–Osserman growth condition (1.2). The existence of the maximal solution $U$ of (1.6) in $\mathbb{R}^N$ has been established by Cheng and Ni (1992), provided that at least a positive entire solution (that is, defined in $\mathbb{R}^N$) exists and $\mathbb{R}^N$ may be approximated by an increasing sequence of smooth bounded sub-domains $(\Omega_n)_{n\geq 1}$ such that $b$ is *positive* on $\partial\Omega_n$. Moreover, $U$ is the unique entire large solution if, in addition, for some $l > 2$ there are two positive constants $C_1$, $C_2$ such that

$$C_1 b(x) \leq |x|^{-l} \leq C_2 b(x) \quad \text{for large } |x|. \tag{1.7}$$

A question that received less attention is whether the above results remain valid under a more general condition on $b$ which allows it to vanish on large parts of $\Omega$ as well as on $\partial\Omega$ (resp., at infinity) if the domain is bounded (resp., unbounded). Among the papers studying this question are Bandle and Marcus (1992*b*), Marcus (1992), Lair and Wood (1999).

Bandle and Marcus (1992*b*) treat the equation (1.6) where $b(x)$ may vanish at a *finite* number of points in $\Omega$ and (if $\Omega$ is an outer domain) it may tend to zero at infinity or be unbounded. When $b(x) = |x|^\nu$ the radially symmetric large solutions in a ball or the complement of a ball are analyzed by studying the corresponding system of ordinary differential equations in the phase plane (obtained after an Emden transformation); to this system the standard results concerning perturbed linear systems (see Hartman (1982)) can be applied. The existence (as well as uniqueness and behavior at the boundary/at infinity) of large solutions in general domains is also procured.

One limitation of the methods given by a number of papers is that they apply when $b > 0$ on $\partial\Omega$ or in a neighborhood of infinity (Cheng and Ni (1992), Bandle and Marcus (1992*b*), Marcus (1992)). Although more general nonlinear terms are involved in Bandle and Marcus (1992*a*) and Lazer and McKenna (1994), $b$ is bounded and bounded away from zero. To remove such a restriction on $b$ unfortunately results in mathematical difficulty.

In §2.1 of Chapter 2 we deal with this issue for elliptic equations of the form

$$\Delta u = b(x)f(u) \quad \text{in } \Omega, \tag{1.8}$$

where $b$ is a smooth non-negative function. The existence of large solutions on bounded domains is established under general assumptions on $f$, while $b(x)$ vanishes in $\Omega$ in a certain way namely, which holds if $b > 0$ on $\partial\Omega$. By suitably adjusting the vanishing condition on $b$ when $\Omega$ is an unbounded domain (possibly, $\mathbb{R}^N$) we obtain the existence of a maximal classical solution $U$ of (1.8). Under an additional assumption on $b$, which is weaker than (1.7) and allows $b$ to vanish at infinity, the maximal solution $U$ is found to be a large solution.

The task of permitting $b$ to be zero on some regions of the domain becomes more important and difficult as the accuracy of the mathematical model describing the real phenomenon increases. This objective is pursued, for instance, in the context of mathematical biology (see e.g., López-Gómez (2000), Du and Ma (2002), Du (2003)).

### PDEs arising in Mathematical Biology

A problem of interest in the study of population biology of one species is Fisher's equation, which was also investigated by Kolmogorov et al. (1937). Assuming that

(a) the species disperse randomly in a bounded environment;

(b) the reproduction of the species follows logistic growth;

(c) the boundary of the environment is hostile to the species;

(d) the diffusion does not depend on the space variable,

then the concentration of the species or the population density satisfies a reaction-diffusion equation, called Fisher's equation after Fisher (1937), subject to initial and boundary conditions (Oruganti et al. (2002)). The stationary version of Fisher's equation is referred to as the diffusive logistic equation. Great attention has been given to this equation and to its more general form

$$\begin{cases} \Delta u + au = b(x)u^p & \text{in } \Omega \subseteq \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where $p > 1$ and $a \in \mathbb{R}$ is a parameter. This equation is a basic population model (Hess (1991)). Many studies related to (1.9) have assumed that the smooth function $b$ is positive and bounded away from zero (cf. García-Melián et al. (1998) and López-Gómez (2000)). In this case (1.9), known as the logistic equation, has been proposed as a model for the population density of a steady-state single species when the domain is surrounded by inhospitable areas (Murray (1993)).

It is known that when $b \in C^{0,\mu}(\overline{\Omega})$ is positive, then (1.9) has a unique positive solution if and only if $a > \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

Not until recently has the case been tackled when $b(x)$ vanishes on some sub-domain of $\Omega$ corresponding to the general problem when the species $u$ is free from crowding effects on some sub-domain of $\Omega$ (Brezis and Oswald (1986), Alama and Tarantello (1996), Ambrosetti and Gámez (1997), Dancer (1996), García-Melián et al. (1998), López-Gómez (2000), Dancer et al. (2003)). Equation (1.9) has also been considered in the context of the prescribed curvature problem on compact manifolds (Kazdan and Warner (1975), Ouyang (1992), and del Pino (1994)).

In mathematical biology, when $b$ is positive on a proper sub-domain $\Omega' \subset \overline{\Omega}' \subset \Omega$ and $b = 0$ on $\overline{\Omega} \setminus \Omega'$, $\Omega$ represents the region inhabited by the species $u$, $a$ measures its birth rate, while $b(x)$ denotes the capacity of $\Omega'$ to support the species $u$ (García-Melián et al. (1998)). Since the unknown $u$ corresponds to the density of the population, only positive solutions of this problem are of interest.

Studies such as Brezis and Oswald (1986), Fraile et al. (1996), Ouyang (1992), Dancer (1996) demonstrate that (1.9) admits positive solutions if and only if

$$\lambda_1(\Omega) < a < \lambda_1(\Omega \setminus \overline{\Omega'}),$$

where $\lambda_1(\Omega \setminus \overline{\Omega'})$ stands for the first Dirichlet eigenvalue of $(-\Delta)$ in $\Omega \setminus \overline{\Omega'}$; moreover, for $a$ in the above range, (1.9) admits a unique positive solution $u_a$; $a \mapsto u_a$ is a continuous map from $(\lambda_1(\Omega), \lambda_1(\Omega \setminus \overline{\Omega'}))$ to $C^{2,\mu}(\overline{\Omega})$, and $\|u_a\|_{L^\infty} \to \infty$ as $a \nearrow \lambda_1(\Omega \setminus \overline{\Omega'})$. For the study of related problems in the whole space we refer to Du (2003) and the references therein.

As mentioned in García-Melián et al. (2001), the understanding of the asymptotics for the logistic equation leads naturally to the study of large solutions. The exact pointwise growth of the positive solutions as $a$ approaches the upper bound $\lambda_1(\Omega \setminus \overline{\Omega'})$ is ascertained by García-Melián et al. (1998): the solutions grow to infinity uniformly on compact subsets of $\Omega \setminus \Omega'$ and they stabilize in $\Omega'$ to the *minimal* solution of the boundary blow-up problem

$$\begin{cases} \Delta u + au = b(x)u^p & \text{in } \Omega', \\ u = \infty & \text{on } \partial\Omega', \end{cases} \tag{1.11}$$

being $a = \lambda_1(\Omega \setminus \overline{\Omega'})$ and $b \equiv 0$ on $\partial\Omega'$ in this precise case.

The main feature of (1.11) is that $b$ *vanishes* on the whole boundary of the domain. The appearance of a vanishing weight $b(x)$ *induces a new phenomenon.* The *critical* combination, manifested near the boundary, between the explosive absorption term $u^p$ and vanishing $b(x)$ greatly influences the qualitative properties of the blow-up solutions.

Demonstration of the existence of a minimal/maximal large solution for equations of the type (1.11) is carried out by García-Melián et al. (2001). They also find the existence of (at least) a classical $C^{2,\mu}(\Omega')$-solution blowing-up on $\partial\Omega'$ for a class of perturbed problems, where the perturbation is of a lower order than $u^p$ at infinity.

The understanding of various models in population dynamics (Du and Huang (1999), Dancer and Du (2002), Du and Guo (2003), Du (2003)) is based on the study of nonlinear elliptic problems of a singular nature which may also exhibit the Neumann/Robin boundary condition. The logistic equation in Du and Huang (1999) features an infinite Dirichlet boundary condition on the interior boundary of the domain (where $b$ vanishes), while a Dirichlet or Neumann/Robin boundary condition is assumed on the exterior boundary.

García-Melián et al. (2001) and Du and Huang (1999), while considering the competing case $0 \cdot \infty$ near the boundary where blow-up arises, restrict $b(x)$ to be positive in the domain. The above studies establish the existence of the minimal/maximal blow-up solution for *any* $a \in \mathbb{R}$. In order to see how the situation changes if $b$ is allowed to vanish on a proper subset of the domain and how nonlinearities other than $u^p$ $(p > 1)$ interact with $b(x)$ raises further difficulties.

The source of such questions is the work Alama and Tarantello (1996), which contains an exhaustive study of the positive solutions to the logistic problem

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

for a wide class of functions $f$ (including $f(u) = u^p$ with $p > 1$), where $a$ is a real parameter and the potential $b$ vanishes in $\Omega$.

López-Gómez (2000) gives the existence of regular and large solutions for a class of nonlinear elliptic boundary value problems of logistic-type, where the non-negative function $b$ can vanish on a finite number of smooth interior sub-domains $\Omega_i$, $i \in \{1, \ldots, r\}$. Let $\sigma_1 \le \sigma_2 \le \ldots \le \sigma_r$ denote the principal eigenvalues of the Laplace operator with Dirichlet boundary conditions in $\Omega_i$. López-Gómez (2000) shows that if $\sigma_i \le a < \sigma_{i+1}$, then there exist positive solutions of the elliptic problem (1.12) in $\Omega \setminus \cup_{k=1}^{i} \overline{\Omega_k}$ going to infinity on $\partial(\cup_{k=1}^{i} \overline{\Omega_k})$, while they satisfy a Dirichlet boundary condition on the rest of the boundary.

In Chapter 3 we establish a necessary and sufficient condition for the existence of large solutions of logistic-type equations, where $b \ge 0$ on $\overline{\Omega}$ is zero on a sub-domain whose boundary is not necessarily smooth. We distinguish between the case of a complete blow-up on the boundary (§3.1) and a partial boundary blow-up (e.g., on an interior boundary) coupled with a Dirichlet, Neumann or Robin boundary condition on the rest of the boundary (e.g., on the exterior boundary) (§3.2 and §3.3).

### 1.2.2 Uniqueness

The subject of boundary blow-up originated with the work of Bieberbach (1916) for the semilinear elliptic equation

$$\Delta u = e^u \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{1.13}$$

where $\Omega$ is a smooth bounded domain. Problems of this type arise in Riemannian geometry; if a Riemannian metric of the form $|ds|^2 = e^{2u(x)}|dx|^2$ has constant Gaussian curvature $-c^2$, then $\Delta u = c^2 e^{2u}$. Bieberbach showed that there exists a unique large solution of (1.13) such that

$$u(x) - \log(d(x)^{-2}) \quad \text{is bounded as } d(x) := \text{dist}\,(x, \partial\Omega) \to 0.$$

Motivated by a problem from mathematical physics, Rademacher (1943) continued this study on smooth bounded domains in $\mathbb{R}^3$. Later, Lazer and McKenna (1993) generalized the results of Bieberbach (1916) and Rademacher (1943) for bounded domains in $\mathbb{R}^N$ satisfying a uniform external sphere condition and for nonlinearities of the type $b(x)e^u$, where $b$ is continuous and *positive* on $\overline{\Omega}$.

The issues of uniqueness and asymptotic behavior of the large solutions near the boundary were first linked by Loewner and Nirenberg (1974). Their philosophy of getting the uniqueness by proving that any large solution blows-up on the boundary at the same rate, has been successfully applied for many classes of problems. Initially, this idea has been investigated in connection with problems from Riemannian geometry. In dimension $N \geq 3$, the notion of Gaussian curvature has to be replaced by scalar curvature. If a metric of the form $|ds|^2 = u(x)^{4/(N-2)}|dx|^2$ has constant scalar curvature $-c^2$, then $u$ satisfies

$$\Delta u = \frac{(N-2)c^2}{4(N-1)} u^{\frac{N+2}{N-2}} \quad \text{in } \Omega. \tag{1.14}$$

Loewner and Nirenberg (1974) described the precise asymptotic behavior at the boundary of large solutions to (1.14) and used this result in order to establish the uniqueness of the solution. Their main result is derived under the assumption that $\partial\Omega$ consists of the disjoint union of finitely compact $C^\infty$ manifolds, each having codimension less than $N/2+1$. More precisely, the uniqueness of a large solution is a consequence of the fact that every large solution $u$ satisfies

$$u(x) = \Gamma(d(x)) + o(\Gamma(d(x))) \quad \text{as } d(x) \to 0, \tag{1.15}$$

where $\Gamma$ is defined by

$$\Gamma(t) = \left[\frac{ct}{\sqrt{N(N-1)}}\right]^{-(N-2)/2}, \quad \text{for all } t > 0. \tag{1.16}$$

Kondrat'ev and Nikishkin (1990) found the leading term of the asymptotic expansion near the boundary of a large solution to $Lu = u^p$ $(p > 1)$ in $\Omega$, where $\partial\Omega$ is a $C^2$-manifold and $L$ is a more general second order elliptic operator than $\Delta$. As a corollary, the uniqueness of the large solution is obtained when $p \geq 3$.

Dynkin (1991) showed that there exist certain relations between hitting probabilities for some Markov processes called superdiffusions and maximal solutions of $\Delta u = u^p$, $1 < p \leq 2$. By means of a probabilistic representation, le Gall (1994) proved a uniqueness result in domains with non-smooth boundary when $p = 2$.

The asymptotic behavior of large solutions near the boundary and the uniqueness of such solutions can be obtained by suitable comparison with singular ODEs (Bandle and Marcus (1992*a*), Lazer and McKenna (1994)). The approach in these works applies to equations of the form

$$\Delta u = f(u) \quad \text{in } \Omega \subset \mathbb{R}^N \text{ a bounded domain}, \tag{1.17}$$

for a general class of nonlinearities (including $f(u) = u^p$ for any $p > 1$ and $f(u) = e^u$). Bandle and Marcus (1992*a*) prove that when $f \in C^1[0,\infty)$ is a positive and non-decreasing function on $(0,\infty)$ with $f(0) = 0$ and

$$\exists \mu > 0 \text{ and } s_0 \geq 1 \text{ such that } f(\tau s) \leq \tau^{\mu+1} f(s) \ \forall \tau \in (0,1) \ \forall s \geq s_0/\tau, \tag{1.18}$$

then any large solution of (1.17) has the same blow-up rate near $\partial\Omega$:

$$\lim_{d(x)\to 0} \frac{u(x)}{Z(d(x))} = 1, \tag{1.19}$$

where $Z$ is a chosen solution of the singular ordinary differential equation

$$\begin{cases} Z''(r) = f(Z(r)), \quad r \in (0, \delta), \text{ for some } \delta > 0, \\ Z(r) \to \infty \text{ as } r \to 0^+. \end{cases} \tag{1.20}$$

If, in addition, $f(\tau s) \leq \tau f(s)$, for all $\tau \in (0, 1)$ and $s > 0$, then uniqueness of the large solution occurs. Lazer and McKenna (1994) consider the case when the $C^1$-function $f$ is either defined and positive on $\mathbb{R}$ or is defined on $[a_0, \infty)$ with $f(a_0) = 0$ and $f(s) > 0$ for $s > a_0$. Their analysis is performed on domains satisfying both a uniform internal sphere condition and a uniform external sphere condition with the same constant $R_1 > 0$. They give conditions for which all large solutions of (1.17) fulfill the stronger asymptotic behavior

$$\lim_{d(x)\to 0} [u(x) - Z(d(x))] = 0, \quad \text{for any } Z \text{ satisfying (1.20)}. \tag{1.21}$$

Namely, $f$ is non-decreasing on its domain and $f'$ is non-decreasing on some neighborhood of infinity such that $\lim_{s\to\infty} f'(s)/\sqrt{F(s)} = \infty$, where $F$ is an anti-derivative of $f$. The existence and uniqueness of large solutions is also ensured.

Whilst the papers by Bandle and Marcus (1992$a$), Lazer and McKenna (1994) are important contributions to the understanding of the existence, uniqueness and asymptotic behavior of large solutions to (1.17), their methods have certain limitations. For instance, apart from examples such as $f(u) = u^p$ $(p > 1)$ or $f(u) = e^u$, it is quite involved to compute a solution of the singular ODE equation (1.20) (chosen in Bandle and Marcus (1992$a$) as $\int_{Z(t)}^{\infty} [\sqrt{2F(s)}]^{-1/2}\, ds = t$). Computationally speaking, a much more convenient formula is desirable. On the other hand, when $f(u) = u^p$ $(p > 1)$, the variation speed of the large solution $u$ of (1.17) is dramatically changed from that corresponding to $f(u) = e^u$. This can be seen from the fact that in the former situation,

$$\lim_{d(x)\to 0} \frac{u(x)}{[d(x)]^{2/(1-p)}} = \left[\frac{p-1}{\sqrt{2(p+1)}}\right]^{2/(1-p)}, \tag{1.22}$$

while in the latter case, the blow-up rate of $u$ is much slower, as shown by

$$\lim_{d(x)\to 0} \frac{u(x)}{\ln[d(x)]^{-2}} = 1. \tag{1.23}$$

Thus, a better understanding of the blow-up phenomenon might be possible by treating power-like nonlinearities separately from those of the exponential type. Recent trends in research support this viewpoint in the context of mathematical biology, where $f(u) = u^p$ ($p > 1$) is of special interest.

The focus of much of the recent literature (Du and Huang (1999), García-Melián et al. (2001), Du and Guo (2003), López-Gómez (2003), Du (2004)) is on the qualitative properties of the large solutions to equations such as

$$\Delta u + au = b(x)u^p \quad \text{in } \Omega \subset \mathbb{R}^N \text{ is a bounded domain,} \tag{1.24}$$

where $a \in \mathbb{R}$ is a parameter, $b \in C^{0,\mu}(\overline{\Omega})$, $0 < \mu < 1$, is a non-negative function, as well as to the boundary value problem

$$\begin{cases} \Delta u + au = b(x)u^p & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \Gamma_{\mathcal{B}} := \partial\Omega \setminus \Gamma_\infty, \end{cases} \tag{1.25}$$

where $\Gamma_\infty$ is a non-empty open and closed subset of $\partial\Omega$ ($\Gamma_\infty \neq \partial\Omega$) and $\mathcal{B}$ denotes any of the Dirichlet, Neumann or Robin boundary operators. A non-negative solution of (1.25) satisfying $u(x) \to \infty$ as $x \to \Gamma_\infty$ is called a *large solution* of (1.25).

The above works have the merit of considering the case when the function $b$ in (1.24) and (1.25) is identically zero on the boundary where the blow-up occurs (that is, on $\Gamma_\infty$, where we understand $\Gamma_\infty = \partial\Omega$ for (1.24)). This is a natural restriction inherited from the logistic equation (see (1.11)).

Determining the effect of the competition $0 \cdot \infty$ between $b(x)$ and $u^p$ on the behavior of the large solution near $\Gamma_\infty$ raises new and interesting challenges in the study of nonlinear PDEs. This issue has only partially been resolved at the cost of imposing a certain decay rate on $b(x)$ near $\Gamma_\infty$.

Du and Huang (1999), García-Melián et al. (2001) show that if

$$\lim_{d(x,\Gamma_\infty)\to 0} \frac{b(x)}{d(x,\Gamma_\infty)^\alpha} = \beta, \quad \text{for some constants } \alpha \geq 0 \text{ and } \beta > 0, \tag{1.26}$$

and $b > 0$ in $\overline{\Omega} \setminus \Gamma_\infty$, then (1.24) and (1.25) admit a unique large solution $u_a$, for each $a \in \mathbb{R}$. The advantage of (1.26) is that it helps determine the dominant term in (1.24) near $\Gamma_\infty$. Based on this, suitable upper and lower solutions near the boundary are built. Using a local argument and some comparison criteria it is found that the singular character of any large solution $u$ at $\Gamma_\infty$ is governed by a uniform rate of explosion, namely

$$\lim_{d(x,\Gamma_\infty)\to 0} \frac{u(x)}{[d(x,\Gamma_\infty)]^{(\alpha+2)/(1-p)}} = \left[\frac{(\alpha+2)(\alpha+p+1)}{\beta(p-1)^2}\right]^{1/(p-1)}. \tag{1.27}$$

Further improvements of this result are given by Du and Guo (2003) and López-Gómez (2003). Local blow-up estimates are deduced by López-Gómez (2003) and Du (2004), whose argument relies on the construction of a family of lower/upper solutions on small annuli with partial boundary blow-up. Du (2004) demonstrates that if $\mathbb{B}$ is an open ball in $\mathbb{R}^N$ such that $x_* \in \Gamma_\infty \cap \mathbb{B}$ and

$$\limsup_{x\to x_*, x\in\Omega} \frac{b(x)}{d(x,\Gamma_\infty)^\alpha} \le \beta, \quad \text{for some constants } \alpha \ge 0 \text{ and } \beta > 0, \tag{1.28}$$

then, for any positive solution of (1.24) in $\Omega \cap \mathbb{B}$, subject to $u|_{\Gamma_\infty \cap \mathbb{B}} = \infty$,

$$\liminf_{x\to x_*, x\in C_{x_*,\omega}} \frac{u(x)}{[d(x,\Gamma_\infty)]^{(\alpha+2)/(1-p)}} \ge \left[\frac{(\alpha+2)(\alpha+p+1)}{\beta(p-1)^2}\right]^{1/(p-1)}, \tag{1.29}$$

holds for all $\omega \in (0, \pi/2)$, where $n_{x_*}$ is the outward unit normal of $\partial\Omega$ at $x_*$ and

$$C_{x_*,\omega} = \{x \in \Omega : \quad \text{angle}(x - x_*, -n_{x_*}) \le \pi/2 - \omega\}.$$

When "lim sup" and "≤" in (1.28) are replaced by "lim inf" and "≥", then one must change these accordingly in (1.29). These local estimates, together with an iteration technique due to Safonov (reproduced by Kim (2002)), are utilized by Du (2004) to relax the uniqueness condition (1.26) to

$$\beta_1 d(x,\Gamma_\infty)^\alpha \le b(x) \le \beta_2 d(x,\Gamma_\infty)^\alpha \text{ for } x \in \Omega,\ d(x,\Gamma_\infty) \le \delta, \tag{1.30}$$

where $\delta > 0, \beta_2 \ge \beta_1 > 0$ and $\alpha \ge 0$ are constants.

The above mentioned papers advance knowledge on the uniqueness and asymptotics of the large solution in the particular setting $f(u) = u^p$ $(p > 1)$ and $b(x)$ which is positive in $\overline{\Omega} \setminus \Gamma_\infty$ and satisfies either (1.26) or (1.30). Their methods take full advantage of the interaction between $u^p$ and $b(x)$, which helps to determine the dominant term of (1.24) near $\Gamma_\infty$. It is demanding, however, to find the corresponding dominant term when nonlinearities of $f$ other than the superlinear powers compete with a *non-negative* function $b$ whose behavior near $\Gamma_\infty$ is not necessarily ruled by (1.30). Thus, to discover the qualitative behavior of the large solutions in a more general framework involving competition $0 \cdot \infty$ near the boundary requires new and effective tools.

The techniques introduced in Chapters 4 and 5, which rely crucially on regular variation theory (§4.1) and its extensions (§5.3.2), will be used to answer the proposed challenge. The uniqueness and blow-up rate of the large solution will be uncovered for nonlinearities varying *regularly* (as a power function) (Chapter 4 and §5.2 of Chapter 5) as well as *rapidly* (as an exponential one) (§5.3 of Chapter 5). Note that the decay rate of $b(x)$ at the boundary $\Gamma_\infty$ is not required to satisfy (1.26) (that is, $\beta$ can be zero or infinity, for any $\alpha > 0$). Local blow-up estimates are also provided, which in the particular case of (1.28) refine the findings of Du (2004) and López-Gómez (2003) by dropping the restriction $x \in C_{x_*,\omega}$ in (1.29).

### 1.2.3 Asymptotic Behavior

Motivated by a close relationship to the uniqueness issue, the asymptotic behavior of the large solutions for equations such as $\Delta u = f(u)$ in $\Omega$ has been of interest in a series of papers (for example Bandle and Essén (1994); Bandle and Marcus (1992*a*, 1995, 1998); del Pino and Letelier (2002); Greco and Porru (1997); Lazer and McKenna (1994)).

The first order approximation turns out to be independent of the geometry of the domain $\Omega$, depending only on the distance function to the boundary $d(x) = \text{dist}(x, \partial\Omega)$. The first term in the asymptotic expansion of the large solution near the boundary for the exponential or power model, namely (1.3) or (1.4), is given by (1.23) or (1.22). The second order effects were first addressed by Lazer and McKenna (1994), who found that for power nonlinearities $f(t) = t^p$ with $p > 3$

$$u(x) - [\gamma d(x)]^{2/(1-p)} = 0, \quad \text{where } \gamma = \frac{p-1}{\sqrt{2(p+1)}},$$

while for the exponential function $f(t) = e^t$,

$$\lim_{d(x) \to 0} \left( u(x) - \ln \frac{2}{[d(x)]^2} \right) = 0.$$

Secondary effects in the asymptotic behavior of large solutions were also studied in Bandle and Marcus (1998), Greco and Porru (1997) under various general assumptions on $f$. The role of the boundary curvature has been underscored by Bandle and Marcus (1998) for balls and annuli, under weak assumptions on the nonlinearity $f$. Further extensions of these results to arbitrary bounded smooth domains can be found in Bandle (2003), del Pino and Letelier (2002) for equations with power nonlinearities and in Bandle (2003) for the exponential nonlinearity.

Based on suitable upper and lower solutions constructed in del Pino and Letelier (2002) and on estimates in Bandle and Marcus (1998), it is proved by Bandle (2003) that for $f(t) = t^p$ $(p > 1)$

$$u(x) = [\gamma d(x)]^{2/(1-p)} \left(1 + \frac{(N-1)H(\sigma(x))}{p+3} d(x) + o(d(x))\right) \quad \text{as } x \to \partial\Omega, \tag{1.31}$$

and for $f(t) = e^t$

$$u(x) = \ln \frac{2}{[d(x)]^2} + (N-1)H(\sigma(x))d(x) + o(d(x)) \quad \text{as } x \to \partial\Omega. \tag{1.32}$$

In the above, $\sigma(x)$ denotes the projection of $x$ to $\partial\Omega$, while $H(\sigma)$ stands for the mean curvature of $\partial\Omega$ at $\sigma$. The asymptotic expansion (1.31) corresponding to general domains is given by del Pino and Letelier (2002) in the case $1 < p < 3$. The uniqueness and explosion rate of the large solution in domains exhibiting a corner has been treated in more generality by Marcus and Véron (1997).

Secondary effects in the blow-up behavior of the solution on arbitrary smooth domains and for a general class of nonlinearities $f$ (including the power and exponential cases) are provided by Bandle and Marcus (2004).

This well developed and focused line of inquiry on the asymptotic expansion of the large solution to $\Delta u = f(u)$ in a smooth bounded domain $\Omega$ proves the dependence of the second explosive term on the curvature of the boundary. However, it is worth investigating whether the same phenomenon manifests itself when a non-negative potential $b(x)$, *identically zero on the boundary* $\partial\Omega$, competes to $f(u)$. In this generality, the query raises significant difficulties even when a first order approximation of the blow-up at the boundary is required (see §1.2.2).

A positive but partial answer to the above question has been given by García-Melián et al. (2001), who establish the two-term asymptotic expansion of the large solution to the logistic equation

$$\Delta u + au = b(x)u^p \quad \text{in } \Omega \subset \mathbb{R}^N,$$

where $\Omega$ is a bounded domain, $p > 1$ and $a$ is a real parameter (more generally, $a$ is a smooth function on $\overline{\Omega}$). Here $b \in C^{0,\mu}(\overline{\Omega})$ is positive in $\Omega$, but vanishes on $\partial\Omega$ as follows

$$b(x) = C_0[d(x)]^\alpha(1 + C_1 d(x) + o(d(x)) \;\text{ as } d(x) \to 0, \tag{1.33}$$

for some constants $C_0, \alpha > 0$ and $C_1 \in \mathbb{R}$. They then prove that the second approximation of the explosion rate of the solution involves both the distance function and the mean curvature $H$ of $\partial\Omega$, namely

$$u(x) = [\widetilde{\gamma} d(x)]^{\frac{\alpha+2}{1-p}} (1 + B(\sigma(x))d(x) + o(d(x))) \quad \text{as } x \to \partial\Omega, \tag{1.34}$$

where $\widetilde{\gamma} = \left(\frac{(p-1)^2 C_0}{(\alpha+2)(\alpha+p+1)}\right)^{1/(\alpha+2)}$ and

$$B(\sigma(x)) = \frac{(N-1)H(\sigma(x)) - C_1(\alpha+p+1)/(p-1)}{\alpha+p+3}.$$

Drawing a parallel between Bandle (2003) and García-Melián et al. (2001), the asymptotic expansion (1.34) can be seen as a refinement of (1.31), since the latter could be recovered if one forced $(C_0, \alpha, C_1) = (1, 0, 0)$ in (1.33).

While the influence of the curvature of the boundary on the second term in the expansion of the large solution for $\Delta u = b(x)f(u)$ in $\Omega$ has been demonstrated when no competition near the boundary arises, the picture is far from being understood otherwise. Could this influence be broken when the decay rate of $b$ is other than that in (1.33) or maybe when $f(u)$ is not $u^p$ ($p > 1$)? It is not yet known how to estimate the second explosive term in the expansion of the solution when $f$ is as general as possible and $b$ vanishes on $\partial\Omega$ without satisfying (1.33).

This query is investigated in Chapter 4 by employing regular variation theory. It is shown that the competition between $f(u)$ and $b(x)$ plays a significant role in eliminating the connection between the second term in the asymptotic expansion of the large solution and the curvature of the boundary. For instance, this happens when $f(u)$ is still $u^p$ but $b(x)$ vanishes on the boundary at a different rate than that in (1.33), alternatively if (1.33) is preserved when $f(u)$ varies regularly without being exactly $u^p$.

## 1.3 Large Solutions for Systems of Equations

The existence or otherwise of solutions for semilinear elliptic systems of the form

$$\begin{cases} \Delta u + f(x,u,v) = 0, & x \in \mathbb{R}^N, \\ \Delta v + g(x,u,v) = 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.35}$$

has recently been the subject of much investigation (for example, Mitidieri et al. (1995), Mitidieri (1996), de Figueiredo and Yang (1998), Qi (1998), Serrin and Zou (1996, 1998*a*,*b*), Yarur (1998)). The case $f(u,v) = v^p$ and $g(u,v) = u^q$ $(p,q > 0)$ in (1.35), that is

$$\begin{cases} \Delta u + v^p = 0, & x \in \mathbb{R}^N, \\ \Delta v + u^q = 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.36}$$

can be thought of as an extension of the Lane–Emden equation $\Delta w + w^p = 0$ in $\mathbb{R}^N$. When the pair $(p,q)$ is above the critical hyperbola

$$1/(p+1) + 1/(q+1) \leq (N-2)/N,$$

then (1.36) admits infinitely many positive radial solutions $(u,v)$ which tend to $(0,0)$ as $|x| \to \infty$, which are called *ground states* (see Serrin and Zou (1998*b*)). This kind of behavior at infinity has been studied in the above cited papers.

Yarur (1998) studies the following system

$$\begin{cases} \Delta u + \alpha(|x|) f(v) = 0, & x \in \mathbb{R}^N \setminus \{0\}, \\ \Delta v + \beta(|x|) g(u) = 0, & x \in \mathbb{R}^N \setminus \{0\}, \end{cases} \tag{1.37}$$

where $f$ and $g$ are increasing Lipschitz continuous on $\mathbb{R}$ with $f(0) = g(0) = 0$ and $\alpha, \beta$ are non-negative $C^1$ functions on $\mathbb{R}^+$ (no smoothness at zero for $(\alpha, \beta)$ is required). She proves the existence of a curve of positive radially symmetric continuous/singular ground states for (1.37) and shows that these curves depend on the conditions imposed on the functions $f$, $g$, $\alpha$ and $\beta$.

While the above papers contribute to the understanding of the nonlinear mechanism of the system leading to ground states, little is known so far about the conditions which favor a *singular* behavior of solutions at infinity.

A solution $(u,v)$ of (1.35) is called an *entire large solution* if it is a classical solution of (1.35) on $\mathbb{R}^N$ such that $u(x) \to \infty$ and $v(x) \to \infty$ as $|x| \to \infty$.

Despite a long history on the topic of blow-up in elliptic equations, only very recently has this been considered for systems of equations (see for example Lair and Shaker (2000), Yang (2003*a*,*b*)).

A major complication when dealing with systems of equations is the lack of a maximum principle. More exactly, although the general theory for second and higher order elliptic equations has much in common there is also a crucial difference. For second order elliptic equations there exists a so-called maximum principle, which has proved a decisive instrument in providing a priori estimates and existence results. The validity of the maximum principle is restricted to second order, scalar, elliptic operators and does not extend in any natural way to systems of second order operators or to higher order scalar equations.

To counter this fact, a common feature appears in the study of elliptic systems: the assumption of radial symmetry on the domain and coefficients. This scenario is favorable to reducing the problem to the study of radially symmetric solutions for ordinary differential equations (ODEs).

Lair and Shaker (2000) focus on the existence of positive radial solutions of

$$\begin{cases} \Delta u = \alpha(x) v^p & \text{in } \mathbb{R}^N, \\ \Delta v = \beta(x) u^q & \text{in } \mathbb{R}^N, \end{cases} \tag{1.38}$$

where $p, q > 0$ and $\alpha, \beta \in C^{0,\mu}_{loc}(\mathbb{R}^N)$ $(0 < \mu < 1)$ are non-negative and radially symmetric functions. Whether these entire solutions are large or bounded depends closely on the nonlinear part of the system, a phenomenon that resembles that noticed by Yarur (1998) in the context of ground states. More exactly, cf. Yarur (1998) if $(f(v), g(u)) = (v^p, u^q)$ in (1.37) and $\alpha$, $\beta$ satisfy

$$\int^{\infty} r\alpha(r)\,dr = \infty \quad \text{and} \quad \int^{\infty} r\beta(r)\,dr = \infty, \tag{1.39}$$

then positive radially symmetric solutions of (1.37) turn out to be ground states. The existence of a solution, near zero, to the Cauchy problem needs

$$\int_0 r\alpha(r)\,dr < \infty \quad \text{and} \quad \int_0 r\beta(r)\,dr < \infty. \tag{1.40}$$

Lair and Shaker (2000) show that if $0 < p, q \leq 1$, then for any pair $(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+$ there exists an entire radial solution of (1.38) with central values $(a, b)$. Thus, $\mathbb{R}^+ \times \mathbb{R}^+ = \mathcal{Z}$, where

$$\mathcal{Z} = \left\{ (a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ \,\middle|\, \begin{array}{l} (\exists) \text{ an entire radial solution of (1.38)} \\ \text{such that } (u(0), v(0)) = (a,b) \end{array} \right\}. \tag{1.41}$$

Moreover, all positive entire radial solutions are *large* when

$$\int_0^\infty r\alpha(r)\,dr = \infty \quad \text{and} \quad \int_0^\infty r\beta(r)\,dr = \infty, \tag{1.42}$$

while they (all of them) become *bounded* provided that

$$\int_0^\infty r\alpha(r)\,dr < \infty \quad \text{and} \quad \int_0^\infty r\beta(r)\,dr < \infty. \tag{1.43}$$

Condition (1.42) can be seen as a reflection of the one found by Lair and Wood (2000) in the scalar case; the existence of entire large solutions to

$$\Delta u = \alpha(x) u^p \quad \text{in } \mathbb{R}^N, \tag{1.44}$$

in the *sub-linear* case ($0 < p \leq 1$) holds if and only if $\alpha$ satisfies (1.42) (see also Lair (2003) where $\alpha$ is not necessarily radial). In the sub-linear case, there are no large solutions for (1.44) on bounded domains, as the Keller–Osserman condition (1.2) fails; for the existence of large solutions to be restored we need $p > 1$.

Lair and Shaker (2000) prove that, in comparison to the case $0 < p, q \leq 1$, the super-linear case $1 < p, q < \infty$ for (1.38) brings significant changes. Precisely, if (1.43) holds, then $\mathcal{Z}$ becomes a closed bounded convex subset of $\mathbb{R}^+ \times \mathbb{R}^+$ and any entire radial solution $(u, v)$ of (1.38) is large provided that

$$(u(0), v(0)) \in \{(a,b) \in \partial\mathcal{Z} |\ a > 0 \ \text{ and } \ b > 0\}.$$

Lair and Shaker (2000) do not treat the case when $0 < p < 1$ and $1 < q < \infty$. More generally, it is desirable to understand what kind of relationship between $f$ and $g$ is conducive to large/bounded entire solutions for systems such as (1.35).

The above questions will be addressed in §2.2 of Chapter 2, where more general nonlinearities are considered in (1.38). The growth rate of the nonlinearities (individually and combined) will determine the structure of $\mathcal{Z}$ in (1.41). The existence of entire large solutions, known from §2.1 to occur in the scalar case under growth conditions of Keller–Osserman type, will manifest to systems, too.

## 1.4 Outline of the Thesis

This dissertation is mostly devoted to the study of large solutions for semilinear elliptic equations of the form

$$\Delta u = f(x,u) \quad \text{in } \Omega \subset \mathbb{R}^N, \tag{1.45}$$

where $\Omega$ is a bounded/unbounded domain with compact boundary or the whole space. More precisely, the thesis develops innovative analytical methods to investigate the qualitative aspects of the blow-up phenomenon in (1.45) arising on the boundary or at infinity. In the latter situation the results play an important role in dealing with semilinear elliptic systems such as

$$\begin{cases} \Delta u = f_1(x,v) & \text{in } \mathbb{R}^N, \\ \Delta v = f_2(x,u) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.46}$$

The purpose of Chapter 2 is two-fold. First, to establish the existence of large solutions to (1.45) when $f(x,u)$ is in the form of $b(x)f(u)$, where $b$ is a *non-negative* smooth function, while $f \in C^1[0,\infty)$ is positive and non-decreasing on $(0,\infty)$ such that $f(0) = 0$ and (1.2) holds (see §2.1). Second, to determine the existence of positive radial entire solutions of (1.46) and classify them either as bounded or large depending on the nonlinear mechanism of (1.46). Under some symmetry assumptions, methods are given to analyze two possible scenarios for nonlinearities $f_1$ and $f_2$. ODEs techniques are used jointly with the findings of §2.1 to extend and complement previous results of Lair and Shaker (2000).

Chapter 3 is dedicated to the existence of large solutions to (1.45) in a bounded domain, where $f(x,u) = b(x)f(u) - au$ (see §3.1). Here $a$ is a real parameter and $b$ is a non-negative function on $\overline{\Omega}$ that *vanishes* on a connected subset $\overline{\Omega_0} \subset \Omega$ and is positive otherwise in $\Omega$. The analysis includes the critical case when $b$ is non-negative (possibly, identically zero) on $\partial\Omega$. A necessary and sufficient condition for the existence of large solutions is provided in terms of the first Dirichlet eigenvalue of the Laplace operator on the zero set of $b$. The assumptions on $f$ are inspired by Alama and Tarantello (1996) who give a corresponding result for the Dirichlet boundary value problem (1.12). In §3.2 we prove the existence of the minimal/maximal positive solution to the above equation in $\Omega \setminus \overline{\Omega}_0$, subject to a Dirichlet, Neumann or Robin boundary condition on $\partial\Omega$ and $u = \infty$ on $\partial\Omega_0$. Using a novel approach, the degenerate situation when $b$ may vanish on $\Omega \setminus \overline{\Omega}_0$ is further discussed under more general conditions on $f$ (see §3.3).

The main aim of Chapters 4 and 5 is to resolve the issues of uniqueness and two-term asymptotic expansion of the large solution near the boundary to the problems studied in Chapter 3. The generality of the framework, coupled with the competition case $0 \cdot \infty$ assumed near the boundary, requires the development of new and effective methods. This need is met in both Chapters 4 and 5, where innovative methods are advanced by establishing inter-disciplinary connections with applied probability. For the first time in this context the regular variation theory (initiated by Karamata and outlined in §4.1) and its extensions due de Haan (presented in §5.3.2) are to be used.

The crucial step advanced in Chapter 4, where regular variation theory is at the fore, is the analysis of the competition between $b(x)$ and $f(u)$ through their variation speed. Karamata's theory, used jointly with a local argument near the boundary, helps determine the blow-up rate of the large solution when previous restrictions in the literature on the decay rate of $b$ are removed and $f$ is varying *regularly* at $\infty$ (imitating a super-linear power). Relying essentially on Karamata's theory, we attest significant changes in the two-term asymptotic expansion of the large solution; in particular, the influence of the curvature of the boundary on the second explosive term (known in the non-competing case) ceases in the competing case even if $f(u) = u^p$, $p > 1$.

The objective of Chapter 5 is dual: first, to refine the uniqueness results given in Chapter 4 by modifying an iterative technique due to Safonov (see §5.2); and second, to establish the uniqueness and blow-up rate of the large solution in the competing case when $f$ varies *rapidly* at $\infty$ (covering exponential models, see §5.3). In both sections, local estimates of the blow-up rate are derived, which improve the findings of López-Gómez (2003) and Du (2004) even for the special classes considered there. The innovative approach presented in §5.3 stresses the interplay between the blow-up topic in PDEs and de Haan's theory in applied probability. The main results on the asymptotic behavior also shed a new light on the non-competing case analyzed by Bandle and Marcus (1992*a*), Lazer and McKenna (1994).

# Chapter 2

# Large and Entire Large Solutions

**"Hold yourself responsible for a higher standard than anybody else expects of you." (Henry Ward Beecher)**

## 2.1 Large Solutions for Elliptic Equations

### 2.1.1 Introduction

We consider the following semilinear elliptic equation

$$\begin{cases} \Delta u = p(x) f(u) & \text{in } \Omega\,, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \Omega\,, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a smooth domain (bounded or possibly unbounded) with compact (possibly empty) boundary. We assume throughout this section that $p$ is a non-negative function such that $p \in C^{0,\alpha}(\overline{\Omega})$ if $\Omega$ is bounded, and $p \in C^{0,\alpha}_{\text{loc}}(\Omega)$, otherwise. The nonlinearity $f$ is assumed to fulfill

$$f \in C^1[0,\infty),\ f' \geq 0,\ f(0) = 0 \text{ and } f > 0 \text{ on } (0,\infty) \tag{2.2}$$

and the Keller–Osserman condition (see Keller (1957); Osserman (1957))

$$\int_1^\infty \frac{dt}{\sqrt{2F(t)}} < \infty\,, \quad \text{where} \quad F(t) = \int_0^t f(s)\,ds. \tag{2.3}$$

The main purpose of section 2.1 is to find properties of *large solutions* of (2.1), that is solutions $u$ satisfying $u(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$ (if $\Omega \not\equiv \mathbb{R}^N$), or $u(x) \to \infty$ as $|x| \to \infty$ (if $\Omega = \mathbb{R}^N$). In the latter case the solution is called an *entire large* solution.

Problems of the type (2.1) were originally studied in a celebrated paper by Loewner and Nirenberg (1974). Their work deals with partial differential equations having a "partial conformal invariance" and is motivated by a concrete problem arising in Riemannian Geometry. More precisely, Loewner and Nirenberg proved the remarkable result that (2.1) has a maximal solution, provided that $\Omega \neq \mathbb{R}^N$, $p \equiv \text{Const.} > 0$ in $\Omega$ and $f(u) = u^{(N+2)/(N-2)}$.

In Bandle and Marcus (1992*a*) and Marcus (1992), problem (2.1) is considered in the special case when $\Omega$ is bounded and $p > 0$ in $\overline{\Omega}$. More precisely, Bandle and Marcus described the precise asymptotic behavior of large solutions near the boundary and established the uniqueness of such solutions, while Marcus obtained existence results for large solutions.

The first main result of this section is an existence theorem for large solutions when $\Omega$ is bounded (see Theorem 1 in Cîrstea and Rădulescu (2002*a*)).

**Theorem 2.1.1.** *Suppose that $\Omega$ is bounded and $p$ satisfies*

$$\left.\begin{array}{l}\textit{for all } x_0 \in \Omega \textit{ with } p(x_0) = 0, \textit{ there is a domain } \Omega_0 \ni x_0 \textit{ such that}\\ \overline{\Omega_0} \subset \Omega \textit{ and } p > 0 \textit{ on } \partial\Omega_0.\end{array}\right\} \tag{2.4}$$

*Then problem (2.1) has a positive large solution.*

*Remark* 2.1.1. Condition (2.4) is weaker than the requirement that $p > 0$ on $\partial\Omega$.

Indeed, the continuity of $p$, the compactness of $\partial\Omega$ and the positivity of $p$ on $\partial\Omega$ imply the existence of some $\delta > 0$ such that $p > 0$ in $\Omega_\delta$, where

$$\Omega_\delta := \{x \in \overline{\Omega};\ \text{dist}\,(x, \partial\Omega) \le \delta\}.$$

Therefore, all the zeros of $p$ are included in $\Omega_0 = \overline{\Omega} \setminus \Omega_\delta \subset\subset \Omega$. Hence $p > 0$ on $\partial\Omega_0$, so that (2.4) is fulfilled.

*Remark* 2.1.2. Theorem 2.1.1 generalizes Theorem 3.1 in Marcus (1992) and Lemma 2.6 in Cheng and Ni (1992), where it is assumed that $p > 0$ on $\partial\Omega$.

The rest of section 2.1 is organized as follows. Subsection 2.1.2 comprises a result that will be repeatedly used, namely Theorem 2.1.2, whose statement can be found in Marcus (1992). The proof of Theorem 2.1.1 is given in §2.1.3. In §2.1.4 and §2.1.5 we make use of Theorem 2.1.1 to find and describe the behavior on the boundary and at infinity of the maximal solution to problem (2.1), where $\Omega$ is an unbounded domain, possibly $\mathbb{R}^N$ (Theorems 2.1.7 and 2.1.10). For the significance of such a study we refer to Dynkin (1991), where it is shown that there exist certain relations between hitting probabilities for superdiffusions and maximal solutions of (2.1) with $f(u) = u^\gamma$, $1 < \gamma \leq 2$. Subsection 2.1.4 gives a necessary condition for the existence of entire large solutions to (2.1) (see Theorem 2.1.9).

### 2.1.2 Boundary Value Problems

The following result, which is mentioned without proof in Marcus (1992), will be applied several times in this section. For the sake of completeness we give here a proof of this theorem (see Theorem 5 in Cîrstea and Rădulescu (2002*a*)).

**Theorem 2.1.2.** *Let $\Omega$ be a bounded domain. Assume that $p \in C^{0,\alpha}(\overline{\Omega})$ is a non-negative function, $f$ satisfies (2.2) and $g : \partial\Omega \to (0, \infty)$ is continuous, then the boundary value problem*

$$\begin{cases} \Delta u = p(x)f(u), & \text{in } \Omega \\ u = g, & \text{on } \partial\Omega \\ u \geq 0, \;\; u \not\equiv 0, & \text{in } \Omega \end{cases} \tag{2.5}$$

*has a unique classical solution, which is positive.*

*Proof.* We first observe that the function $u^+(x) = n$ is a super-solution of problem (2.5), provided that $n$ is sufficiently large. To find a positive sub-solution, we look for an arbitrary positive solution to the following auxiliary problem

$$\Delta v = \Phi(r) \qquad \text{in } A(\underline{r}, \overline{r}) = \{x \in \mathbb{R}^N;\; \underline{r} < |x| < \overline{r}\} \tag{2.6}$$

where

$$\underline{r} = \inf\{\tau > 0;\; \partial B(0,\tau) \cap \overline{\Omega} \neq \emptyset\}, \quad \overline{r} = \sup\{\tau > 0;\; \partial B(0,\tau) \cap \overline{\Omega} \neq \emptyset\}$$

$$\Phi(r) = \max_{|x|=r} p(x) \quad \text{for any } r \in [\underline{r}, \overline{r}].$$

The function

$$v(r) = 1 + \int_{\underline{r}}^{r} \sigma^{1-N} \left( \int_0^\sigma \tau^{N-1} \Phi(\tau)\, d\tau \right) d\sigma, \quad \underline{r} \leq r \leq \overline{r}$$

verifies equation (2.6). The assumptions on $f$ and $g$ imply that

$$g_0 := \min_{\partial\Omega} g > 0 \quad \text{and} \quad \lim_{z \searrow 0} \int_z^{g_0} \frac{dt}{f(t)} = \infty.$$

This will be used to justify the existence of a positive number $c$ such that

$$\max_{\partial\Omega} v = \int_c^{g_0} \frac{dt}{f(t)}. \tag{2.7}$$

Next, we define the function $u_-$ such that

$$v(x) = \int_c^{u_-(x)} \frac{dt}{f(t)}, \qquad \forall x \in \Omega. \tag{2.8}$$

It turns out that $u_-$ is a positive sub-solution of problem (2.5). Indeed, we have

$$u_- \in C^2(\Omega) \cap C(\overline{\Omega}) \quad \text{and} \quad u_- \geq c \text{ in } \Omega.$$

On the one hand, from (2.6), (2.8) and (2.2) it follows that

$$\begin{aligned} p(x) \leq \Delta v(x) &= \frac{1}{f(u_-(x))} \Delta u_-(x) + \left(\frac{1}{f}\right)'(u_-(x))|\nabla u_-(x)|^2 \\ &\leq \frac{1}{f(u_-(x))} \Delta u_-(x) \qquad \text{in } \Omega, \end{aligned}$$

which yields

$$\Delta u_-(x) \geq p(x) f(u_-(x)) \qquad \text{in } \Omega.$$

On the other hand, taking into account (2.7) and (2.8), we find

$$u_-(x) \leq g(x) \qquad \forall x \in \partial\Omega.$$

So, we have proved that $u_-$ is a positive sub-solution to problem (2.5), therefore, this problem has at least a positive solution $u$. Furthermore, taking into account the regularity of $p$ and $f$, a standard boot-strap argument based on Schauder and Hölder regularity shows that $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Let us now assume that $u_1$ and $u_2$ are arbitrary solutions of (2.5). In order to prove the uniqueness, it is enough to show that $u_1 \geq u_2$ in $\Omega$. Denote

$$\omega := \{x \in \Omega;\ u_1(x) < u_2(x)\}$$

and suppose that $\omega \neq \emptyset$. Then the function $\tilde{u} = u_1 - u_2$ satisfies

$$\begin{cases} \Delta\tilde{u} = p(x)(f(u_1) - f(u_2)), & \text{in } \omega \\ \tilde{u} = 0, & \text{on } \partial\omega\,. \end{cases} \tag{2.9}$$

Since $f$ is non-decreasing and $p \geq 0$, it follows by (2.9) that $\tilde{u}$ is a super-harmonic function in $\omega$ which vanishes on $\partial\omega$. Thus, by the maximum principle, either $\tilde{u} \equiv 0$ or $\tilde{u} > 0$ in $\omega$, which yields a contradiction. Thus $u_1 \geq u_2$ in $\Omega$.

We give in what follows an alternative proof for the uniqueness. Let $u_1$, $u_2$ be two arbitrary solutions of problem (2.5). As above, it is enough to show that $u_1 \geq u_2$ in $\Omega$. Fix $\varepsilon > 0$. We claim that

$$u_2(x) \leq u_1(x) + \varepsilon(1 + |x|^2)^{-1/2} \qquad \text{for any } x \in \Omega. \tag{2.10}$$

Suppose the contrary. Since (2.10) is obviously fulfilled on $\partial\Omega$, we deduce that

$$\max_{x\in\overline{\Omega}} \left\{u_2(x) - u_1(x) - \varepsilon(1 + |x|^2)^{-1/2}\right\}$$

is achieved in $\Omega$. At that point we have

$$\begin{aligned} 0 &\geq \Delta\left(u_2(x) - u_1(x) - \varepsilon(1 + |x|^2)^{-1/2}\right) \\ &= p(x)\left(f(u_2(x)) - f(u_1(x))\right) - \varepsilon\Delta(1 + |x|^2)^{-1/2} \\ &= p(x)\left(f(u_2(x)) - f(u_1(x))\right) + \varepsilon(N-3)(1 + |x|^2)^{-3/2} \\ &\quad + 3\,\varepsilon(1 + |x|^2)^{-5/2} > 0\,, \end{aligned}$$

which is a contradiction. Since $\varepsilon > 0$ is chosen arbitrarily, inequality (2.10) implies that $u_2 \leq u_1$ in $\Omega$. □

We point out that the hypothesis that $f$ is differentiable at the origin is essential in order to find a *positive* solution to problem (2.5). Indeed, consider $\Omega = B_1$, and $f(u) = u^{(\beta-2)/\beta}$, where $\beta > 2$. Choose $p \equiv 1$ and $g \equiv C$ on $\partial B_1$, where $C = (\beta^2 + (N-2)\beta)^{-\beta/2}$. For this choice of $\Omega$, $p$, $f$ and $g$, the function $u(r) = Cr^{\beta}$, $0 \leq r \leq 1$, is the unique solution of problem (2.5), but $u(0) = 0$.

Under the hypotheses on $f$ made in the statement of Theorem 2.1.2, except $f$ is of class $C^1$ at the origin (but $f \in C^{0,\alpha}$ at $u = 0$), problem (2.5) has a unique solution which may vanish in $\Omega$. For this purpose it is sufficient to choose, as a sub-solution in the above proof, the function $u_- = 0$.

**Proposition 2.1.3.** *Let $\Omega = B(0, R)$ for some $R > 0$ and let $p$ be radially symmetric in $\Omega$, then (2.1), subject to the Dirichlet boundary condition*

$$u = c \ (const.) > 0 \quad \text{on } \partial\Omega, \tag{2.11}$$

*has a unique solution $u_c$, which, moreover, is positive and radially symmetric.*

*Proof.* By Theorem 2.1.2, problem (2.1)+(2.11) has a unique solution $u_c$, which is positive. If $u_c$ were not radially symmetric, then a different solution could be obtained by rotating it, which would contradict the uniqueness of the solution. □

*Remark* 2.1.3. Any possible large solution of (2.1) is *positive* in $\Omega$, whenever $\Omega$ is a bounded domain or the whole space.

Indeed, assume that $u(x_0) = 0$ for some $x_0 \in \Omega$. Since $u$ is a large solution we can find a smooth domain $\omega \subset\subset \Omega$ such that $x_0 \in \omega$ and $u > 0$ on $\partial\omega$. Thus, by Theorem 2.1.2, the problem

$$\begin{cases} \Delta\zeta = p(x)f(\zeta) & \text{in } \omega, \\ \zeta = u & \text{on } \partial\omega, \\ \zeta \geq 0 & \text{in } \omega \end{cases}$$

has a unique solution, which is positive. By uniqueness, $\zeta = u$ in $\omega$, which is a contradiction. This shows that any large solution of (2.1) cannot vanish in $\Omega$.

### 2.1.3 Existence Results on Bounded Domains

We assume that $\Omega$ is bounded throughout §2.1.3. By Keller (1957) and Osserman (1957), problem (2.1) with $p \equiv 1$ has large solutions if and only if $f$ fulfills (2.3). Next we infer that (2.3) is necessary for the existence of large solutions to (2.1).

**Lemma 2.1.4.** *If (2.2) holds, then the Keller–Osserman condition (2.3) is necessary for the existence of large solutions of (2.1).*

*Proof.* Suppose, a priori, that (2.1) has a large solution $u_\infty$. For any $n \geq 1$, consider the problem

$$\begin{cases} \Delta u = \|p\|_\infty f(u) & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega. \end{cases}$$

By Theorem 2.1.2, this problem has a unique solution, say $u_n$, which is positive in $\overline{\Omega}$. By the maximum principle

$$0 < u_n \leq u_{n+1} \leq u_\infty \quad \text{in } \Omega, \quad \forall n \geq 1.$$

Thus, for every $x \in \Omega$, it makes sense to define $\overline{u}(x) = \lim_{n\to\infty} u_n(x)$. Since $(u_n)$ is uniformly bounded on every compact set $\omega \subset\subset \Omega$, standard elliptic regularity implies that $\overline{u}$ is a large solution of the problem $\Delta u = \|p\|_\infty f(u)$ in $\Omega$. □

**Example 2.1.1.** Typical nonlinearities satisfying (2.2) and (2.3) are:

i) $f(u) = e^u - 1$;
ii) $f(u) = u^p,\ p > 1$;
iii) $f(u) = u[\ln(u+1)]^p,\ p > 2$.

For nonlinearities, as in the above example, the following result holds (Lemma 1 in Cîrstea and Rădulescu (2002*a*)).

**Lemma 2.1.5.** *Assume that conditions* (2.2) *and* (2.3) *are fulfilled, then*

$$\int_1^\infty \frac{dt}{f(t)} < \infty\,. \tag{2.12}$$

*Proof.* Fix $R > 0$ and denote $B = B(0,R)$. By Theorem 2.1.2, the boundary value problem

$$\begin{cases} \Delta u_n = f(u_n), & \text{in } B \\ u_n = n, & \text{on } \partial B \\ u_n \geq 0, u_n \not\equiv 0, & \text{in } B \end{cases} \tag{2.13}$$

has a unique positive solution. Since $f$ is non-decreasing, it follows by the maximum principle that $u_n(x)$ increases with $n$, for any fixed $x \in B$.

We first claim that $(u_n)$ is uniformly bounded in every compact sub-domain of $B$. Indeed, let $K \subset B$ be any compact set and $d := \text{dist}\,(K, \partial B)$, then

$$0 < d \leq \text{dist}\,(x, \partial B), \quad \forall x \in K. \tag{2.14}$$

By Proposition 1 of Bandle and Marcus (1992*a*), there exists a continuous, non-increasing function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$u_n(x) \leq \mu(\text{dist}\,(x, \partial B)), \quad \forall x \in K.$$

The claim now follows from (2.14). Thus, for every $x \in B$ we can define

$$u(x) := \lim_{n\to\infty} u_n(x).$$

We next show that $u$ is a classical large solution of

$$\Delta u = f(u) \quad \text{in } B. \tag{2.15}$$

Fix $x_0 \in B$ and let $r > 0$ be such that $\overline{B(x_0, r)} \subset B$. Let $\Psi \in C^\infty(B)$ be such that $\Psi \equiv 1$ in $\overline{B(x_0, r/2)}$ and $\Psi \equiv 0$ in $B \setminus B(x_0, r)$. We have

$$\Delta(\Psi u_n) = 2\nabla\Psi \cdot \nabla u_n + p_n,$$

where $p_n = u_n\Delta\Psi + \Psi\Delta u_n$. Since $(u_n)$ is uniformly bounded on $\overline{B(x_0, r)}$ and $f$ is non-decreasing on $[0, \infty)$, it follows that $\|p_n\|_\infty \le C$, where $C$ is a constant independent of $n$. From now on, using the same argument given in the proof of Lemma 3 in Lair and Shaker (1997), we find that $(u_n)$ converges in $C^{2,\alpha}(\overline{B(x_0, r_1)})$, for some $r_1 > 0$. Since $x_0 \in B$ is arbitrary, this shows that $u \in C^2(B)$ and $u$ is a positive solution of (2.15). Moreover, by the Gidas–Ni–Nirenberg Theorem, $u$ is radially symmetric in $B$, namely $u(x) = u(r)$, $r = |x|$, and $u$ satisfies in the $r$ variable the equation

$$u''(r) + \frac{N-1}{r}u'(r) = f(u(r)), \qquad 0 < r < R.$$

This equation can be rewritten as follows

$$(r^{N-1}u'(r))' = r^{N-1}f(u(r)), \qquad 0 < r < R. \tag{2.16}$$

Integrating (2.16) from 0 to $r$ we obtain

$$u'(r) = r^{1-N}\int_0^r s^{N-1}f(u(s))\,ds, \qquad 0 < r < R.$$

Hence $u$ is a non-decreasing function and

$$u'(r) \le r^{1-N}f(u(r))\int_0^r s^{N-1}\,ds = \frac{r}{N}f(u(r)), \qquad 0 < r < R. \tag{2.17}$$

Similarly, $u_n$ is non-decreasing on $(0, R)$, for any $n \ge 1$.

In order to show that $u$ is a large solution of (2.15), it remains to prove that $u(r) \to \infty$ as $r \nearrow R$. Assume the contrary, then there exists $C > 0$ such that $u(r) < C$ for all $0 \le r < R$. Let $N_1 \ge 2C$ be fixed. The monotonicity of $u_{N_1}$ and the fact that $u_{N_1}(r) \to N_1$ as $r \nearrow R$ imply the existence of some $r_1 \in (0, R)$ such that $C \le u_{N_1}(r)$, for $r \in [r_1, R)$. Hence

$$C \le u_{N_1}(r) \le u_{N_1+1}(r) \le \cdots \le u_n(r) \le u_{n+1}(r) \le \cdots \quad \forall n \ge N_1, \quad \forall r \in [r_1, R).$$

Passing to the limit as $n \to \infty$, we obtain $u(r) \ge C$ for all $r \in [r_1, R)$, which is a contradiction.

Integrating (2.17) on $(0, r)$ and taking $r \nearrow R$ we find

$$\int_{u(0)}^{\infty} \frac{1}{f(t)}\, dt \le \frac{R^2}{2N}.$$

The conclusion of Lemma 2.1.5 is therefore proved. □

*Proof of Theorem* 2.1.1. By Theorem 2.1.2, the boundary value problem

$$\begin{cases} \Delta v_n = p(x) f(v_n), & \text{in } \Omega \\ v_n = n, & \text{on } \partial\Omega \\ v_n \ge 0, \ \ v_n \not\equiv 0, & \text{in } \Omega \end{cases} \tag{2.18}$$

has a unique positive solution, for any $n \ge 1$.

We now claim that

(a) for all $x_0 \in \Omega$ there exist an open set $\mathcal{O} \subset\subset \Omega$ containing $x_0$ and $M_0 = M_0(x_0) > 0$ such that $v_n \le M_0$ in $\mathcal{O}$, for any $n \ge 1$;

(b) $\lim_{x \to \partial\Omega} v(x) = \infty$, where $v(x) = \lim_{n \to \infty} v_n(x)$.

We first remark that the sequence $(v_n)$ is non-decreasing. Indeed, by Theorem 2.1.2, the boundary value problem

$$\begin{cases} \Delta \zeta = \|p\|_\infty f(\zeta), & \text{in } \Omega \\ \zeta = 1, & \text{on } \partial\Omega \\ \zeta > 0, & \text{in } \Omega \end{cases}$$

has a unique solution. Then, by the maximum principle,

$$0 < \zeta \leq v_1 \leq \cdots \leq v_n \leq \cdots \qquad \text{in } \Omega\,. \tag{2.19}$$

We also observe that (a) and (b) are sufficient to conclude the proof. In fact, assertion (a) shows that the sequence $(v_n)$ is uniformly bounded on every compact subset of $\Omega$. Standard elliptic regularity arguments (see the proof of Lemma 3 in Lair and Shaker (1997)) show that $v$ is a solution of problem (2.1). Then, by (2.19) and (b), it follows that $v$ is the desired solution.

To prove (a) we distinguish two cases :

*Case* 2.1.1. $p(x_0) > 0$.

By the continuity of $p$, there exists a ball $B = B(x_0, r) \subset\subset \Omega$ such that

$$m_0 := \min\{p(x);\; x \in \overline{B}\} > 0.$$

Let $w$ be a positive solution of problem

$$\begin{cases} \Delta w = m_0 f(w), & \text{in } B \\ w(x) \to \infty, & \text{as } x \to \partial B. \end{cases} \tag{2.20}$$

The existence of $w$ follows by Theorem III in Keller (1957). By the maximum principle it follows that $v_n \leq w$ in $B$. Furthermore, $w$ is bounded in $\overline{B(x_0, r/2)}$. Setting $M_0 = \sup\limits_{\mathcal{O}} w$, where $\mathcal{O} = B(x_0, r/2)$, we obtain (a).

*Case* 2.1.2. $p(x_0) = 0$.

Our hypothesis (2.4) and the boundedness of $\Omega$ imply the existence of a domain $\mathcal{O} \subset\subset \Omega$ which contains $x_0$ such that $p > 0$ on $\partial\mathcal{O}$. The case 2.1.1 shows that for any $x \in \partial\mathcal{O}$ there exist a ball $B(x, r_x)$ strictly contained in $\Omega$ and a constant $M_x > 0$ such that $v_n \leq M_x$ on $B(x, r_x/2)$, for any $n \geq 1$. Since $\partial\mathcal{O}$ is compact, it follows that it may be covered by a finite number of such balls, say $B(x_i, r_{x_i}/2)$, $i = 1, \cdots, k_0$. Setting $M_0 = \max\{M_{x_1}, \cdots, M_{x_{k_0}}\}$ we have $v_n \leq M_0$ on $\partial\mathcal{O}$, for any $n \geq 1$. By the maximum principle we obtain $v_n \leq M_0$ in $\mathcal{O}$ and (a) follows.

Let us now consider the problem

$$\begin{cases} -\Delta z = p(x) & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \\ z \geq 0,\; z \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{2.21}$$

From Theorem 1 in Brezis and Oswald (1986) we infer that (2.21) has a unique solution. Moreover, by the maximum principle, this solution is positive in $\Omega$.

We first observe that for proving (b) it is sufficient to show that

$$\int_{v(x)}^{\infty} \frac{dt}{f(t)} \leq z(x) \quad \text{for any } x \in \Omega. \tag{2.22}$$

By Lemma 1, the left hand-side of (2.22) is well defined in $\Omega$. Fix $\varepsilon > 0$. Since $v_n = n$ on $\partial\Omega$, there is $n_1 = n_1(\varepsilon)$ such that

$$\int_{v_n(x)}^{\infty} \frac{dt}{f(t)} \leq \varepsilon(1+R^2)^{-1/2} \leq z(x)+\varepsilon(1+|x|^2)^{-1/2} \qquad \forall x \in \partial\Omega\,,\ \forall n \geq n_1\,, \tag{2.23}$$

where $R > 0$ is chosen so that $\overline{\Omega} \subset B(0, R)$.

In order to prove (2.22), it is enough to show that

$$\int_{v_n(x)}^{\infty} \frac{dt}{f(t)} \leq z(x) + \varepsilon(1 + |x|^2)^{-1/2} \qquad \forall x \in \Omega\,,\ \forall n \geq n_1. \tag{2.24}$$

Indeed, putting $n \to \infty$ in (2.24) we deduce (2.22), since $\varepsilon > 0$ is arbitrarily chosen. Assume now, by contradiction, that (2.24) fails, then

$$\max_{x\in\overline{\Omega}} \left\{ \int_{v_n(x)}^{\infty} \frac{dt}{f(t)} - z(x) - \varepsilon(1 + |x|^2)^{-1/2} \right\} > 0.$$

Using (2.23) we see that the point where the maximum is achieved must lie in $\Omega$. At this point, say $x_0$, we have

$$\begin{aligned}
0 &\geq \Delta \left( \int_{v_n(x)}^{\infty} \frac{dt}{f(t)} - z(x) - \varepsilon(1 + |x|^2)^{-1/2} \right)_{|x=x_0} \\
&= \left( -\frac{1}{f(v_n)} \Delta v_n - \left(\frac{1}{f}\right)'(v_n) \cdot |\nabla v_n|^2 - \Delta z(x) - \varepsilon\Delta(1 + |x|^2)^{-1/2} \right)_{|x=x_0} \\
&= \left( -p(x) - \left(\frac{1}{f}\right)'(v_n) \cdot |\nabla v_n|^2 + p(x) - \varepsilon\Delta(1 + |x|^2)^{-1/2} \right)_{|x=x_0} \\
&= \left( -\left(\frac{1}{f}\right)'(v_n) \cdot |\nabla v_n|^2 + \varepsilon(N - 3)(1 + |x|^2)^{-3/2} + 3\,\varepsilon(1 + |x|^2)^{-5/2} \right)_{|x=x_0} \\
&> 0.
\end{aligned}$$

This contradiction shows that inequality (2.24) holds. This completes the proof of Theorem 2.1.1. □

**Corollary 2.1.6.** *Let $\Omega = B(0, R)$ for some $R > 0$. If $p$ is radially symmetric in $\Omega$ and $p_{|\partial\Omega} > 0$, then there exists a radial large solution of (2.1).*

*Proof.* The large solution constructed in the proof of Theorem 2.1.1 will be radially symmetric by virtue of Proposition 2.1.3. □

### 2.1.4 Existence of Entire Large Solutions

Our next purpose is to prove the existence of an entire maximal solution for (2.1), under more general hypotheses than in Cheng and Ni (1992). They investigate the structure of all positive solutions of (2.1) in the special case when $f(u) = u^\gamma$, $\gamma > 1$; they also establish the existence of the maximal classical solution $U$ of (2.1), under the hypotheses that this equation possesses at least a positive entire solution and there is a sequence of smooth bounded domains $(\Omega_n)_{n\geq1}$ such that, for any $n \geq 1$,

$$\overline{\Omega_n} \subseteq \Omega_{n+1}, \quad \mathbb{R}^N = \cup_{n=1}^{\infty}\Omega_n, \quad p > 0 \text{ on } \partial\Omega_n. \tag{2.25}$$

Cheng and Ni (1992) prove that the maximal solution $U$ is the unique entire large solution of problem (2.1), under the additional restriction that for some $l > 2$ there exist two positive constants $C_1$, $C_2$ such that

$$C_1 p(x) \leq |x|^{-l} \leq C_2 p(x) \quad \text{for large } |x|\,. \tag{2.26}$$

Our result in the case $\Omega = \mathbb{R}^N$ is the following (Theorem 2 in Cîrstea and Rădulescu (2002*a*)).

**Theorem 2.1.7.** *Assume that $\Omega = \mathbb{R}^N$ and problem (2.1) has at least a solution. Suppose that $p$ satisfies the condition*

$$\left.\begin{array}{l} \text{There exists a sequence of smooth bounded domains } (\Omega_n)_{n\geq1}, \text{ where} \\ \overline{\Omega}_n \subset \Omega_{n+1},\ \mathbb{R}^N = \cup_{n=1}^{\infty}\Omega_n, \text{ and (2.4) holds in } \Omega_n, \text{ for any } n \geq 1. \end{array}\right\} \tag{2.27}$$

*Then there exists a maximal classical solution $U$ of (2.1).*

*If $p$ verifies the additional condition*

$$\int_0^\infty r\Phi(r)\,dr < \infty\,, \quad \text{where } \Phi(r) = \max\{p(x) :\ |x| = r\}, \tag{2.28}$$

*then $U$ is an entire large solution.*

By Remark 2.1.1, it follows that condition (2.27) (resp., (2.28)) is weaker than the assumption (2.25) (resp., (2.26)) imposed by Cheng and Ni (1992).

*Remark* 2.1.4. If $p$ is radially symmetric in $\mathbb{R}^N$ and not identically zero at infinity, then (2.27) is fulfilled.

Indeed, we can find an increasing sequence of positive numbers $(R_n)_{n\geq 1}$ such that $R_n \to \infty$ and $p > 0$ on $\partial B(0, R_n)$, for any $n \geq 1$, therefore, $(p_1)'$ is satisfied on $\Omega_n = B(0, R_n)$.

We provide below an example of $p \geq 0$ that vanishes in every neighborhood of infinity, while hypotheses (2.27) and (2.28) are fulfilled.

**Example 2.1.2.** Let $p$ be given by

$$\begin{cases} p(r) = 0 & \text{for} \quad r = |x| \in [n - 1/3, n + 1/3], \;\; n \geq 1; \\ p(r) > 0 & \text{in} \quad \mathbb{R}_+ \setminus \displaystyle\bigcup_{n=1}^{\infty} [n - 1/3, n + 1/3]; \\ p \in C^1[0, \infty) & \text{and} \quad \displaystyle\max_{r \in [n, n+1]} p(r) = \frac{2}{n^2(2n+1)}. \end{cases}$$

Of course, (2.27) is fulfilled by choosing $\Omega_n = B(0, n + 1/2)$. On the other hand, condition (2.28) is also satisfied since

$$\begin{aligned} \int_1^{\infty} r\Phi(r)\,dr &= \sum_{n=1}^{\infty} \int_n^{n+1} rp(r)\,dr \\ &\leq \sum_{n=1}^{\infty} \int_n^{n+1} \frac{2}{n^2(2n+1)} r\,dr = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty. \end{aligned}$$

*Proof of Theorem* 2.1.7. By Theorem 2.1.1, the boundary value problem

$$\begin{cases} \Delta v_n = p(x) f(v_n), & \text{in } \Omega_n \\ v_n(x) \to \infty\,, & \text{as } x \to \partial\Omega_n \\ v_n > 0\,, & \text{in } \Omega_n \end{cases} \tag{2.29}$$

has a solution. Since $\overline{\Omega}_n \subset \Omega_{n+1}$ we can apply, for each $n \geq 1$, the maximum principle (in the same manner as in the uniqueness proof of Theorem 2.1.2) to find $v_n \geq v_{n+1}$ in $\Omega_n$. Since $\mathbb{R}^N = \cup_{n=1}^{\infty} \Omega_n$ and $\overline{\Omega_n} \subset \Omega_{n+1}$, it follows that for every $x_0 \in \mathbb{R}^N$ there exists $n_0 = n_0(x_0)$ such that $x_0 \in \Omega_n$ for all $n \geq n_0$. By the monotonicity of the sequence $(v_n(x_0))_{n\geq n_0}$ we can define $U(x_0) = \lim_{n\to\infty} v_n(x_0)$. By applying the standard bootstrap argument (see Theorem 1 in Lair and Shaker (1996)) we find that $U \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}^N)$ and $\Delta U = p(x)f(U)$ in $\Omega$.

We now prove that $U$ is the maximal solution of problem (2.1). Indeed, let $u$ be an arbitrary solution of (2.1). Applying again the maximum principle we obtain $v_n \geq u$ in $\Omega_n$, for all $n \geq 1$. By the definition of $U$, we have $U \geq u$ in $\mathbb{R}^N$.

We point out that $U$ is independent of the choice of the sequence of domains $\Omega_n$ and the number of solutions of problem (2.29). This follows easily by the uniqueness of the maximal solution.

We suppose, in addition, that $p$ satisfies (2.28) and we shall prove that $U$ blows-up at infinity. To this aim, it is sufficient to find a positive function $w \in C(\mathbb{R}^N)$ such that $U \geq w$ in $\mathbb{R}^N$ and $w(x) \to \infty$ as $|x| \to \infty$.

We first observe that (2.28) implies that

$$K = \int_0^\infty r^{1-N}\left(\int_0^r \sigma^{N-1}\Phi(\sigma)\,d\sigma\right)dr < \infty. \tag{2.30}$$

Indeed, for all $R > 0$ we have

$$\begin{aligned}\int_0^R r^{1-N}\left(\int_0^r \sigma^{N-1}\Phi(\sigma)\,d\sigma\right)dr &= \frac{1}{2-N}\int_0^R \frac{d}{dr}(r^{2-N})\left(\int_0^r \sigma^{N-1}\Phi(\sigma)\,d\sigma\right)dr\\ &= \frac{R^{2-N}}{2-N}\int_0^R \sigma^{N-1}\Phi(\sigma)\,d\sigma\\ &\quad - \frac{1}{2-N}\int_0^R r\Phi(r)\,dr\\ &\leq \frac{1}{N-2}\int_0^\infty r\Phi(r)\,dr < \infty.\end{aligned}$$

Using (2.30) and the maximum principle we obtain that the problem

$$\begin{cases} -\Delta z = \Phi(r), & r = |x| < \infty,\\ z(x) \to 0 & \text{as } |x| \to \infty\end{cases}$$

has a unique positive radial solution, which is given by

$$z(r) = K - \int_0^r \sigma^{1-N}\left(\int_0^\sigma \tau^{N-1}\Phi(\tau)\,d\tau\right)d\sigma, \qquad \forall r \geq 0.$$

Let $w$ be the positive function defined implicitly by

$$z(x) = \int_{w(x)}^{\infty} \frac{dt}{f(t)}, \qquad \forall x \in \mathbb{R}^N. \tag{2.31}$$

Assumption (2.2) and L'Hospital rule yield

$$\lim_{t \searrow 0} \frac{f(t)}{t} = \lim_{t \searrow 0} f'(t) = f'(0) \in [0, \infty)\,,$$

which implies the existence of some $\delta > 0$ such that

$$\frac{f(t)}{t} < f'(0) + 1 \quad \text{for all } 0 < t < \delta.$$

Thus, for every $s \in (0, \delta)$, we have

$$\int_s^{\delta} \frac{dt}{f(t)} > \frac{1}{f'(0)+1} \int_s^{\delta} \frac{dt}{t} = \frac{1}{f'(0)+1} (\ln \delta - \ln s).$$

It follows that $\lim_{s \searrow 0} \int_s^{\delta} \frac{dt}{f(t)} = \infty$, which provides the possibility to define $w$ as in (2.31).

We claim that $w \leq v_n$ in $\Omega_n$ for all $n \geq 1$. Obviously this inequality is true on $\partial\Omega_n$. Using the same arguments as in the proof of the inequality (2.10) (with $\Omega$ replaced by $\Omega_n$) we obtain that for any $\varepsilon > 0$ and $n \geq 1$ we have

$$w(x) \leq v_n(x) + \varepsilon(1 + |x|^2)^{-1/2} \qquad \text{in } \Omega_n$$

and the claim follows. Consequently, $U \geq w$ in $\mathbb{R}^N$ and, by (2.31), $w(x) \to \infty$ as $|x| \to \infty$. This completes the proof of Theorem 2.1.7. □

**Corollary 2.1.8.** *Let $\Omega \equiv \mathbb{R}^N$. Assume that $p$ is radially symmetric in $\mathbb{R}^N$, not identically zero at infinity such that (2.28) is fulfilled, then (2.1) has a radial entire large solution.*

*Proof.* By Remark 2.1.4 and Corollary 2.1.6, the entire large solution constructed as in the proof of Theorem 2.1.7 will be radially symmetric. □

#### 2.1.4.1 A Necessary Condition

By Lemma 2.1.5 we know that if (2.2) and (2.3) are satisfied then (2.12) holds. We establish here that (2.12) is a necessary condition for the existence of entire large solutions to (2.1) when $p$ satisfies (2.28) (cf. Theorem 4 in Cîrstea and Rădulescu (2002*a*)). Note that $f$ is not assumed to satisfy (2.3), while the regularity we impose on $p$ is weaker than before.

**Theorem 2.1.9.** *Assume that $p \in C(\mathbb{R}^N)$ is a non-negative and non-trivial function which fulfills (2.28). Let $f$ be a function satisfying (2.2), then condition*

$$\int_1^\infty \frac{dt}{f(t)} < \infty \tag{2.32}$$

*is necessary for the existence of entire large solutions to (2.1).*

*Proof.* Let $u$ be an entire large solution of problem (2.1). Define

$$\bar{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \left( \int_a^{u(x)} \frac{dt}{f(t)} \right) dS = \frac{1}{\omega_N} \int_{|\xi|=1} \left( \int_a^{u(r\xi)} \frac{dt}{f(t)} \right) dS,$$

where $\omega_N$ denotes the surface area of the unit sphere in $\mathbb{R}^N$ and $a$ is chosen such that $a \in (0, u_0)$, where $u_0 = \inf_{\mathbb{R}^N} u > 0$. By the divergence theorem we have

$$\begin{aligned}
\bar{u}'(r) &= \frac{1}{\omega_N} \int_{|\xi|=1} \frac{1}{f(u(r\xi))} \nabla u(r\xi) \cdot \xi \, dS = \frac{1}{\omega_N r^N} \int_{|y|=r} \frac{1}{f(u(y))} \nabla u(y) \cdot y \, dS \\
&= \frac{1}{\omega_N r^N} \int_{|y|=r} \nabla \left( \int_a^{u(y)} \frac{dt}{f(t)} \right) \cdot y \, dS = \frac{1}{\omega_N r^{N-1}} \int_{|y|=r} \frac{\partial}{\partial \nu} \left( \int_a^{u(y)} \frac{dt}{f(t)} \right) dS \\
&= \frac{1}{\omega_N r^{N-1}} \int_{B(0,r)} \Delta \left( \int_a^{u(x)} \frac{dt}{f(t)} \right) dx.
\end{aligned}$$

Since $u$ is a positive classical solution it follows that

$$|\bar{u}'(r)| \le Cr \to 0 \qquad \text{as } r \to 0\,.$$

On the other hand, we have

$$\begin{aligned}
\omega_N \left( R^{N-1} \bar{u}'(R) - r^{N-1} \bar{u}'(r) \right) &= \int_D \Delta \left( \int_a^{u(x)} \frac{1}{f(t)} \, dt \right) dx \\
&= \int_r^R \left( \int_{|x|=z} \Delta \left( \int_a^{u(x)} \frac{dt}{f(t)} \right) dS \right) dz,
\end{aligned}$$

where $D := \{x \in \mathbb{R}^N : \; r < |x| < R\}$. Dividing by $R - r$ and letting $R \to r$, we arrive at

$$\begin{aligned}\omega_N(r^{N-1}\bar{u}'(r))' &= \int_{|x|=r} \Delta\left(\int_a^{u(x)} \frac{dt}{f(t)}\right) dS = \int_{|x|=r} \operatorname{div}\left(\frac{1}{f(u(x))}\nabla u(x)\right) dS \\ &= \int_{|x|=r} \left[\left(\frac{1}{f}\right)'(u(x)) \cdot |\nabla u(x)|^2 + \frac{1}{f(u(x))}\Delta u(x)\right] dS \\ &\le \int_{|x|=r} \frac{p(x)f(u(x))}{f(u(x))}\, dS \le \omega_N r^{N-1}\Phi(r).\end{aligned}$$

Integrating the above inequality, we get

$$\bar{u}(r) \le \bar{u}(0) + \int_0^r \sigma^{1-N}\left(\int_0^\sigma \tau^{N-1}\Phi(\tau)\, d\tau\right) d\sigma \qquad \forall r \ge 0. \tag{2.33}$$

Since (2.28) implies (2.30), we have

$$\bar{u}(r) \le \bar{u}(0) + K \qquad \forall r \ge 0.$$

Thus $\bar{u}$ is bounded and assuming that (2.32) is not fulfilled it follows that $u$ cannot be a large solution. □

### 2.1.5 Existence Results on Unbounded Domains $\Omega \neq \mathbb{R}^N$

We now consider the case in which $\Omega$ is unbounded, but $\Omega \neq \mathbb{R}^N$; we say that a large solution $u$ of (2.1) is *regular* if $u$ tends to zero at infinity. Theorem 3.1 of Marcus (1992) gives the existence of regular large solutions to problem (2.1) by assuming that there exist $\gamma > 1$ and $\beta > 0$ such that

$$\liminf_{t \to 0} f(t)t^{-\gamma} > 0 \qquad \text{and} \qquad \liminf_{|x| \to \infty} p(x)|x|^\beta > 0.$$

The large solution constructed in Marcus (1992) is the smallest large solution of problem (2.1). In the next result we show that problem (2.1) admits a maximal classical solution $U$ and that $U$ blows-up at infinity if $\Omega = \mathbb{R}^N \setminus \overline{B(0,R)}$ (Theorem 3 in Cîrstea and Rădulescu (2002*a*)).

**Theorem 2.1.10.** *Suppose that $\Omega \neq \mathbb{R}^N$ is unbounded and that problem* (2.1) *has at least a solution. Assume that $p$ satisfies condition* (2.27) *in $\Omega$, then there exists a maximal classical solution $U$ of problem* (2.1).

*If* $\Omega = \mathbb{R}^N \setminus \overline{B(0,R)}$ *and* $p$ *satisfies the additional condition* (2.28)*, with* $\Phi(r) = 0$ *for* $r \in [0,R]$*, then the maximal solution* $U$ *is a large solution that blows-up at infinity.*

*Remark* 2.1.5. By Theorem 2.1.10 and the result of Marcus, in the case $\Omega = \mathbb{R}^N \setminus \overline{B(0,R)}$, problem (2.1) admits large solutions tending to zero or to infinity as $|x| \to \infty$ (*regular* or *normal* large solutions).

*Proof of Theorem* 2.1.10*.* We argue in a similar manner as in the proof of Theorem 2.1.7, but with some changes due to the fact that $\Omega \neq \mathbb{R}^N$.

Let $(\Omega_n)_{n\geq 1}$ be the sequence of bounded smooth domains given by condition (2.27). For $n \geq 1$ fixed, let $v_n$ be a positive solution of problem (2.29) and recall that $v_n \geq v_{n+1}$ in $\Omega_n$. Set $U(x) = \lim_{n\to\infty} v_n(x)$, for every $x \in \Omega$. With the same arguments as in Theorem 2.1.7 we find $U$ is a classical solution to (2.1) and $U$ is the maximal solution. Hence the first part of Theorem 2.1.10 is proved.

For the second part, in which $\Omega = \mathbb{R}^N \setminus \overline{B(0,R)}$, we suppose (2.28) with $\Phi(r) = 0$ for $r \in [0,R]$. To prove that $U$ is a normal large solution it is enough to show the existence of a positive function $w \in C(\mathbb{R}^N \setminus \overline{B(0,R)})$ such that

$$\begin{cases} w \leq U & \text{in } \mathbb{R}^N \setminus \overline{B(0,R)}, \\ w(x) \to \infty & \text{as } |x| \to \infty \ \text{ and as } |x| \searrow R. \end{cases}$$

We proceed as in the proof of Theorem 2.1.7, but $z$ is now the unique positive radial solution of

$$\begin{cases} -\Delta z = \Phi(r), & \text{if } |x| = r > R \\ z(x) \to 0 & \text{as } |x| \to \infty \\ z(x) \to 0 & \text{as } |x| \searrow R. \end{cases}$$

The uniqueness of $z$ follows by the maximum principle. Moreover,

$$\begin{aligned} z(r) = \left( \frac{1}{R^{N-2}} - \frac{1}{r^{N-2}} \right) \int_R^\infty \sigma^{1-N} \left( \int_0^\sigma \tau^{N-1}\Phi(\tau)\, d\tau \right) d\sigma \\ - \frac{1}{R^{N-2}} \int_R^r \sigma^{1-N} \left( \int_0^\sigma \tau^{N-1}\Phi(\tau)\, d\tau \right) d\sigma. \end{aligned}$$

This completes the proof. □

## 2.2 Entire Large Solutions for Elliptic Systems

The results of this section are included in Cîrstea and Rădulescu (2002*b*).

### 2.2.1 Introduction

Consider the following semilinear elliptic system

$$\begin{cases} \Delta u = p(x)g(v) & \text{in } \mathbb{R}^N, \\ \Delta v = q(x)f(u) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.34}$$

where $N \geq 3$ and $p, q \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$ $(0 < \alpha < 1)$ are non-negative and radially symmetric functions. Throughout section 2.2 we assume that $f, g \in C^{0,\beta}_{\text{loc}}[0,\infty)$ $(0 < \beta < 1)$ are positive and non-decreasing on $(0,\infty)$.

We are concerned here with the existence of positive *entire large solutions* of (2.34), that is positive classical solutions which satisfy $u(x) \to \infty$ and $v(x) \to \infty$ as $|x| \to \infty$. Set $\mathbb{R}^+ = (0,\infty)$ and define

$$\mathcal{G} = \left\{ (a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ \;\middle|\; \begin{array}{l} (\exists) \text{ an entire radial solution of (2.34)} \\ \text{so that } (u(0), v(0)) = (a,b) \end{array} \right\}. \tag{2.35}$$

The case of pure powers in the nonlinearities was treated by Lair and Shaker (2000). They proved that $\mathcal{G} = \mathbb{R}^+ \times \mathbb{R}^+$ if $f(t) = t^\gamma$ and $g(t) = t^\theta$ for $t \geq 0$ with $0 < \gamma, \theta \leq 1$. Moreover, they established that all positive entire radial solutions of (2.34) are *large* provided that

$$\int_0^\infty tp(t)\, dt = \infty, \qquad \int_0^\infty tq(t)\, dt = \infty. \tag{2.36}$$

If, in turn

$$\int_0^\infty tp(t)\, dt < \infty, \qquad \int_0^\infty tq(t)\, dt < \infty \tag{2.37}$$

then all positive entire radial solutions of (2.34) are *bounded*.

When both $\gamma$ and $\theta$ are greater than 1 and (2.37) is satisfied, then the structure of $\mathcal{G}$ changes from $\mathbb{R}^+ \times \mathbb{R}^+$ to a closed bounded convex subset of $\mathbb{R}^+ \times \mathbb{R}^+$ (cf. Lair and Shaker (2000)). The existence of entire large solutions is also proven.

## 2.2.2 Main Results

Our purpose is to generalize the above results to a larger class of systems.

We establish whether the entire radial solutions of (2.34) are *bounded* or *large* based on the behavior of $f$ and $g$ at infinity, combined with (2.36) or (2.37). In Theorem 2.2.1 we analyse the case when $f$ and $g$ are related by (2.38), while in Theorem 2.2.2 we assume that $f, g \in C^1[0,\infty)$ satisfy (2.39) and (2.40).

### 2.2.2.1 First Scenario: Theorem 2.2.1

We first consider the situation that $g \circ f$ has a sub-linear growth and obtain the following (Theorem 1 in Cîrstea and Rădulescu (2002*b*)).

**Theorem 2.2.1.** *Assume that*

$$\lim_{t\to\infty} \frac{g(cf(t))}{t} = 0 \quad \textit{for all } c > 0, \tag{2.38}$$

*then* $\mathcal{G} = \mathbb{R}^+ \times \mathbb{R}^+$. *Moreover, the following hold:*

i) *If $p$ and $q$ satisfy (2.36), then all positive entire radial solutions of (2.34) are large.*

ii) *If $p$ and $q$ satisfy (2.37), then all positive entire radial solutions of (2.34) are bounded.*

*Furthermore, if $f, g$ are locally Lipschitz continuous on $(0,\infty)$ and $(u,v)$, $(\tilde{u},\tilde{v})$ denote two positive entire radial solutions of (2.34), then there exists a positive constant $C$ such that for all $r \in [0,\infty)$*

$$\max\{|u(r)-\tilde{u}(r)|, |v(r)-\tilde{v}(r)|\} \le C \max\{|u(0)-\tilde{u}(0)|, |v(0)-\tilde{v}(0)|\}.$$

*Remark* 2.2.1. This result improves Theorem 1 in Lair and Shaker (2000), where $f(t) = t^\gamma$ and $g(t) = t^\theta$ for $t \ge 0$ with $0 < \gamma, \theta < 1$. Note that even in this case (2.38) is more relaxed, as it allows a combination of super-linear and sub-linear powers in the nonlinearities as long as $\gamma\theta < 1$.

#### 2.2.2.2 Second Scenario: Theorem 2.2.2

If $f$ and $g$ satisfy the stronger regularity $f, g \in C^1[0,\infty)$, then we drop the assumption (2.38) and require, in turn,

$$f(0) = g(0) = 0, \ \liminf_{u\to\infty} \frac{f(u)}{g(u)} =: \sigma > 0 \tag{2.39}$$

and the Keller–Osserman condition

$$\int_1^\infty \frac{dt}{\sqrt{G(t)}} < \infty, \ \text{where } G(t) = \int_0^t g(s)\, ds. \tag{2.40}$$

By (2.39) and (2.40) we see that $f$ satisfies condition (2.40), too.

For the significance of (2.40) in the study of large solutions to the scalar case, we refer to §2.1.3.

Set $\eta = \min\{p, q\}$. If $\eta$ is not identically zero at infinity and assumption (2.37) holds, then we prove

**Property 1.** $\mathcal{G} \neq \emptyset$ (see Lemma 2.2.5).

**Property 2.** $\mathcal{G}$ is *bounded* (see Lemma 2.2.7).

**Property 3.** $F(\mathcal{G}) \subset \mathcal{G}$ (see Lemma 2.2.8), where

$$F(\mathcal{G}) = \{(a,b) \in \partial\mathcal{G} \mid a > 0 \text{ and } b > 0\}.$$

For $(c,d) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}$, define

$$R_{c,d} = \sup\left\{ r > 0 \;\middle|\; \begin{array}{l} (\exists) \text{ a radial solution of (2.34) in } B(0,r) \\ \text{so that } (u(0), v(0)) = (c,d) \end{array} \right\}. \tag{2.41}$$

**Property 4.** $0 < R_{c,d} < \infty$ provided that $\nu = \max\{p(0), q(0)\} > 0$ (see Lemma 2.2.9).

Our main result in this case is (see Theorem 2 in Cîrstea and Rădulescu (2002*b*)):

**Theorem 2.2.2.** *Let $f, g \in C^1[0,\infty)$ satisfy (2.39) and (2.40). Assume that (2.37) holds, $\eta$ is not identically zero at infinity and $\nu > 0$, then any entire radial solution $(u, v)$ of (2.34) with $(u(0), v(0)) \in F(\mathcal{G})$ is large.*

*Remark* 2.2.2. We generalize Theorem 3 in Lair and Shaker (2000), where $f(t) = t^\gamma$ and $g(t) = t^\theta$ with $\gamma, \theta > 1$.

*Remark* 2.2.3. We point out that the behavior of $f$ and $g$ at infinity plays a crucial role in the existence of entire large solutions of (2.34). More precisely, if (2.37) holds then $\mathcal{G}$, defined by (2.35), is $\mathbb{R}^+ \times \mathbb{R}^+$ in the framework of Theorem 2.2.1, where all entire radial solutions are *bounded*, in contrast to Theorem 2.2.2 where $\mathcal{G}$ is *bounded* and the existence of entire large solutions is ensured.

The rest of the section 2.2 is organized as follows. In §2.2.3 we present some auxiliary results. Subsection 2.2.4 comprises the proof of Theorem 2.2.1, while §2.2.5 gives the proof of Theorem 2.2.2 along with that of Properties 1–4.

### 2.2.3 Preliminary Results

A condition equivalent to (2.36) is given below (cf. Lemma 2 in Cîrstea and Rădulescu (2002*b*)).

**Lemma 2.2.3.** *Condition* (2.36) *holds if and only if*

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = \infty$$

*where*

$$A(r) \equiv \int_0^r t^{1-N} \int_0^t s^{N-1} p(s)\, ds\, dt,$$
$$B(r) \equiv \int_0^r t^{1-N} \int_0^t s^{N-1} q(s)\, ds\, dt, \qquad \forall r > 0.$$

*Proof.* Indeed, for any $r > 0$

$$\begin{aligned} A(r) &= \frac{1}{N-2}\left[\int_0^r tp(t)\, dt - \frac{1}{r^{N-2}}\int_0^r t^{N-1}p(t)\, dt\right] \\ &\le \frac{1}{N-2}\int_0^r tp(t)\, dt. \end{aligned} \tag{2.42}$$

On the other hand,

$$\begin{aligned} \int_0^r tp(t)\, dt - \frac{1}{r^{N-2}}\int_0^r t^{N-1}p(t)\, dt &= \frac{1}{r^{N-2}}\int_0^r \left(r^{N-2} - t^{N-2}\right) tp(t)\, dt \\ &\ge \frac{1}{r^{N-2}}\left[r^{N-2} - \left(\frac{r}{2}\right)^{N-2}\right]\int_0^{r/2} tp(t)\, dt. \end{aligned}$$

This combined with (2.42) yields

$$\frac{1}{N-2}\int_0^r tp(t)\,dt \geq A(r) \geq \frac{1}{N-2}\left[1-\left(\frac{1}{2}\right)^{N-2}\right]\int_0^{r/2} tp(t)\,dt.$$

Our conclusion follows now by letting $r \to \infty$. □

If (2.37) holds, then a continuous dependence on the initial data is valid for bounded positive entire radial solutions of (2.34) (Lemma 3 in Cîrstea and Rădulescu (2002*b*)).

**Lemma 2.2.4.** *Assume that condition* (2.37) *holds. Let $f$ and $g$ be locally Lipschitz continuous functions on* $(0,\infty)$*. If* $(u,v)$ *and* $(\tilde{u},\tilde{v})$ *denote two bounded positive entire radial solutions of* (2.34)*, then there exists a positive constant $C$ such that for all* $r \in [0,\infty)$

$$\max\{|u(r)-\tilde{u}(r)|, |v(r)-\tilde{v}(r)|\} \leq C\, \max\{|u(0)-\tilde{u}(0)|, |v(0)-\tilde{v}(0)|\}.$$

*Proof.* We first see that radial solutions of (2.34) are solutions of the ordinary differential equations system

$$\begin{cases} u''(r) + \dfrac{N-1}{r}u'(r) = p(r)\,g(v(r)), & r > 0 \\ v''(r) + \dfrac{N-1}{r}v'(r) = q(r)\,f(u(r)), & r > 0. \end{cases} \tag{2.43}$$

Define $K = \max\{|u(0)-\tilde{u}(0)|, |v(0)-\tilde{v}(0)|\}$. Integrating the first equation of (2.43), we get

$$u'(r) - \tilde{u}'(r) = r^{1-N}\int_0^r s^{N-1}p(s)(g(v(s)) - g(\tilde{v}(s)))\,ds.$$

Hence

$$|u(r)-\tilde{u}(r)| \leq K + \int_0^r t^{1-N}\int_0^t s^{N-1}p(s)|g(v(s)) - g(\tilde{v}(s))|\,ds\,dt. \tag{2.44}$$

Since $(u,v)$ and $(\tilde{u},\tilde{v})$ are bounded entire radial solutions of (2.34) we have

$$\begin{aligned} |g(v(r)) - g(\tilde{v}(r))| &\leq m|v(r)-\tilde{v}(r)| \qquad \text{for any } r \in [0,\infty) \\ |f(u(r)) - f(\tilde{u}(r))| &\leq m|u(r)-\tilde{u}(r)| \qquad \text{for any } r \in [0,\infty), \end{aligned}$$

where $m$ denotes a Lipschitz constant for both functions $f$ and $g$. Therefore, using (2.44) we find

$$|u(r) - \tilde{u}(r)| \leq K + m \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) |v(s) - \tilde{v}(s)| \, ds \, dt. \tag{2.45}$$

Arguing as above, but now with the second equation of (2.43), we obtain

$$|v(r) - \tilde{v}(r)| \leq K + m \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) |u(s) - \tilde{u}(s)| \, ds \, dt. \tag{2.46}$$

Define

$$X(r) = K + m \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) |v(s) - \tilde{v}(s)| \, ds \, dt.$$

$$Y(r) = K + m \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) |u(s) - \tilde{u}(s)| \, ds \, dt.$$

It is clear that $X$ and $Y$ are non-decreasing functions with $X(0) = Y(0) = K$. By a simple calculation together with (2.45) and (2.46) we obtain

$$\begin{aligned} (r^{N-1} X')'(r) &= m r^{N-1} p(r) |v(r) - \tilde{v}(r)| \leq m r^{N-1} p(r) Y(r) \\ (r^{N-1} Y')'(r) &= m r^{N-1} q(r) |u(r) - \tilde{u}(r)| \leq m r^{N-1} q(r) X(r). \end{aligned} \tag{2.47}$$

Since $Y$ is non-decreasing, we have

$$\begin{aligned} X(r) &\leq K + m Y(r) A(r) \leq K + \frac{m}{N-2} Y(r) \int_0^r t p(t) \, dt \\ &\leq K + m C_p Y(r) \end{aligned} \tag{2.48}$$

where $C_p = (1/(N-2)) \int_0^\infty t p(t) \, dt$. Using (2.48) in the second inequality of (2.47) we find

$$(r^{N-1} Y')'(r) \leq m r^{N-1} q(r) (K + m C_p Y(r)).$$

Integrating twice this inequality from 0 to $r$, we obtain

$$Y(r) \leq K(1 + m C_q) + \frac{m^2}{N-2} C_p \int_0^r t q(t) Y(t) \, dt,$$

where $C_q = (1/(N-2)) \int_0^\infty t q(t) \, dt$. From Gronwall's inequality, we deduce

$$Y(r) \leq K(1 + m C_q) e^{\frac{m^2}{N-2} C_p \int_0^r t q(t) \, dt} \leq K(1 + m C_q) e^{m^2 C_p C_q}$$

and similarly for $X$. The conclusion now follows from the above inequality, (2.45) and (2.46). □

### 2.2.4 Proof of Theorem 2.2.1

Since the radial solutions of (2.34) are solutions of the ordinary differential equations system (2.43) it follows that the radial solutions of (2.34) with $u(0) = a > 0$, $v(0) = b > 0$ satisfy

$$u(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s)\, g(v(s))\, ds\, dt, \qquad r \geq 0. \tag{2.49}$$

$$v(r) = b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s)\, f(u(s))\, ds\, dt, \qquad r \geq 0. \tag{2.50}$$

Define $v_0(r) = b$ for all $r \geq 0$. Let $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ be two sequences of functions given by

$$u_k(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s)\, g(v_{k-1}(s))\, ds\, dt, \qquad r \geq 0.$$

$$v_k(r) = b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s)\, f(u_k(s))\, ds\, dt, \qquad r \geq 0.$$

Since $v_1(r) \geq b$, we find $u_2(r) \geq u_1(r)$ for all $r \geq 0$. This implies $v_2(r) \geq v_1(r)$ which further produces $u_3(r) \geq u_2(r)$ for all $r \geq 0$. Proceeding at the same manner we conclude that

$$u_k(r) \leq u_{k+1}(r) \quad \text{and} \quad v_k(r) \leq v_{k+1}(r), \qquad \forall r \geq 0 \text{ and } k \geq 1.$$

We now prove that the non-decreasing sequences $(u_k(r))_{k\geq 1}$ and $(v_k(r))_{k\geq 1}$ are bounded from above on bounded sets. Indeed, we have

$$u_k(r) \leq u_{k+1}(r) \leq a + g(v_k(r))A(r), \qquad \forall r \geq 0 \tag{2.51}$$

and

$$v_k(r) \leq b + f\left(u_k(r)\right) B(r), \qquad \forall r \geq 0. \tag{2.52}$$

Let $R > 0$ be arbitrary. By (2.51) and (2.52) we find

$$u_k(R) \leq a + g\left(b + f\left(u_k(R)\right) B(R)\right) A(R), \qquad \forall k \geq 1$$

or, equivalently,

$$1 \leq \frac{a}{u_k(R)} + \frac{g\left(b + f\left(u_k(R)\right) B(R)\right)}{u_k(R)} A(R), \qquad \forall k \geq 1. \tag{2.53}$$

By the monotonicity of $(u_k(R))_{k\geq 1}$, there exists $\lim_{k\to\infty} u_k(R) := L(R)$. We claim that $L(R)$ is finite. Assume the contrary, then, by taking $k \to \infty$ in (2.53) and using (2.38) we obtain a contradiction. Since $u_k'(r),\ v_k'(r) \geq 0$ we get that the map $(0,\infty) \ni R \to L(R)$ is non-decreasing on $(0,\infty)$ and

$$u_k(r) \leq u_k(R) \leq L(R), \qquad \forall r \in [0,R],\ \forall k \geq 1. \tag{2.54}$$

$$v_k(r) \leq b + f\left(L(R)\right) B(R), \qquad \forall r \in [0,R],\ \forall k \geq 1. \tag{2.55}$$

It follows that there exists $\lim_{R\to\infty} L(R) = \overline{L} \in (0,\infty]$ and the sequences $(u_k(r))_{k\geq 1}$, $(v_k(r))_{k\geq 1}$ are bounded above on bounded sets. Thus, we can define

$$u(r) := \lim_{k\to\infty} u_k(r) \quad \text{and} \quad v(r) := \lim_{k\to\infty} v_k(r) \text{ for all } r \geq 0.$$

By standard elliptic regularity theory we obtain that $(u,v)$ is a positive entire solution of (2.34) with $u(0) = a$ and $v(0) = b$.

We now assume that, in addition, condition (2.37) is fulfilled. According to Lemma 2.2.3 we have that $\lim_{r\to\infty} A(r) = \overline{A} < \infty$ and $\lim_{r\to\infty} B(r) = \overline{B} < \infty$. Passing to the limit as $k \to \infty$ in (2.53) we find

$$1 \leq \frac{a}{L(R)} + \frac{g\left(b + f\left(L(R)\right) B(R)\right)}{L(R)} A(R) \leq \frac{a}{L(R)} + \frac{g(b + f\left(L(R)\right) \overline{B})}{L(R)} \overline{A}.$$

Letting $R \to \infty$ and using (2.38) we deduce $\overline{L} < \infty$. Thus, taking into account (2.54) and (2.55), we obtain

$$u_k(r) \leq \overline{L} \quad \text{and} \quad v_k(r) \leq b + f(\overline{L})\overline{B}, \qquad \forall r \geq 0,\ \forall k \geq 1.$$

So, we have found upper bounds for $(u_k(r))_{k\geq 1}$ and $(v_k(r))_{k\geq 1}$ which are independent of $r$. Thus, the solution $(u,v)$ is bounded from above. This shows that any solution of (2.49) and (2.50) will be bounded from above provided (2.37) holds. Thus, we can apply Lemma 2.2.4 to achieve the second assertion of ii).

Let us now drop the condition (2.37) and assume that (2.36) is fulfilled. In this case, Lemma 2.2.3 tells us that $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = \infty$. Let $(u,v)$ be an entire positive radial solution of (2.34). Using (2.49) and (2.50) we obtain

$$u(r) \geq a + g(b)\, A(r), \qquad \forall r \geq 0.$$

$$v(r) \geq b + f(a)\, B(r), \qquad \forall r \geq 0.$$

Taking $r \to \infty$ we get that $(u,v)$ is an entire large solution. This concludes the proof of Theorem 2.2.1. $\square$

### Examples

We now give some examples of nonlinearities $f$ and $g$ which satisfy the assumptions of Theorem 2.2.1 (see Dalmasso (2000)).

1. Let
$$f(t) = \sum_{j=1}^{l} a_j t^{\gamma_j}, \quad g(t) = \sum_{k=1}^{m} b_k t^{\theta_j} \quad \text{for } t > 0$$
with $a_j$, $b_k$, $\gamma_j$, $\theta_k > 0$ and $f(t) = g(t) = 0$ for $t \le 0$. Assume that $\gamma\theta < 1$, where
$$\gamma = \max_{1\le j\le l} \gamma_j, \quad \theta = \max_{1\le k\le m} \theta_k.$$

2. Let
$$f(t) = (1+t^2)^{\gamma/2} \quad \text{and} \quad g(t) = (1+t^2)^{\theta/2} \quad \text{for } t \in \mathbb{R}$$
with $\gamma$, $\theta > 0$ and $\gamma\theta < 1$.

3. Let
$$f(t) = \begin{cases} t^\gamma & \text{if } 0 \le t \le 1, \\ t^\theta & \text{if } t \ge 1, \end{cases}$$
and
$$g(t) = \begin{cases} t^\theta & \text{if } 0 \le t \le 1, \\ t^\gamma & \text{if } t \ge 1, \end{cases}$$
with $\gamma$, $\theta > 0$, $\gamma\theta < 1$ and $f(t) = g(t) = 0$ for $t \le 0$.

4. Let $g(t) = t$ for $t \in \mathbb{R}$, $f(t) = 0$ for $t \le 0$ and
$$f(t) = t\left(-\ln\left(\left(\frac{2}{\pi}\right)\arctan t\right)\right)^\gamma \quad \text{for } t > 0$$
where $\gamma \in (0, 1/2)$.

## 2.2.5 Proof of Theorem 2.2.2

Let $f, g \in C^1[0,\infty)$ satisfy (2.39) and (2.40). Suppose that $\eta$ is not identically zero at infinity and (2.37) holds. We first give the proofs of Properties 1–4 (see Lemmas 4–7 in Cîrstea and Rădulescu (2002*b*)) which are the main tools used to deduce Theorem 2.2.2.

**Lemma 2.2.5.** $\mathcal{G} \neq \emptyset$.

*Proof.* By Corollary 2.1.8, the problem

$$\Delta\psi = (p+q)(x)(f+g)(\psi) \quad \text{in } \mathbb{R}^N,$$

has a positive radial entire large solution. Since $\psi$ is radial, we have

$$\psi(r) = \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}(p+q)(s)(f+g)(\psi(s))\, ds\, dt, \quad \forall r \geq 0.$$

We claim that $(0, \psi(0)] \times (0, \psi(0)] \subseteq \mathcal{G}$. To prove this, fix $0 < a, b \leq \psi(0)$ and let $v_0(r) \equiv b$ for all $r \geq 0$. Define the sequences $(u_k)_{k\geq 1}$ and $(v_k)_{k\geq 1}$ by

$$u_k(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(v_{k-1}(s))\, ds\, dt, \quad \forall r \in [0, \infty), \quad \forall k \geq 1, \tag{2.56}$$

$$v_k(r) = b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(u_k(s))\, ds\, dt, \quad \forall r \in [0, \infty), \quad \forall k \geq 1. \tag{2.57}$$

We first see that $v_0 \leq v_1$ which produces $u_1 \leq u_2$. Consequently, $v_1 \leq v_2$ which further yields $u_2 \leq u_3$. With the same arguments, we obtain that $(u_k)$ and $(v_k)$ are non-decreasing sequences. Since $\psi'(r) \geq 0$ and $b = v_0 \leq \psi(0) \leq \psi(r)$ for all $r \geq 0$ we find

$$\begin{aligned} u_1(r) &\leq a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(\psi(s))\, ds\, dt \\ &\leq \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}(p+q)(s)(f+g)(\psi(s))\, ds\, dt = \psi(r). \end{aligned}$$

Thus $u_1 \leq \psi$. It follows that

$$\begin{aligned} v_1(r) &\leq b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(\psi(s))\, ds\, dt \\ &\leq \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}(p+q)(s)(f+g)(\psi(s))\, ds\, dt = \psi(r). \end{aligned}$$

Similar arguments show that

$$u_k(r) \leq \psi(r) \quad \text{and} \quad v_k(r) \leq \psi(r) \quad \forall r \in [0, \infty), \quad \forall k \geq 1.$$

Thus, $(u_k)$ and $(v_k)$ converge and $(u, v) = \lim_{k\to\infty}(u_k, v_k)$ is an entire radial solution of (2.34) such that $(u(0), v(0)) = (a, b)$. This completes the proof. □

An easy consequence of the above result is

**Corollary 2.2.6.** *If* $(a, b) \in \mathcal{G}$, *then* $(0, a] \times (0, b] \subseteq \mathcal{G}$.

*Proof.* Indeed, the process used before can be repeated by taking

$$u_k(r) = a_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(v_{k-1}(s))\, ds\, dt, \quad \forall r \in [0, \infty), \quad \forall k \geq 1,$$
$$v_k(r) = b_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(u_k(s))\, ds\, dt, \quad \forall r \in [0, \infty), \quad \forall k \geq 1,$$

where $0 < a_0 \leq a$, $0 < b_0 \leq b$ and $v_0(r) \equiv b_0$ for all $r \geq 0$.

Letting $(U, V)$ be the entire radial solution of (2.34) with central values $(a, b)$ we obtain as in Lemma 2.2.5,

$$u_k(r) \leq u_{k+1}(r) \leq U(r), \quad \forall r \in [0, \infty), \quad \forall k \geq 1,$$
$$v_k(r) \leq v_{k+1}(r) \leq V(r), \quad \forall r \in [0, \infty), \quad \forall k \geq 1.$$

Set $(u, v) = \lim_{k\to\infty}(u_k, v_k)$. We see that $u \leq U$, $v \leq V$ on $[0, \infty)$ and $(u, v)$ is an entire radial solution of (2.34) with central values $(a_0, b_0)$. This shows that $(a_0, b_0) \in \mathcal{G}$, so that our assertion is proved. □

**Lemma 2.2.7.** $\mathcal{G}$ *is bounded.*

*Proof.* Set $0 < \lambda < \min\{\sigma, 1\}$ and let $\delta = \delta(\lambda)$ be large enough so that

$$f(t) \geq \lambda g(t), \qquad \forall t \geq \delta. \tag{2.58}$$

Since $\eta$ is radially symmetric and not identically zero at infinity, we can assume $\eta > 0$ on $\partial B(0, R)$ for some $R > 0$. Corollary 2.1.6 ensures the existence of a positive large solution $\zeta$ of the problem

$$\Delta\zeta = \lambda\eta(x) g\left(\frac{\zeta}{2}\right) \qquad \text{in } B(0, R).$$

Arguing by contradiction, let us assume that $\mathcal{G}$ is not bounded, then, there exists $(a, b) \in \mathcal{G}$ such that $a + b > \max\{2\delta, \zeta(0)\}$. Let $(u, v)$ be the entire radial solution of (2.34) such that $(u(0), v(0)) = (a, b)$. Since $u(x) + v(x) \geq a + b > 2\delta$ for all $x \in \mathbb{R}^N$, by (2.58), we find

$$f(u(x)) \geq f\left(\frac{u(x) + v(x)}{2}\right) \geq \lambda g\left(\frac{u(x) + v(x)}{2}\right) \qquad \text{if } u(x) \geq v(x)$$

and

$$g(v(x)) \geq g\left(\frac{u(x)+v(x)}{2}\right) \geq \lambda g\left(\frac{u(x)+v(x)}{2}\right) \qquad \text{if } v(x) \geq u(x).$$

It follows that

$$\begin{aligned}\Delta(u+v) &= p(x)g(v) + q(x)f(u) \geq \eta(x)(g(v)+f(u)) \\ &\geq \lambda\eta(x)g\left(\frac{u+v}{2}\right) \qquad \text{in } \mathbb{R}^N.\end{aligned}$$

On the other hand, $\zeta(x) \to \infty$ as $|x| \to R$ and $u, v \in C^2(\overline{B(0,R)})$. Thus, by the maximum principle, we conclude that $u+v \leq \zeta$ in $B(0,R)$. But this is impossible since $u(0)+v(0) = a+b > \zeta(0)$. □

**Lemma 2.2.8.** $F(\mathcal{G}) \subset \mathcal{G}$.

*Proof.* Let $(a,b) \in F(\mathcal{G})$. We claim that $(a-1/n_0, b-1/n_0) \in \mathcal{G}$ provided $n_0 \geq 1$ is large enough so that $\min\{a,b\} > 1/n_0$. Indeed, if this is not true, by Corollary 2.2.6

$$D := \left[a - \frac{1}{n_0}, \infty\right) \times \left[b - \frac{1}{n_0}, \infty\right) \subseteq (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}.$$

So, we can find a small ball $B$ centered in $(a,b)$ such that $B \subset\subset D$, i.e., $B \cap \mathcal{G} = \emptyset$, but this will contradict the choice of $(a,b)$. Consequently, there exists $(u_{n_0}, v_{n_0})$ an entire radial solution of (2.34) such that $(u_{n_0}(0), v_{n_0}(0)) = (a-1/n_0, b-1/n_0)$. Thus, for any $n \geq n_0$, we can define

$$\begin{aligned}u_n(r) &= a - \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(v_n(s))\, ds\, dt, \qquad r \geq 0, \\ v_n(r) &= b - \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(u_n(s))\, ds\, dt, \qquad r \geq 0.\end{aligned}$$

Using Corollary 2.2.6 once more, we conclude that $(u_n)_{n\geq n_0}$ and $(v_n)_{n\geq n_0}$ are non-decreasing sequences. We now prove that $(u_n)$ and $(v_n)$ converge on $\mathbb{R}^N$. To this aim, let $x_0 \in \mathbb{R}^N$ be arbitrary, but $\eta$ is not identically zero at infinity so that, for some $R_0 > 0$, we have $\eta > 0$ on $\partial B(0,R_0)$ and $x_0 \in B(0,R_0)$.

Since $\sigma = \liminf_{u\to\infty} \frac{f(u)}{g(u)} > 0$, we find $\tau \in (0,1)$ such that

$$f(t) \geq \tau g(t), \qquad \forall t \geq \frac{a+b}{2} - \frac{1}{n_0}.$$

Therefore, on the set where $u_n \geq v_n$, we have

$$f(u_n) \geq f\left(\frac{u_n + v_n}{2}\right) \geq \tau g\left(\frac{u_n + v_n}{2}\right).$$

Similarly, on the set where $u_n \leq v_n$, we have

$$g(v_n) \geq g\left(\frac{u_n + v_n}{2}\right) \geq \tau g\left(\frac{u_n + v_n}{2}\right).$$

It follows that, for any $x \in \mathbb{R}^N$,

$$\begin{aligned}\Delta(u_n + v_n) &= p(x)g(v_n) + q(x)f(u_n) \geq \eta(x)[g(v_n) + f(u_n)] \\ &\geq \tau\eta(x)g\left(\frac{u_n + v_n}{2}\right).\end{aligned}$$

On the other hand, by Corollary 2.1.6, there exists a positive large solution of

$$\Delta\zeta = \tau\eta(x)g\left(\frac{\zeta}{2}\right) \qquad \text{in } B(0, R_0).$$

The maximum principle yields $u_n + v_n \leq \zeta$ in $B(0, R_0)$. So, it makes sense to define $(u(x_0), v(x_0)) = \lim_{n\to\infty}(u_n(x_0), v_n(x_0))$. Since $x_0$ is arbitrary, the functions $u$, $v$ exist on $\mathbb{R}^N$. Hence $(u, v)$ is an entire radial solution of (2.34) with central values $(a, b)$, i.e., $(a, b) \in \mathcal{G}$. □

**Lemma 2.2.9.** *If, in addition,* $\nu = \max\{p(0), q(0)\} > 0$, *then* $0 < R_{c,d} < \infty$ *where* $R_{c,d}$ *is defined by* (2.41).

*Proof.* Since $\nu > 0$ and $p, q \in C[0, \infty)$, there exists $\epsilon > 0$ such that $(p+q)(r) > 0$ for all $0 \leq r < \epsilon$. Let $0 < R < \epsilon$ be arbitrary. By Corollary 2.1.6, there exists a positive radial large solution of the problem

$$\Delta\psi_R = (p + q)(x)(f + g)(\psi_R) \qquad \text{in } B(0, R).$$

Moreover, for any $0 \leq r < R$,

$$\psi_R(r) = \psi_R(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}(p + q)(s)(f + g)(\psi_R(s))\, ds\, dt.$$

It is clear that $\psi_R'(r) \geq 0$. Thus, we find

$$\psi_R'(r) = r^{1-N} \int_0^r s^{N-1}(p + q)(s)(f + g)(\psi_R(s))\, ds \leq C(f + g)(\psi_R(r))$$

where $C > 0$ is a positive constant such that $\int_0^\epsilon (p+q)(s)\,ds \le C$.

Since $f+g$ satisfies (2.2) and (2.3), we may then invoke Lemma 2.1.5 to conclude

$$\int_1^\infty \frac{dt}{(f+g)(t)} < \infty.$$

Therefore, we get

$$-\frac{d}{dr}\int_{\psi_R(r)}^\infty \frac{ds}{(f+g)(s)} = \frac{\psi_R'(r)}{(f+g)(\psi_R(r))} \le C \qquad \text{for any } 0 < r < R.$$

Integrating from 0 to $R$ and recalling that $\psi_R(r) \to \infty$ as $r \nearrow R$, we obtain

$$\int_{\psi_R(0)}^\infty \frac{ds}{(f+g)(s)} \le CR.$$

Letting $R \searrow 0$ we conclude that

$$\lim_{R\searrow 0}\int_{\psi_R(0)}^\infty \frac{ds}{(f+g)(s)} = 0.$$

This implies that $\psi_R(0) \to \infty$ as $R \searrow 0$. So, there exists $0 < \tilde{R} < \epsilon$ such that $0 < c, d \le \psi_{\tilde{R}}(0)$. Set

$$u_k(r) = c + \int_0^r t^{1-N}\int_0^t s^{N-1}p(s)g(v_{k-1}(s))\,ds\,dt, \quad \forall r \in [0,\infty),\ \forall k \ge 1 \quad (2.59)$$

$$v_k(r) = d + \int_0^r t^{1-N}\int_0^t s^{N-1}q(s)f(u_k(s))\,ds\,dt, \quad \forall r \in [0,\infty),\ \forall k \ge 1 \quad (2.60)$$

where $v_0(r) = d$ for all $r \in [0,\infty)$. As in Lemma 2.2.5, we find that $(u_k)$ resp., $(v_k)$ are non-decreasing and

$$u_k(r) \le \psi_{\tilde{R}}(r) \quad \text{and} \qquad v_k(r) \le \psi_{\tilde{R}}(r), \qquad \forall r \in [0,\tilde{R}),\ \forall k \ge 1.$$

Thus, for any $r \in [0,\tilde{R})$, there exists $(u(r), v(r)) = \lim_{k\to\infty}(u_k(r), v_k(r))$ which is, moreover, a radial solution of (2.34) in $B(0,\tilde{R})$ such that $(u(0), v(0)) = (c,d)$. This shows that $R_{c,d} \ge \tilde{R} > 0$. By the definition of $R_{c,d}$ we also derive

$$\lim_{r\nearrow R_{c,d}} u(r) = \infty \quad \text{and} \quad \lim_{r\nearrow R_{c,d}} v(r) = \infty. \qquad (2.61)$$

On the other hand, since $(c,d) \notin \mathcal{G}$, we conclude that $R_{c,d}$ is finite. □

**Proof of Theorem 2.2.2 completed.**

Let $(a, b) \in F(\mathcal{G})$ be arbitrary. By Lemma 2.2.8, $(a, b) \in \mathcal{G}$ so that we can define $(U, V)$ an entire radial solution of (2.34) with $(U(0), V(0)) = (a, b)$. Obviously, for any $n \geq 1$, $(a+1/n, b+1/n) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}$. By Lemma 2.2.9, $R_{a+1/n,b+1/n}$ (in short, $R_n$) defined by (2.41) is a positive number. Let $(U_n, V_n)$ be the radial solution of (2.34) in $B(0, R_n)$ with the central values $(a + 1/n, b + 1/n)$. Thus,

$$U_n(r) = a + \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g(V_n(s))\, ds\, dt, \qquad \forall r \in [0, R_n), \quad (2.62)$$
$$V_n(r) = b + \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f(U_n(s))\, ds\, dt, \qquad \forall r \in [0, R_n). \quad (2.63)$$

In view of (2.61) we have

$$\lim_{r \nearrow R_n} U_n(r) = \infty \quad \text{and} \quad \lim_{r \nearrow R_n} V_n(r) = \infty, \quad \forall n \geq 1.$$

We claim that $(R_n)_{n \geq 1}$ is a non-decreasing sequence. Indeed, if $(u_k)$, $(v_k)$ denote the sequences of functions defined by (2.59) and (2.60) with $c = a + 1/(n+1)$ and $d = b + 1/(n+1)$, then

$$\begin{aligned} u_k(r) \leq u_{k+1}(r) \leq U_n(r),, \qquad \forall r \in [0, R_n),\ \forall k \geq 1 \\ v_k(r) \leq v_{k+1}(r) \leq V_n(r), \qquad \forall r \in [0, R_n),\ \forall k \geq 1. \end{aligned} \quad (2.64)$$

This implies that $(u_k(r))_{k \geq 1}$ and $(v_k(r))_{k \geq 1}$ converge for any $r \in [0, R_n)$. Moreover, $(U_{n+1}, V_{n+1}) = \lim_{k \to \infty}(u_k, v_k)$ is a radial solution of (2.34) in $B(0, R_n)$ with central values $(a + 1/(n+1), b + 1/(n+1))$. By the definition of $R_{n+1}$, it follows that $R_{n+1} \geq R_n$ for any $n \geq 1$.

Set $R := \lim_{n \to \infty} R_n$ and let $0 \leq r < R$ be arbitrary, then, there exists $n_1 = n_1(r)$ such that $r < R_n$ for all $n \geq n_1$. From (2.64) we see that $U_{n+1} \leq U_n$ (resp., $V_{n+1} \leq V_n$) on $[0, R_n)$ for all $n \geq 1$. So, there exists $\lim_{n \to \infty}(U_n(r), V_n(r))$ which, by (2.62) and (2.63), is a radial solution of (2.34) in $B(0, R)$ with central values $(a, b)$. Consequently,

$$\lim_{n \to \infty} U_n(r) = U(r) \quad \text{and} \quad \lim_{n \to \infty} V_n(r) = V(r) \quad \text{for any } r \in [0, R). \qquad (2.65)$$

Since $U_n'(r) \geq 0$, from (2.63) we find

$$V_n(r) \leq b + \frac{1}{n} + f(U_n(r)) \int_0^\infty t^{1-N} \int_0^t s^{N-1} q(s)\, ds\, dt.$$

This yields

$$V_n(r) \le C_1 U_n(r) + C_2 f(U_n(r)) \tag{2.66}$$

where $C_1$ is an upper bound of $(V(0)+1/n)/(U(0)+1/n)$ and

$$C_2 = \int_0^\infty t^{1-N} \int_0^t s^{N-1} q(s)\, ds\, dt \le \frac{1}{N-2} \int_0^\infty s q(s)\, ds < \infty.$$

Define $h(t) = g(C_1 t + C_2 f(t))$ for $t \ge 0$. It is easy to check that $h$ satisfies (2.2) and (2.3). So, by Lemma 2.1.5 we can define

$$\Gamma(s) = \int_s^\infty \frac{dt}{h(t)}, \qquad \text{for all } s > 0.$$

But $U_n$ verifies

$$\Delta U_n = p(x) g(V_n)$$

which combined with (2.66) implies

$$\Delta U_n \le p(x) h(U_n).$$

A simple calculation shows that

$$\begin{aligned}\Delta\Gamma(U_n) &= \Gamma'(U_n)\Delta U_n + \Gamma''(U_n)|\nabla U_n|^2 = \frac{-1}{h(U_n)}\Delta U_n + \frac{h'(U_n)}{[h(U_n)]^2}|\nabla U_n|^2 \\ &\ge \frac{-1}{h(U_n)} p(r) h(U_n) = -p(r)\end{aligned}$$

which we rewrite as

$$\left(r^{N-1}\frac{d}{dr}\Gamma(U_n)\right)' \ge -r^{N-1}p(r) \qquad \text{for any } 0 < r < R_n.$$

Fix $0 < r < R$. Then $r < R_n$ for all $n \ge n_1$ provided $n_1$ is large enough. Integrating the above inequality over $[0, r]$, we get

$$\frac{d}{dr}\Gamma(U_n) \ge -r^{1-N} \int_0^r s^{N-1} p(s)\, ds.$$

Integrating this new inequality over $[r, R_n]$ we obtain

$$-\Gamma(U_n(r)) \ge -\int_r^{R_n} t^{1-N} \int_0^t s^{N-1} p(s)\, ds\, dt, \qquad \forall n \ge n_1,$$

since $U_n(r) \to \infty$ as $r \nearrow R_n$ implies $\Gamma(U_n(r)) \to 0$ as $r \nearrow R_n$. Therefore,

$$\Gamma(U_n(r)) \leq \int_r^{R_n} t^{1-N} \int_0^t s^{N-1} p(s)\, ds\, dt, \qquad \forall n \geq n_1.$$

Letting $n \to \infty$ and using (2.65) we find

$$\Gamma(U(r)) \leq \int_r^{R} t^{1-N} \int_0^t s^{N-1} p(s)\, ds\, dt,$$

or, equivalently

$$U(r) \geq \Gamma^{-1}\left(\int_r^{R} t^{1-N} \int_0^t s^{N-1} p(s)\, ds\, dt\right).$$

Passing to the limit as $r \nearrow R$ and using $\lim_{s \searrow 0} \Gamma^{-1}(s) = \infty$ we deduce

$$\lim_{r \nearrow R} U(r) \geq \lim_{r \nearrow R} \Gamma^{-1}\left(\int_r^{R} t^{1-N} \int_0^t s^{N-1} p(s)\, ds\, dt\right) = \infty.$$

But $(U, V)$ is an entire solution so that we conclude that $R = \infty$ and $\lim_{r\to\infty} U(r) = \infty$. Since (2.37) holds and $V'(r) \geq 0$ we find

$$\begin{aligned} U(r) &\leq a + g(V(r)) \int_0^{\infty} t^{1-N} \int_0^t s^{N-1} p(s)\, ds\, dt \\ &\leq a + g(V(r)) \frac{1}{N-2} \int_0^{\infty} t p(t)\, dt, \qquad \forall r \geq 0. \end{aligned}$$

We deduce $\lim_{r\to\infty} V(r) = \infty$, otherwise we obtain that $\lim_{r\to\infty} U(r)$ is finite, a contradiction. Consequently, $(U, V)$ is an entire large solution of (2.34). This concludes our proof. □

# Chapter 3

# Large Solutions for Logistic-type Equations: Existence

**"Man's mind stretched to a new idea never goes back to its original dimensions." (Oliver Wendell Holmes)**

The next chapters deal with the qualitative properties of the large solutions to semilinear elliptic equations of the type (2.1) on a smooth bounded domain, where a linear perturbation is introduced. We are interested in equations of the form

$$\Delta u + au = b(x)f(u) \quad \text{in } \Omega, \tag{3.1}$$

where $b(x) \geq 0$ will, in particular, satisfy (2.4). The zero set of $b(x)$ is specified to be the closure of an interior sub-domain $\Omega_0$ such that $b$ is positive on $\Omega \setminus \overline{\Omega_0}$.

The purpose of this chapter is to analyze the effect of the linear perturbation on the existence of large solutions. Section §3.1 is dedicated to the situation of a complete boundary blow-up. In §3.2 and §3.3 the definition of a large solution is given for boundary value problems to accommodate the case when blow-up occurs partially on the boundary and a Dirichlet, Neumann or Robin boundary condition arises on some other parts of the boundary of the domain. Among the new tools used in Chapter 2 distinguish some comparison principles and a result due to Alama and Tarantello (1996) about an equivalent condition for the existence of positive solutions of (3.1), subject to $u = 0$ on $\partial\Omega$. Each section is structured such that to allow, as much as possible, an independent reading. Thus, some comparison principles are given in different places in a form which serves the purpose and then refined when the need arises.

# 3.1 Pure Boundary Blow-up Problems

## 3.1.1 Introduction

Consider the semilinear elliptic equation

$$\Delta u + au = b(x)f(u) \qquad \text{in } \Omega, \tag{3.2}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 3$), $a$ is a real parameter and $b \in C^{0,\mu}(\overline{\Omega})$, $0 < \mu < 1$, satisfies $b \geq 0$ and $b \not\equiv 0$ in $\Omega$.

Let $\Omega_0$ denote the interior of the set where $b$ vanishes in $\Omega$, that is

$$\Omega_0 := \text{int}\,\{x \in \Omega : \; b(x) = 0\}. \tag{3.3}$$

Suppose, throughout §3.1, that $\Omega_0$ is connected, $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ on $\Omega \setminus \overline{\Omega}_0$.

Assume that $f \in C^1[0, \infty)$ satisfies

$$f \geq 0 \text{ and } f(u)/u \text{ is increasing on } (0, \infty). \tag{3.4}$$

Following Alama and Tarantello (1996), define by $H_\infty$ the Dirichlet Laplacian on the set $\Omega_0 \subset \Omega$ as the unique self-adjoint operator associated with the quadratic form $\xi(u) = \int_\Omega |\nabla u|^2\, dx$ with form domain

$$H^1_D(\Omega_0) = \{u \in H^1_0(\Omega) : \; u(x) = 0 \quad \text{for a.e. } x \in \Omega \setminus \Omega_0\}.$$

If $\partial\Omega_0$ satisfies an exterior cone condition, then $H^1_D(\Omega_0)$ coincides with $H^1_0(\Omega_0)$ and $H_\infty$ is the classical Laplace operator with Dirichlet condition on $\partial\Omega_0$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of $H_\infty$ in $\Omega_0$ ($\lambda_{\infty,1} = +\infty$ if $\Omega_0 = \emptyset$).

Define $\mu_0 := \lim_{u \searrow 0} f(u)/u$ and $\mu_\infty := \lim_{u \to \infty} f(u)/u$.

The results of Alama and Tarantello rely on the existence of a principal eigenvalue for the operator $-\Delta + \mu b$ in the limiting cases $\mu = \mu_0$ and $\mu = \mu_\infty$. Denote by $\lambda_1(\mu_0)$ (resp., $\lambda_1(\mu_\infty)$) the first eigenvalue of $H_{\mu_0} = -\Delta + \mu_0 b$ (resp., $H_{\mu_\infty} = -\Delta + \mu_\infty b$) in $H^1_0(\Omega)$. Recall that $\lambda_1(+\infty) = \lambda_{\infty,1}$.

Theorem A (bis) of Alama and Tarantello (1996) (see also del Pino (1994), Umezu and Taira (1999)) asserts that problem (3.2) subject to the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \tag{3.5}$$

has a positive solution $u_a$ if and only if $a \in (\lambda_1(\mu_0), \lambda_1(\mu_\infty))$. Moreover, $u_a$ is the unique positive solution for (3.2)+(3.5).

*Notation* 1. We denote by $(E_a)$ the combination of (3.2)+(3.5).

The main goal of section 3.1 is to give a corresponding necessary and sufficient condition, but for the existence of *large* (or *blow-up*) solutions of (3.2). We say that a solution $u$ of (3.2) is *large* if $u \geq 0$ in $\Omega$ and $u(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$.

### 3.1.2 Existence of Large Solutions

Recall that when $a \equiv 0$, $b \equiv 1$ and $f$ is assumed to fulfill (2.2) (which is weaker than (3.4)), then the Keller–Osserman condition (2.3), namely

$$\int_1^\infty \frac{dt}{\sqrt{2F(t)}} < \infty\,, \quad \text{where} \quad F(t) = \int_0^t f(s)\,ds \tag{3.6}$$

is necessary and sufficient for the existence of large solutions to (3.2). Moreover, Remark 3.1.2 in §3.1.5 shows that if (3.4) holds, then (3.2) can have large solutions only if (3.6) is fulfilled. In this context, we find the maximal interval for the parameter $a$ that ensures the existence of large solutions to problem (3.2).

Our main result of section 3.1 is the following (see Theorem 1.1 in Cîrstea and Rădulescu (2003*b*)).

**Theorem 3.1.1.** *Assume that $f$ satisfies conditions* (3.4) *and* (3.6)*. Then problem* (3.2) *possesses large solutions if and only if $a \in (-\infty, \lambda_{\infty,1})$. Moreover, in this case, any large solution is positive.*

We point out that the framework of Theorem 3.1.1 includes the case when $b$ vanishes at some points on $\partial\Omega$, or even if $b \equiv 0$ on $\partial\Omega$.

The above result also applies to problems on Riemannian manifolds if $\Delta$ is replaced by the Laplace–Beltrami operator

$$\Delta_B = \frac{1}{\sqrt{c}}\frac{\partial}{\partial x_i}\left(\sqrt{c}\, a_{ij}(x)\frac{\partial}{\partial x_i}\right), \qquad c := \det\,(a_{ij})\,,$$

with respect to the metric $ds^2 = c_{ij}\,dx_i dx_j$, where $(c_{ij})$ is the inverse of $(a_{ij})$. In this case Theorem 3.1.1 applies to concrete problems arising in Riemannian geometry. For instance, (cf. Loewner and Nirenberg (1974)) if $\Omega$ is replaced by the standard $N$–sphere $(S^N, g_0)$, $\Delta$ is the Laplace-Beltrami operator $\Delta_{g_0}$, $a =$

$N(N-2)/4$, and $f(u) = (N-2)/[4(N-1)]\, u^{(N+2)/(N-2)}$, we find the prescribing scalar curvature equation on $S^N$.

The structure of section 3.1 is as follows. In §3.1.3 we establish a strong maximum principle based on an improved form of Theorem 2.1.2. Subsection 3.1.4 comprises a comparison principle, while §3.1.5 is concerned with the Keller–Osserman condition. The proof of Theorem 3.1.1 is given in §3.1.6.

### 3.1.3 A Strong Maximum Principle

We present here a refined form of Theorem 2.1.2 (see Theorem A.1 in Cîrstea and Rădulescu (2002*c*)).

**Theorem 3.1.2.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Assume that $0 \not\equiv p \in C^{0,\mu}(\overline{\Omega})$ is non-negative and $f \in C^1[0,\infty)$ is a positive, non-decreasing function on $(0,\infty)$ such that $f(0) = 0$. If $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ is non-negative, then the boundary value problem*

$$\begin{cases} \Delta u = p(x) f(u) & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases} \tag{3.7}$$

*has a unique classical solution, which is positive in $\Omega$.*

*Remark* 3.1.1. The conclusion of Theorem 3.1.2 has been established by Theorem 2.1.2 when $\Phi$ is assumed to be positive on $\partial\Omega$. Our approach for proving the positivity of the solution was essentially based on this assumption and it fails when the zero set of $\Phi$ is non-empty.

Under the same assumptions on $p$ and $f$ as in Theorem 3.1.2, we have

**Corollary 3.1.3 (Strong maximum principle).** *Let $\Omega$ be a non-empty domain in $\mathbb{R}^N$. If $u$ is a non-negative classical solution of the equation $\Delta u = p(x) f(u)$ in $\Omega$, then the following alternative holds: either $u \equiv 0$ in $\Omega$ or $u$ is positive in $\Omega$.*

*Proof.* If $u \not\equiv 0$ in $\Omega$, then there exists $x_0 \in \Omega$ such that $u(x_0) > 0$. We claim that $u > 0$ in $\Omega$. Arguing by contradiction, let us assume that $u(x_1) = 0$ for some $x_1 \in \Omega$. Let $\omega \subset\subset \Omega$ be a smooth bounded domain such that $x_1 \in \omega$ and $x_0 \in \partial\omega$. Set $p_0 := 1 + \sup_\omega p > 0$ and consider the problem

$$\begin{cases} \Delta v = p_0 f(v) & \text{in } \omega, \\ v = u \not\equiv 0 & \text{on } \partial\omega, \\ v \geq 0 & \text{in } \omega. \end{cases} \tag{3.8}$$

By Theorem 3.1.2, this problem has a unique solution $v_0$, which is positive in $\omega$. It is clear that 0 (resp., $u$) is a sub-solution (resp., super-solution) for (3.8). So, there exists a solution $v_1$ of (3.8) satisfying $0 \leq v_1 \leq u$. By uniqueness, we deduce $v_1 = v_0 > 0$ in $\omega$. Hence, $u \geq v_0 > 0$ in $\omega$, but this is impossible since $u(x_1) = 0$. □

**Corollary 3.1.4.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. If $u_1$ is a non-negative classical solution of the equation $\Delta u + au = p(x)f(u)$ in $\Omega$ such that $u_1 \not\equiv 0$ on $\partial\Omega$, then $u_1$ is positive in $\Omega$.*

*Proof.* Let $\Phi \in C^{0,\mu}(\partial\Omega)$ be such that $\Phi \not\equiv 0$ and $0 \leq \Phi \leq u_1$ on $\partial\Omega$. Consider the problem

$$\begin{cases} \Delta u = |a|u + \|p\|_\infty f(u) & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega, \\ u \geq 0 & \text{in } \Omega. \end{cases} \tag{3.9}$$

By Theorem 3.1.2, this problem has a unique solution, say $u_0$, which is positive in $\Omega$. But $u_1$ is a super-solution for problem (3.9), so $u_1 \geq u_0 > 0$ in $\Omega$ and the claim is proved. □

*Proof of Theorem* 3.1.2. We first observe that $u_- = 0$ is a sub-solution of (3.7), while $u^+ = n$ is a super-solution of (3.7) if $n$ is large enough. Hence problem (3.7) has at least a solution $u_\Phi$.

Taking into account the regularity of $p$ and $f$, a standard boot-strap argument based on Schauder and Hölder regularity shows that $u_\Phi \in C^2(\Omega) \cap C(\overline{\Omega})$. The fact that $u_\Phi$ is the unique classical solution to (3.7) follows in the same way as in Theorem 2.1.2.

In what follows we give two proofs for the positivity of $u_\Phi$: the first one relies essentially on Theorem 1.20 in Díaz (1985), while the second proof offers a much easier and direct approach.

*First proof.* Set $M := \max_{\overline{\Omega}} p$. Let $u_*$ be the unique non-negative classical solution of the problem

$$\begin{cases} \Delta u_* = Mf(u_*) & \text{in } \Omega, \\ u_* = \Phi & \text{on } \partial\Omega. \end{cases}$$

To conclude that $u_\Phi > 0$ in $\Omega$ it is enough to show that $u_\Phi \geq u_* > 0$ in $\Omega$. Since $f \in C^1[0,\infty)$, we have

$$\lim_{u\to 0^+} \frac{u^2}{F(u)} = \lim_{u\to 0^+} \frac{2u}{f(u)} = \frac{2}{f'(0)} > 0 \tag{3.10}$$

which implies that $\int_{0^+}^1 \frac{du}{\sqrt{F(u)}} = \infty$. By applying Theorem 1.20 in Díaz (1985), we conclude that $u_* > 0$ in $\Omega$.

We now prove that $u_\Phi \geq u_*$ in $\Omega$. To this aim, fix $\varepsilon > 0$. We claim that

$$u_*(x) \leq u_\Phi(x) + \varepsilon(1+|x|^2)^{-1/2} \quad \text{for any } x \in \Omega. \tag{3.11}$$

Assume the contrary. Since $u_{*\,|\partial\Omega} = u_{\Phi\,|\partial\Omega} = \Phi$ we infer that

$$\max_{x\in\overline{\Omega}} \{u_*(x) - u_\Phi(x) - \varepsilon(1+|x|^2)^{-1/2}\}$$

is achieved in $\Omega$. At that point we have

$$\begin{aligned} 0 &\geq \Delta\left(u_*(x) - u_\Phi(x) - \varepsilon(1+|x|^2)^{-1/2}\right) \\ &= Mf(u_*(x)) - p(x)f(u_\Phi(x)) - \varepsilon\Delta(1+|x|^2)^{-1/2} \\ &\geq p(x)\left(f(u_*(x)) - f(u_\Phi(x))\right) + \varepsilon(N-3)(1+|x|^2)^{-3/2} \\ &\quad + 3\,\varepsilon(1+|x|^2)^{-5/2} > 0\,, \end{aligned}$$

which is a contradiction. Since $\varepsilon > 0$ is chosen arbitrarily, inequality (3.11) implies that $u_\Phi \geq u_*$ in $\Omega$. □

*Second proof.* Since $\Phi \not\equiv 0$, there exists $x_0 \in \Omega$ such that $u_\Phi(x_0) > 0$. To conclude that $u_\Phi > 0$ in $\Omega$ it is sufficient to prove that $u_\Phi > 0$ on $B(x_0;\overline{r})$ where $\overline{r} = \operatorname{dist}(x_0, \partial\Omega)$. Without loss of generality, we can assume that $x_0 = 0$. By the continuity of $u_\Phi$, there exists $\underline{r} \in (0,\overline{r})$ such that $u_\Phi(x) > 0$ for all $x$ with $|x| \leq \underline{r}$. So, $\min_{|x|=\underline{r}} u_\Phi(x) =: \rho > 0$. We define

$$M := \max_{\overline{\Omega}} p, \quad \eta := \int_\rho^{\rho+1} \frac{dt}{f(t)} \quad \text{and} \quad \nu(\varepsilon) := \int_\varepsilon^{\rho+1} \frac{dt}{f(t)} \quad \text{for } 0 < \varepsilon < \rho.$$

It remains to show that $u_\Phi > 0$ in $A(\underline{r}, \overline{r})$, where

$$A(\underline{r}, \overline{r}) := \{x \in \mathbb{R}^N : \underline{r} < |x| < \overline{r}\}.$$

To this aim, we need the following lemma (see Lemma A.1 in Cîrstea and Rădulescu (2002*c*)).

**Lemma 3.1.5.** *For $\varepsilon > 0$ small enough, the problem*

$$\begin{cases} -\Delta v = M & \text{in } A(\underline{r}, \overline{r}), \\ v(x) = \eta & \text{as } |x| = \underline{r}, \\ v(x) = \nu(\varepsilon) & \text{as } |x| = \overline{r}, \end{cases} \tag{3.12}$$

*has a unique solution, which is increasing in $A(\underline{r}, \overline{r})$.*

*Proof.* By the maximum principle, the problem (3.12) has a unique solution. Moreover, $v$ is radially symmetric in $A(\underline{r}, \overline{r})$, namely $v(x) = v(r)$, $r = |x|$. The function $v$ satisfies

$$v''(r) + \frac{N-1}{r} v'(r) = -M, \quad \underline{r} < r < \overline{r}.$$

Integrating this relation twice, we find

$$v(r) = -\frac{M}{2N} r^2 - \frac{C_1}{N-2} r^{2-N} + C_2, \quad \underline{r} < r < \overline{r},$$

where $C_1$ and $C_2$ are real constants. The boundary conditions $v(\underline{r}) = \eta$ and $v(\overline{r}) = \nu(\varepsilon)$ imply that

$$C_1 = \left(\nu(\varepsilon) - \eta + \frac{M}{2N}(\overline{r}^2 - \underline{r}^2)\right) \frac{N-2}{\underline{r}^{2-N} - \overline{r}^{2-N}}.$$

From (3.10) we deduce $\nu(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Thus, taking $\varepsilon > 0$ sufficiently small, $C_1$ becomes large enough to ensure that $v'(r) > 0$ for all $r \in (\underline{r}, \overline{r})$. $\square$

Set $\varepsilon > 0$ sufficiently small such that the conclusion of Lemma 3.1.5 holds. Let $\underline{u}$ be the function defined implicitly as follows

$$\int_{\underline{u}(x)+\varepsilon}^{\rho+1} \frac{dt}{f(t)} = v(x) \quad \text{for all } x \in A(\underline{r}, \overline{r}). \tag{3.13}$$

It is easy to check that

$$\begin{cases} \Delta \underline{u} \geq M f(\underline{u}+\varepsilon) \geq p(x) f(\underline{u}) & \text{in } A(\underline{r}, \overline{r}), \\ \underline{u}(x) = \rho - \varepsilon < u_\Phi(x) & \text{as } |x| = \underline{r}, \\ \underline{u}(x) = 0 \leq u_\Phi(x) & \text{as } |x| = \overline{r}. \end{cases}$$

Using the maximum principle (as in the proof of (3.11)) we get $\underline{u} \leq u_\Phi$ in $A(\underline{r}, \overline{r})$. By (3.13) and Lemma 3.1.5 we infer that $\underline{u}$ decreases in $A(\underline{r}, \overline{r})$. Thus, $\underline{u} > 0$ in $A(\underline{r}, \overline{r})$. This completes the proof of Theorem 3.1.2. □

The positiveness of the solution in Theorem 3.1.2 follows essentially by the assumption that $f \in C^1$ on $[0, \infty)$. We show in what follows that if $f$ is not differentiable at the origin, then problem (3.7) has a unique solution that is not necessarily positive in $\Omega$. However, in this case, the positiveness of the solution may depend on $c$ and on the geometry of $\Omega$. Indeed, let us consider the problem

$$\begin{cases} \Delta u = \sqrt{u} & \text{in } \Omega\,, \\ u \geq 0 & \text{in } \Omega\,, \\ u = c & \text{on } \partial\Omega, \end{cases} \tag{3.14}$$

where $c > 0$ is a constant.

To justify the uniqueness, let $u_1$, $u_2$ be two solutions of (3.14). It is sufficient to show that $u_1 \leq u_2$ in $\Omega$. Set

$$\omega = \{x \in \Omega;\ u_1(x) > u_2(x)\}$$

and assume that $\omega \neq \emptyset$, then

$$\Delta(u_1 - u_2) = \sqrt{u_1} - \sqrt{u_2} > 0 \quad \text{in } \omega$$

and $u_1 - u_2 = 0$ on $\partial\omega$. The maximum principle implies that $u_1 - u_2 \leq 0$ in $\omega$, which yields a contradiction.

The existence of a solution follows after observing that $u_- = 0$ (resp. $u_+ = c$) is a sub-solution (resp. super-solution) for our problem.

The following example illustrates that in certain situations the unique solution of the problem (3.14) may vanish.

**Example 3.1.1.** Set $\Omega = B(0,1) \subset \mathbb{R}^N$ and $w(x) = a|x|^4$. If $c \le 1/(4N+8)^2$, let us choose $a$ so that $c \le a \le 1/(4N+8)^2$. It follows that

$$\begin{cases} \Delta w = (4N+8)a|x|^2 \le \sqrt{a}|x|^2 = \sqrt{w} & \text{in } \Omega\,, \\ w = a \ge c & \text{on } \partial\Omega. \end{cases}$$

This means that $w$ is a super-solution of (3.14). Since $w(0) = 0$ then, necessarily, $u(0) = 0$.

The next example shows that in some cases, depending on $c$ and on $\operatorname{diam}\Omega$, the unique solution of (3.14) is positive.

**Example 3.1.2.** Suppose that $\Omega$ can be included in a ball $B(x_0, R)$ with $R \le R_c := 2\sqrt[4]{c}\sqrt{N+2}$. Define $w(x) = a|x-x_0|^4$, where $a$ is chosen so that $\sqrt{c}/R^2 \ge \sqrt{a} \ge 1/(4N+8)$, then $w$ satisfies

$$\begin{cases} \Delta w = (4N+8)a|x-x_0|^2 \ge \sqrt{a}|x-x_0|^2 = \sqrt{w} & \text{in } \Omega\,, \\ w = a|x-x_0|^4 \le c & \text{on } \partial\Omega \end{cases}$$

which shows that $w$ is a sub-solution of (3.14). We conclude that $u(x) \ge w(x) > 0$, for any $x \in \Omega\backslash\{x_0\}$.

If $\operatorname{diam}\Omega < 2R \le 2R_c$, there exist two points $x_0$ and $x_1$ such that $\Omega$ can be included in each of the balls $B(x_0, R)$ and $B(x_1, R)$. Using the previous conclusion, we have

$$u(x) \ge a\max\{|x-x_0|^4, |x-x_1|^4\} \ge a\left|\frac{x_1-x_0}{2}\right|^4 > 0.$$

Choosing $a = c/R^4$, $|x_1 - x_0| = 2R - \operatorname{diam}\Omega$ and $R = R_c$, we find

$$u(x) \ge \frac{c}{R^4}\left(\frac{2R - \operatorname{diam}\,\Omega}{2}\right)^4 = c\left(1 - \frac{\operatorname{diam}\,\Omega}{2R}\right)^4 > 0, \quad \forall x \in \Omega.$$

### 3.1.4 A Comparison Principle

We give a comparison principle (see Lemma 1 in Cîrstea and Rădulescu (2003*b*)), which plays an important role in the proof of Theorem 3.1.1.

**Lemma 3.1.6.** *Let $\omega \subset \mathbb{R}^N$ be a smooth bounded domain. Assume that $f$ is continuous on $(0,\infty)$, $f(u)/u$ is increasing on $(0,\infty)$, and $p$, $q$, $r$ are $C^{0,\mu}$-functions* *on $\overline{\omega}$ such that $r \geq 0$ and $p > 0$ in $\omega$. Let $u_1$, $u_2 \in C^2(\omega)$ be positive functions such that*

$$\Delta u_1 + qu_1 - pf(u_1) + r \leq 0 \leq \Delta u_2 + qu_2 - pf(u_2) + r \text{ in } \omega \tag{3.15}$$

$$\limsup_{\text{dist }(x,\partial\omega)\to 0}(u_2 - u_1)(x) \leq 0, \tag{3.16}$$

*then $u_1 \geq u_2$ in $\omega$.*

*Proof.* We use the same method as in the proof of Lemma 1.1 in Marcus and Véron (1997) (see also Lemma 2.1 in Du and Huang (1999)), that goes back to Benguria et al. (1981).

By (3.15) we obtain, for any non-negative function $\phi \in H^1(\omega)$ with compact support in $\omega$,

$$\begin{aligned} &\int_\omega (\nabla u_1 \cdot \nabla\phi - qu_1\phi + pf(u_1)\phi - r\phi)\, dx \\ &\qquad \geq 0 \geq \int_\omega (\nabla u_2 \cdot \nabla\phi - qu_2\phi + pf(u_2)\phi - r\phi)\, dx\,. \end{aligned} \tag{3.17}$$

Let $\varepsilon_1 > \varepsilon_2 > 0$ and denote

$$\omega(\varepsilon_1, \varepsilon_2) = \{x \in \omega : \; u_2(x) + \varepsilon_2 > u_1(x) + \varepsilon_1\}$$
$$v_i = (u_i + \varepsilon_i)^{-1}\left((u_2+\varepsilon_2)^2 - (u_1+\varepsilon_1)^2\right)^+, \quad i = 1, 2.$$

Notice that $v_i \in H^1(\omega)$ and, by (3.16), it has compact support in $\omega$. Using (3.17) with $\phi = v_i$ and taking into account that $v_i$ vanishes outside $\omega(\varepsilon_1, \varepsilon_2)$, we find

$$\begin{aligned} \int_{\omega(\varepsilon_1,\varepsilon_2)} (\nabla u_1 \cdot \nabla v_1 - \nabla u_2 \cdot \nabla v_2)\, dx &\geq \int_{\omega(\varepsilon_1,\varepsilon_2)} p(f(u_2)v_2 - f(u_1)v_1)\, dx \\ &+ \int_{\omega(\varepsilon_1,\varepsilon_2)} q(u_1v_1 - u_2v_2)\, dx + \int_{\omega(\varepsilon_1,\varepsilon_2)} r(v_1 - v_2)\, dx\,. \end{aligned} \tag{3.18}$$

A simple computation shows that the integral in the left-hand side of (3.18) equals

$$-\int_{\omega(\varepsilon_1,\varepsilon_2)} \left( \left|\nabla u_2 - \frac{u_2+\varepsilon_2}{u_1+\varepsilon_1}\nabla u_1\right|^2 + \left|\nabla u_1 - \frac{u_1+\varepsilon_1}{u_2+\varepsilon_2}\nabla u_2\right|^2 \right) dx \leq 0\,.$$

Passing to the limit as $0 < \varepsilon_2 < \varepsilon_1 \to 0$, the first term on the right hand-side of (3.18) converges to

$$\int_{\omega(0,0)} p\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_2^2 - u_1^2)\, dx\,,$$

the second term goes to 0, while the third one converges to

$$\int_{\omega(0,0)} \frac{r(u_2-u_1)^2(u_2+u_1)}{u_1u_2}\,dx \geq 0\,.$$

Hence we avoid a contradiction only in the case when $\omega(0,0)$ has measure 0, which means that $u_1 \geq u_2$ on $\omega$. □

### 3.1.5 On the Keller–Osserman Condition

*Remark* 3.1.2. If (3.4) holds, then problem (3.2) can have large solutions only if $f$ satisfies the Keller–Osserman condition (3.6).

*Proof.* Suppose, a priori, that problem (3.2) has a large solution $u_\infty$. Set $\tilde f(u) = |a|u + \|b\|_\infty f(u)$ for $u \geq 0$. Notice that $\tilde f \in C^1[0,\infty)$ satisfies (2.2). For any $n \geq 1$, consider the problem

$$\begin{cases} \Delta u = \tilde f(u) & \text{in } \Omega\,,\\ u = n & \text{on } \partial\Omega\,,\\ u \geq 0 & \text{in } \Omega\,. \end{cases}$$

By Theorem 3.1.2, this problem has a unique solution, say $u_n$, which is positive in $\overline{\Omega}$. Applying Lemma 3.1.6 for $q \equiv -|a|$, $p \equiv \|b\|_\infty$, $r \equiv 0$ and $\omega = \Omega$, we obtain

$$0 < u_n \leq u_{n+1} \leq u_\infty \quad \text{in } \Omega, \quad \forall n \geq 1.$$

Thus, for every $x \in \Omega$, we can define $\bar u(x) = \lim_{n\to\infty} u_n(x)$. Moreover, since $(u_n)$ is uniformly bounded on every compact subset of $\Omega$, standard elliptic regularity arguments show that $\bar u$ is a positive large solution of the problem $\Delta u = \tilde f(u)$. It follows that $\tilde f$ satisfies the Keller–Osserman condition (3.6). Then, by (3.4), $\mu_\infty := \lim_{u\to\infty} f(u)/u > 0$ which yields $\lim_{u\to\infty} \tilde f(u)/f(u) = |a|/\mu_\infty + \|b\|_\infty < \infty$. Consequently, our claim follows. □

*Remark* 3.1.3. If $f$ satisfies (3.4) and (3.6), then

$$\mu_\infty := \lim_{u\to\infty} \frac{f(u)}{u} = \lim_{u\to\infty} f'(u) = \infty.$$

Indeed, by l'Hospital's rule, $\lim_{u\to\infty} F(u)/u^2 = \mu_\infty/2$. But, by (3.6), we deduce $\mu_\infty = \infty$. Then, by (3.4) we find $f'(u) \geq f(u)/u$ for any $u > 0$, which shows that $\lim_{u\to\infty} f'(u) = \infty$.

Nonlinearities $f$ as in Remark 3.1.3 are illustrated by

(i) $f(u) = e^u - 1$; $f(u) = e^u \ln(u+1)$; $f(u) = e^{e^u} - e$;

(ii) $f(u) = u^p$, $f(u) = u^p \ln(u+1)$ with $p > 1$;

(iii) $f(u) = u[\ln(u+1)]^p$ with $p > 2$.

We shall provide an equivalent criterion to the Keller–Osserman condition (3.6). To this aim, a significant role is played by the set $\mathcal{G}$ defined by

$$\mathcal{G} = \left\{ g \,\middle|\, \begin{array}{l} g \in C^2(0,\delta),\ \text{for some } \delta > 0,\ \ g'' > 0 \text{ on } (0,\delta), \\ \lim\limits_{t \searrow 0} g(t) = \infty \text{ and there exists } \lim\limits_{t \searrow 0} g'(t)/g''(t) \end{array} \right\}. \tag{3.19}$$

Note that $\mathcal{G} \not\equiv \emptyset$. We see, for example, that $e^\Theta \subset \mathcal{G}$ where

$$\Theta = \left\{ \theta : \quad \theta \in C^2(0,\infty),\quad \theta \text{ is convex on } (0,\infty) \ \text{ and } \ \lim_{t \searrow 0} \theta(t) = \infty \right\}.$$

Obviously, $\Theta \not\equiv \emptyset$. Let $\theta \in \Theta$ be arbitrary. Since $\theta'$ is non-decreasing on $(0,\infty)$ and $\lim_{t \searrow 0} \theta(t) = \infty$, it follows that $\lim_{t \searrow 0} \theta'(t) = -\infty$, then,

$$\left| \frac{\theta'(t)}{(\theta'(t))^2 + \theta''(t)} \right| \le \frac{1}{|\theta'(t)|} \to 0 \quad \text{as } t \searrow 0,$$

which proves that $e^\theta \in \mathcal{G}$.

*Remark* 3.1.4. We have $\lim\limits_{t \searrow 0} g(t)/g''(t) = \lim\limits_{t \searrow 0} g'(t)/g''(t) = 0$, for any $g \in \mathcal{G}$.

Indeed, let $g \in \mathcal{G}$ be arbitrary, then

$$\lim_{t \searrow 0} g'(t) = -\infty, \quad \lim_{t \searrow 0} \ln g(t) = \infty \quad \text{and} \quad \lim_{t \searrow 0} \ln |g'(t)| = \infty. \tag{3.20}$$

L'Hospital's rule and (3.20) imply that $\lim_{t \searrow 0} g(t)/g'(t) = \lim_{t \searrow 0} g'(t)/g''(t) = 0$.

*Notation* 2. We denote by (3.4') the case when (3.4) is fulfilled and there exists $\lim_{u \to \infty} (F/f)'(u) := \gamma$.

We see that when (3.4') is satisfied, then $\gamma \ge 0$; if, in addition, (3.6) holds, then $\gamma \le 1/2$ (cf. Lemma 4.1 in Cîrstea and Rădulescu (2002*c*)).

**Lemma 3.1.7.** *If* (3.4') *is satisfied, then the following hold:*

(i) $\gamma \geq 0$;

(ii) $\gamma \leq 1/2$ *provided that* (3.6) *is also fulfilled.*

*Proof.* (i) If we suppose that $\gamma < 0$, then there exists $s_1 > 0$ such that $(F/f)'(u) \leq \gamma/2 < 0$ for any $u \geq s_1$. Integrating this inequality over $(s_1, \infty)$, we obtain a contradiction. Thus, $\gamma \geq 0$.

(ii) Let (3.6) be satisfied. By the definition of $\gamma$, $\lim_{u\to\infty} \frac{F(u)f'(u)}{f^2(u)} = 1 - \gamma$. By Remark 3.1.3 and L'Hospital's rule, we obtain

$$\lim_{u\to\infty} \frac{F(u)}{f^2(u)} \overset{\frac{\infty}{\infty}}{=} \lim_{u\to\infty} \frac{1}{2f'(u)} = 0$$

and

$$0 \leq \lim_{u\to\infty} \frac{\sqrt{F(u)}/f(u)}{\int_u^\infty \frac{ds}{\sqrt{F(s)}}} \overset{\frac{0}{0}}{=} -\frac{1}{2} + \lim_{u\to\infty} \frac{F(u)f'(u)}{f^2(u)} = \frac{1}{2} - \gamma. \tag{3.21}$$

This concludes the proof. □

Next, we use the class $\mathcal{G}$ introduced by (3.19) to deduce an equivalent condition to (3.6) (cf. Lemma 4.2 in Cîrstea and Rădulescu (2002*c*)).

**Lemma 3.1.8.** *If* (3.4') *is satisfied, then the Keller–Osserman growth condition* (3.6) *holds if and only if*

$$\lim_{t\searrow 0} \frac{tf(g(t))}{g''(t)} = \infty, \quad \textit{for some function } g \in \mathcal{G}. \tag{3.22}$$

*A. Necessary condition.* Since (3.6) holds, we can define a positive function $g$ as follows

$$\int_{g(t)}^\infty \frac{ds}{\sqrt{F(s)}} = t^\vartheta \quad \text{for all } t > 0, \quad \text{where } \vartheta \in (3/2, \infty) \text{ is arbitrary.} \tag{3.23}$$

Obviously, $g \in C^2(0,\infty)$ and $\lim_{t\searrow 0} g(t) = \infty$.

We show that $g \in \mathcal{G}$ satisfies (3.22). We divide our argument into three steps:

*Step* 3.1.1. $\lim_{t\searrow 0} g'(t)t^{1-2\vartheta}/f(g(t)) = \vartheta(\gamma - 1/2)$.

We derive twice relation (3.23) and obtain

$$g'(t) = -\vartheta t^{\vartheta-1}\sqrt{F(g(t))} \tag{3.24}$$

respectively

$$\begin{aligned} g''(t) &= \frac{\vartheta-1}{t}g'(t) + \frac{\vartheta^2}{2}t^{2\vartheta-2}f(g(t)) \\ &= \frac{\vartheta^2}{2}t^{2\vartheta-2}f(g(t))\left(\frac{2(\vartheta-1)}{\vartheta^2}\frac{g'(t)}{t^{2\vartheta-1}f(g(t))}+1\right). \end{aligned} \tag{3.25}$$

By (3.21) and (3.24), we find

$$\begin{aligned} \lim_{t\searrow 0}\frac{g'(t)}{t^{2\vartheta-1}f(g(t))} &= \lim_{t\searrow 0}\frac{-\vartheta t^{\vartheta-1}\sqrt{F(g(t))}}{t^{2\vartheta-1}f(g(t))} = \lim_{t\searrow 0} -\vartheta\frac{\sqrt{F(g(t))}/f(g(t))}{\int_{g(t)}^{\infty}\frac{ds}{\sqrt{F(s)}}} \\ &= \lim_{u\to\infty} -\vartheta\frac{\sqrt{F(u)}/f(u)}{\int_{u}^{\infty}\frac{ds}{\sqrt{F(s)}}} = \vartheta\left(\gamma-\frac{1}{2}\right). \end{aligned}$$

*Step* 3.1.2. $g''>0$ on $(0,\delta)$ for $\delta$ small enough.

Since $\gamma \geq 0$, by Step 3.1.1, we get

$$\lim_{t\searrow 0}\frac{2(\vartheta-1)}{\vartheta^2}\frac{g'(t)}{t^{2\vartheta-1}f(g(t))} = \frac{2(\vartheta-1)}{\vartheta}\left(\gamma-\frac{1}{2}\right) \geq \frac{1}{\vartheta}-1 > -1. \tag{3.26}$$

In view of (3.25), the assertion of this step follows.

*Step* 3.1.3. $\lim_{t\searrow 0} g'(t)/g''(t) = 0$ and $\lim_{t\searrow 0} tf(g(t))/g''(t) = \infty$.

Using (3.25) and (3.26), we obtain

$$\lim_{t\searrow 0}\frac{g'(t)}{g''(t)} = \lim_{t\searrow 0}\frac{2t}{\vartheta^2}\frac{g'(t)}{t^{2\vartheta-1}f(g(t))}\frac{1}{\frac{2(\vartheta-1)}{\vartheta^2}\frac{g'(t)}{t^{2\vartheta-1}f(g(t))}+1} = 0.$$

For any $t \in (0,\delta)$ ($\delta > 0$ as in Step 3.1.2), we infer that

$$\frac{tf(g(t))}{g''(t)} = \frac{tf(g(t))}{\frac{\vartheta-1}{t}g'(t) + \frac{\vartheta^2}{2}t^{2\vartheta-2}f(g(t))} \geq \frac{tf(g(t))}{\frac{\vartheta^2}{2}t^{2\vartheta-2}f(g(t))} = \frac{2}{\vartheta^2 t^{2\vartheta-3}}.$$

Sending $t$ to 0, the claim of Step 3.1.3 is proved.

*B. Sufficient condition.* Choose $g \in \mathcal{G}$ so that (3.22) is fulfilled. We have

$$\lim_{t\searrow 0}\frac{(g'(t))^2}{F(g(t))} = 2\lim_{t\searrow 0}\frac{g''(t)}{f(g(t))} = 0.$$

We choose $\delta > 0$ small enough such that $g'(s) < 0$ and $g''(s) > 0$ for all $s \in (0, \delta)$. It follows that

$$\int_{g(\delta)}^{\infty} \frac{dt}{\sqrt{F(t)}} = \lim_{t \searrow 0} \int_{g(\delta)}^{g(t)} \frac{ds}{\sqrt{F(s)}} = \lim_{t \searrow 0} \int_{t}^{\delta} \frac{-g'(s)\, ds}{\sqrt{F(g(s))}}$$
$$\leq \delta \sup_{t \in (0,\delta)} \frac{-g'(t)}{\sqrt{F(g(t))}} < \infty.$$

Hence, the growth condition (3.6) holds. □

### 3.1.6 Proof of the Main Result

*A. Necessary condition.* Let $u_\infty$ be a large solution of problem (3.2). We claim that $u_\infty$ is positive. Indeed, since $u_\infty(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$, there exists a smooth open set $\omega \subset\subset \Omega$ such that $u_\infty > 0$ on $\Omega \setminus \omega$. So, it is enough to show that $u_\infty > 0$ in $\overline{\omega}$. To this aim, set $M_0 := 1 + \sup_\omega b > 0$ and consider the problem

$$\begin{cases} \Delta u = |a| u + M_0 f(u) & \text{in } \omega\,, \\ u = u_\infty & \text{on } \partial\omega\,, \\ u \geq 0 & \text{in } \omega\,. \end{cases} \tag{3.27}$$

By Theorem 3.1.2, this problem has a unique solution $u_0$ and, moreover, $u_0 > 0$ in $\overline{\omega}$. Since $u_\infty$ is a super-solution for problem (3.27), we infer that $u_\infty \geq u_0 > 0$ in $\overline{\omega}$. This shows that any large solution of (3.2) is positive.

Suppose that $\lambda_{\infty,1}$ is finite. Arguing by contradiction, let us assume that $a \geq \lambda_{\infty,1}$. Set $\lambda \in (\lambda_1(\mu_0), \lambda_{\infty,1})$ and denote by $u_\lambda$ the unique positive solution of problem $(E_a)$ with $a = \lambda$. We have

$$\begin{cases} \Delta(M u_\infty) + \lambda_{\infty,1}(M u_\infty) \leq b(x) f(M u_\infty) & \text{in } \Omega\,, \\ M u_\infty = \infty & \text{on } \partial\Omega\,, \\ M u_\infty \geq u_\lambda & \text{in } \Omega\,, \end{cases}$$

where $M := \max\{\max_{\overline{\Omega}} u_\lambda / \min_{\overline{\Omega}} u_\infty;\, 1\}$. By the sub and super-solutions method, we conclude that problem $(E_a)$ with $a = \lambda_{\infty,1}$ has at least a positive solution (between $u_\lambda$ and $M u_\infty$). But this is a contradiction. So, necessarily, $a \in (-\infty, \lambda_{\infty,1})$. □

*B. Sufficient condition.* This will be proved with the aid of several results (see Lemmas 2 and 3 in Cîrstea and Rădulescu (2003*b*)). In the rest of §3.1.6 we assume that $f$ satisfies (3.4) and (3.6).

**Lemma 3.1.9.** *Let $\omega$ be a smooth bounded domain in $\mathbb{R}^N$. Assume that $p$, $q$, $r$ are $C^{0,\mu}$-functions on $\overline{\omega}$ such that $r \geq 0$ and $p > 0$ in $\overline{\omega}$, then, for any non-negative function $0 \not\equiv \Phi \in C^{0,\mu}(\partial\omega)$, the boundary value problem*

$$\begin{cases} \Delta u + q(x)u = p(x)f(u) - r(x) & \text{in } \omega, \\ u > 0 & \text{in } \omega, \\ u = \Phi & \text{on } \partial\omega, \end{cases} \tag{3.28}$$

*has a unique solution.*

*Proof.* By Lemma 3.1.6, problem (3.28) has at most one solution. The existence of a positive solution will be obtained by device of sub and super-solutions.

Set $m := \inf_\omega p > 0$. Define $\bar{f}(u) = mf(u) - \|q\|_\infty u - \bar{r}$, where $\bar{r} := \sup_\omega r + 1 > 0$. Let $t_0$ be the unique positive solution of the equation $\bar{f}(u) = 0$. By Remark 3.1.3, we infer that $\lim_{u\to\infty} \bar{f}(u)/f(u) = m > 0$. Combining this with (3.6), we conclude that the function $\varphi(w) = \bar{f}(w + t_0)$ defined for $w \geq 0$ satisfies the assumptions of Theorem III in Keller (1957). It follows that there exists a positive large solution for the equation $\Delta w = \varphi(w)$ in $\omega$. Thus the function $\bar{u}(x) = w(x) + t_0$, for all $x \in \omega$, is a positive large solution of the problem

$$\Delta u + \|q\|_\infty u = mf(u) - \bar{r} \quad \text{in } \omega. \tag{3.29}$$

By Theorem 3.1.2, the boundary value problem

$$\begin{cases} \Delta u = \|q\|_\infty u + \|p\|_\infty f(u) & \text{in } \omega, \\ u > 0 & \text{in } \omega, \\ u = \Phi & \text{on } \partial\omega, \end{cases}$$

has a unique classical solution $\underline{u}$. By Lemma 3.1.6, we find $\underline{u} \leq \overline{u}$ in $\omega$ and $\underline{u}$ (resp., $\overline{u}$) is a positive sub-solution (resp., super-solution) of problem (3.28). It follows that (3.28) has a unique solution. □

Under the assumptions of Lemma 3.1.9, we obtain the following result that generalizes Lemma 1.3 in Marcus and Véron (1997).

**Corollary 3.1.10.** *There exists a positive large solution of the problem*

$$\Delta u + q(x)u = p(x)f(u) - r(x) \qquad \text{in } \omega. \tag{3.30}$$

*Proof.* Set $\Phi = n$ and let $u_n$ be the unique solution of (3.28). By Lemma 3.1.6, $u_n \leq u_{n+1} \leq \overline{u}$ in $\omega$, where $\overline{u}$ denotes a large solution of (3.29). It follows that $\lim_{n\to\infty} u_n(x) = u_\infty(x)$ exists and is a positive large solution of (3.30). Furthermore, every positive large solution of (3.30) dominates $u_\infty$, that is the solution $u_\infty$ is the *minimal large solution.* This follows from the definition of $u_\infty$ and Lemma 3.1.6. □

**Lemma 3.1.11.** *If* $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ *is a non-negative function and* $b > 0$ *on* $\partial\Omega$*, then the boundary value problem*

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega, \end{cases} \tag{3.31}$$

*has a solution if and only if* $a \in (-\infty, \lambda_{\infty,1})$*. Moreover, in this case, the solution is unique.*

*Proof.* The first part follows with the same arguments as in the proof of Theorem 3.1.1 (necessary condition).

For the sufficient condition, fix $a < \lambda_{\infty,1}$ and let $\lambda_{\infty,1} > \lambda_* > \max\{a, \lambda_1(\mu_0)\}$. Let $u_*$ be the unique positive solution of $(E_a)$ with $a = \lambda_*$.

Let $\Omega_i$ $(i = 1, 2)$ be sub-domains of $\Omega$ such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ and $\Omega \setminus \overline{\Omega}_1$ is smooth. We define $u_+ \in C^2(\Omega)$ a positive function in $\Omega$ such that $u_+ \equiv u_\infty$ on $\Omega \setminus \Omega_2$ and $u_+ \equiv u_*$ on $\Omega_1$. Here $u_\infty$ denotes a positive large solution of (3.30) for $p(x) = b(x)$, $r(x) = 0$, $q(x) = a$ and $\omega = \Omega \setminus \overline{\Omega}_1$. Using Remark 3.1.3 and the fact that $b_0 := \inf_{\Omega_2\setminus\Omega_1} b > 0$, it is easy to check that if $C > 0$ is large enough, then $\overline{v}_\Phi = Cu_+$ satisfies

$$\begin{cases} \Delta\overline{v}_\Phi + a\overline{v}_\Phi \leq b(x)f(\overline{v}_\Phi) & \text{in } \Omega, \\ \overline{v}_\Phi = \infty & \text{on } \partial\Omega, \\ \overline{v}_\Phi \geq \max\limits_{\partial\Omega} \Phi & \text{in } \Omega. \end{cases}$$

By Theorem 3.1.2, there exists a unique classical solution $\underline{v}_\Phi$ of the problem

$$\begin{cases} \Delta \underline{v}_\Phi = |a| \underline{v}_\Phi + \|b\|_\infty f(\underline{v}_\Phi) & \text{in } \Omega, \\ \underline{v}_\Phi > 0 & \text{in } \Omega, \\ \underline{v}_\Phi = \Phi & \text{on } \partial\Omega. \end{cases}$$

It is clear that $\underline{v}_\Phi$ is a positive sub-solution of (3.31) and $\underline{v}_\Phi \leq \max_{\partial\Omega} \Phi \leq \overline{v}_\Phi$ in $\Omega$. Therefore, by the sub and super-solutions method, problem (3.31) has at least a solution $v_\Phi$ between $\underline{v}_\Phi$ and $\overline{v}_\Phi$. Next, the uniqueness of solution to (3.31) can be obtained by using essentially the same technique as in Theorem 1 of Brezis and Oswald (1986) or Appendix II of Brezis and Kamin (1992). □

*Proof of Theorem 3.1.1 completed.* Fix $a < \lambda_{\infty,1}$. Two cases may occur:

*Case* 3.1.1. $b > 0$ on $\partial\Omega$.

Denote by $v_n$ the unique solution of (3.31) with $\Phi \equiv n$. For $\Phi \equiv 1$, set $v := \underline{v}_\Phi$ and $V := \overline{v}_\Phi$, where $\underline{v}_\Phi$ and $\overline{v}_\Phi$ are defined in the proof of Lemma 3.1.11. The sub and super-solutions method, together with the uniqueness of solution of (3.31), shows that $v \leq v_n \leq v_{n+1} \leq V$ in $\Omega$. Hence, $v_\infty(x) := \lim_{n\to\infty} v_n(x)$ exists and it is a large solution of (3.2).

*Case* 3.1.2. $b \geq 0$ on $\partial\Omega$.

Let $z_n$ $(n \geq 1)$ be the unique solution of (3.28) for $p \equiv b + 1/n$, $r \equiv 0$, $q \equiv a$, $\Phi \equiv n$ and $\omega = \Omega$. By Lemma 3.1.6, $(z_n)$ is non-decreasing. Moreover, $(z_n)$ is uniformly bounded on every compact subdomain of $\Omega$. Indeed, if $K \subset \Omega$ is an arbitrary compact set, then $d := \operatorname{dist}(K, \partial\Omega) > 0$. Choose $\delta \in (0, d)$ small enough so that $\overline{\Omega}_0 \subset C_\delta$, where

$$C_\delta = \{x \in \Omega : \ \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

Since $b > 0$ on $\partial C_\delta$, Case 3.1.1 allows us to define $z_+$ as a large solution of (3.2) for $\Omega = C_\delta$. Using Lemma 3.1.6 for $p \equiv b + 1/n$, $r \equiv 0$, $q \equiv a$ and $\omega = C_\delta$, we obtain $z_n \leq z_+$ in $C_\delta$, for all $n \geq 1$. So, $(z_n)$ is uniformly bounded on $K$. By the monotonicity of $(z_n)$, we conclude that $z_n \to \underline{z}$ in $L^\infty_{\text{loc}}(\Omega)$. Finally, standard elliptic regularity arguments lead to $z_n \to \underline{z}$ in $C^{2,\alpha}_{\text{loc}}(\Omega)$. This completes the proof of Theorem 3.1.1. □

## 3.2 Mixed Boundary Value Problems

### 3.2.1 Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$. We denote by $\mathcal{B}$ either the *Dirichlet* boundary operator $\mathcal{D}u := u$ or the *Neumann/Robin* boundary operator $\mathcal{R}u = \frac{\partial u}{\partial \nu} + \beta(x)u$. Here $\nu$ is the outward unit normal to $\partial\Omega$ and $\beta \geq 0$ is in $C^{1,\mu}(\partial\Omega)$, $0 < \mu < 1$.

Our main aim of section 3.2 is to study the existence of *large solutions* to the boundary value problem

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega \setminus \overline{\Omega}_0\,, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.32}$$

where $a$ is a real parameter, $b \in C^{0,\mu}(\overline{\Omega})$ is non-negative and $f \in C^1[0,\infty)$.

By a *large solution* of (3.32) we mean any non-negative $C^2(\overline{\Omega} \setminus \overline{\Omega}_0)$-solution of (3.32) that satisfies $u(x) \to \infty$ as $x \in \Omega \setminus \overline{\Omega}_0$ and $d(x) := \operatorname{dist}(x, \Omega_0) \to 0$.

Let $\Omega_0$ (given by (3.3)) be non-empty, connected and with smooth boundary. We assume, throughout section 3.2, that $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ on $\overline{\Omega} \setminus \overline{\Omega}_0$.

The existence and uniqueness of large solutions of (3.32) has been recently treated by Du and Huang (1999) in the particular case $f(u) = u^p$ $(p > 1)$.

Our purpose is to extend the existence result in Du and Huang (1999) to the case of much more general nonlinearities of Keller–Osserman type.

Note that the Robin condition $\mathcal{R} = 0$ relates essentially to heat flow problems in a body with constant temperature in the surrounding medium. More generally, if $\alpha$ and $\beta$ are smooth functions on $\partial\Omega$ such that $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, then the boundary condition $Bu = \alpha\frac{\partial u}{\partial \nu} + \beta u = 0$ represents the exchange of heat at the surface of the reactant by Newtonian cooling. Moreover, the boundary condition $Bu = 0$ is called isothermal (Dirichlet) condition if $\alpha \equiv 0$, and it becomes an adiabatic (Neumann) condition if $\beta \equiv 0$. An intuitive meaning of the condition $\alpha + \beta > 0$ on $\partial\Omega$ is that, for the diffusion process described by problem (3.2), either the reflection phenomenon or the absorption phenomenon may occur at each point of the boundary.

### 3.2.2 Existence of Large Solutions

The main result of section 3.2 establishes the existence of large solutions to (3.32) when $f$ satisfies (3.6) and (3.4') (in Notation 2). More exactly, we have (see Theorem 1.2 in Cîrstea and Rădulescu (2002*c*)):

**Theorem 3.2.1.** *Let* (3.4') *and* (3.6) *hold, then, for any* $a \in \mathbb{R}$, *problem* (3.32) *has a minimal (resp., maximal) large solution* $\underline{U}_a$ *(resp.,* $\overline{U}_a$*).*

*Remark* 3.2.1. This result generalizes Theorem 2.4 in Du and Huang (1999), where $f(u) = u^p$ with $p > 1$.

The rest of section 3.2 is organized as follows. In §3.2.3 we give a comparison principle, which extends the previous result of §3.1.4 to the case of mixed boundary value problems. Subsection 3.2.4 proves the existence and uniqueness of positive solutions to (3.32), subject to a non-homogeneous Dirichlet boundary condition on $\partial\Omega_0$. The proof of Theorem 3.2.1 is presented in §3.2.5.

### 3.2.3 A Comparison Principle

The next result (see Lemma 2.3 in Cîrstea and Rădulescu (2002*c*)) extends Lemma 2.1 in Du and Huang (1999).

**Lemma 3.2.2.** *Assume that* $\omega \subset\subset \Omega$ *and* $p \in C^{0,\mu}(\overline{\Omega} \setminus \omega)$ *is a positive function in* $\Omega \setminus \overline{\omega}$. *If* $u_1$, $u_2 \in C^2(\overline{\Omega} \setminus \overline{\omega})$ *are positive functions in* $\Omega \setminus \overline{\omega}$ *and*

$$\Delta u_1 + au_1 - p(x)f(u_1) \leq 0 \leq \Delta u_2 + au_2 - p(x)f(u_2) \quad \text{in } \Omega \setminus \overline{\omega} \tag{3.33}$$

$$\mathcal{B}u_1 \geq 0 \geq \mathcal{B}u_2 \quad \text{on } \partial\Omega\,; \qquad \limsup_{\text{dist}\,(x,\partial\omega)\to 0} (u_2 - u_1)(x) \leq 0\,, \tag{3.34}$$

*then* $u_1 \geq u_2$ *on* $\overline{\Omega} \setminus \overline{\omega}$.

*Proof.* We distinguish 2 cases:

*Case* 3.2.1. $\mathcal{B} = \mathcal{D}$.

The assertion is an easy consequence of Lemma 3.1.6.

*Case* 3.2.2. $\mathcal{B} = \mathcal{R}$.

Let $\phi_1$, $\phi_2$ be two non-negative $C^2$-functions on $\overline{\Omega} \setminus \omega$ vanishing near $\partial\omega$. Multiplying in (3.33) the first inequality (resp., the second one) by $\phi_1$ (resp., $\phi_2$) and applying integration by parts, together with (3.34), we deduce

$$\begin{aligned} -\int_{\widetilde{\Omega}} (\nabla u_2 \cdot \nabla \phi_2 - \nabla u_1 \cdot \nabla \phi_1)\, dx - \int_{\partial\Omega} \beta(x)(u_2\phi_2 - u_1\phi_1)\, dS(x) \\ \geq \int_{\widetilde{\Omega}} p(x)(f(u_2)\phi_2 - f(u_1)\phi_1)\, dx + a\int_{\widetilde{\Omega}} (u_1\phi_1 - u_2\phi_2)\, dx, \end{aligned} \tag{3.35}$$

where $\widetilde{\Omega} := \Omega \setminus \overline{\omega}$. Let $\varepsilon_1 > \varepsilon_2 > 0$ and denote

$$\Omega_+(\varepsilon_1, \varepsilon_2) = \{x \in \widetilde{\Omega} : \ u_2(x) + \varepsilon_2 > u_1(x) + \varepsilon_1\}.$$
$$v_i = (u_i + \varepsilon_i)^{-1} \left((u_2 + \varepsilon_2)^2 - (u_1 + \varepsilon_1)^2\right)^+, \quad i = 1, 2.$$

Since $v_i$ can be approximated closely in the $H^1 \cap L^\infty$-topology on $\overline{\Omega} \setminus \omega$ by non-negative $C^2$-functions vanishing near $\partial\omega$, it follows that (3.35) holds for $v_i$ taking place of $\phi_i$. Since $v_i$ vanishes outside the set $\Omega_+(\varepsilon_1, \varepsilon_2)$, relation (3.35) becomes

$$\begin{aligned} -\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} (\nabla u_2 \cdot \nabla v_2 - \nabla u_1 \cdot \nabla v_1)\, dx - \int_{\partial\Omega} \beta(x)(u_2v_2 - u_1v_1)\, dS(x) \\ \geq \int_{\Omega_+(\varepsilon_1,\varepsilon_2)} p(x)(f(u_2)v_2 - f(u_1)v_1)\, dx + a\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} (u_1v_1 - u_2v_2)\, dx. \end{aligned} \tag{3.36}$$

As $\varepsilon_1 \to 0$ (recall that $\varepsilon_1 > \varepsilon_2 > 0$) the second term on the left hand-side of (3.36) converges to 0. From now on, the course of the proof is the same as in Lemma 3.1.6. This completes the proof. □

### 3.2.4 Auxiliary Results

In what follows we establish the existence of positive solutions to (3.32), subject to a non-homogeneous Dirichlet boundary condition on $\partial\Omega_0$ (see Lemma 5.1 in Cîrstea and Rădulescu (2002*c*)).

**Lemma 3.2.3.** *If (3.4) and (3.6) hold, then for any positive function $\Phi \in C^{2,\mu}(\partial\Omega_0)$ and $a \in \mathbb{R}$ the problem*

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega \setminus \overline{\Omega}_0\,, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega\,, \\ u = \Phi & \text{on } \partial\Omega_0\,, \end{cases} \tag{3.37}$$

*has a unique positive solution.*

*Proof.* By Lemma 3.2.2, (3.37) has at most a positive solution. To prove the existence of a positive solution to (3.37), we shall use the sub and super-solutions method.

Let $\omega \subset\subset \Omega_0$ be such that the first Dirichlet eigenvalue of $(-\Delta)$ in the smooth domain $\Omega_0 \setminus \overline{\omega}$ is greater than $a$. Let $p \in C^{0,\mu}(\overline{\Omega})$ be such that $p(x) = b(x)$ for $x \in \overline{\Omega} \setminus \Omega_0$, $p(x) = 0$ for $x \in \overline{\Omega}_0 \setminus \omega$ and $p(x) > 0$ for $x \in \omega$. By virtue of Lemma 3.1.11, problem

$$\begin{cases} \Delta u + au = p(x)f(u) & \text{in } \Omega, \\ u = 1 & \text{on } \partial\Omega, \end{cases}$$

has a unique positive solution $u_1$.

We choose $\Omega_1$ and $\Omega_2$ two sub-domains of $\Omega$ so that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$.

Define $u^* \in C^2(\overline{\Omega} \setminus \Omega_0)$ so that $u^* \equiv 1$ on $\overline{\Omega} \setminus \Omega_2$, $u^* \equiv u_1$ on $\overline{\Omega}_1 \setminus \Omega_0$ and $m_* := \min_{\overline{\Omega}\setminus\Omega_0} u^* > 0$.

*Claim* 3.2.1. *For $\ell \geq 1$ large enough, $\ell u^*$ is a super-solution for problem* (3.37).

We first observe that for $x \in \overline{\Omega}_1 \setminus \overline{\Omega}_0$ and $\ell \geq 1$,

$$-\Delta(\ell u^*) = \ell a u_1 - \ell p(x) f(u_1) \geq a(\ell u^*) - b(x) f(\ell u^*). \tag{3.38}$$

Denote by $M^* := \sup_{\Omega\setminus\Omega_1} (au^* + \Delta u^*)$ and $b_0 := \min_{\overline{\Omega}\setminus\Omega_1} b > 0$. By Remark 3.1.3, there exists $\ell_1 \geq 1$ such that $f(\ell m_*) \geq \ell M^*/b_0$ for all $\ell \geq \ell_1$.

For $x \in \Omega \setminus \overline{\Omega}_1$ and $\ell \geq \ell_1$ we have

$$b(x) f(\ell u^*) \geq b_0 f(\ell m_*) \geq \ell(au^* + \Delta u^*)$$

which can be rewritten as

$$-\Delta(\ell u^*) \geq a(\ell u^*) - b(x) f(\ell u^*) \quad \text{for } x \in \Omega \setminus \overline{\Omega}_1 \ \text{ and } \ell \geq \ell_1. \tag{3.39}$$

By (3.38) and (3.39), it follows that

$$-\Delta(\ell u^*) \geq a(\ell u^*) - b(x) f(\ell u^*) \quad \text{in } \Omega \setminus \overline{\Omega}_0, \ \text{ for any } \ \ell \geq \ell_1.$$

On the other hand,

$$\mathcal{B}(\ell u^*) \geq \ell \min\{1, \min_{x\in\partial\Omega} \beta(x)\} \geq 0 \quad \text{on } \partial\Omega, \ \text{ for every } \ell > 0.$$

By taking $\ell \geq \max\,\{\max_{\partial\Omega_0} \Phi/m_*; \ell_1\}$, the claim follows.

Set $\overline{b} := \sup_\Omega b$. By Theorem 3.1.2, the boundary value problem

$$\begin{cases} \Delta u_* = \overline{b} f(u_*) + |a| u_* & \text{in } \Omega \setminus \overline{\Omega}_0\,, \\ u_* = 0 & \text{on } \partial\Omega\,, \\ u_* = \Phi & \text{on } \partial\Omega_0\,, \end{cases} \tag{3.40}$$

has a unique non-negative solution, which is positive in $\Omega \setminus \overline{\Omega}_0$. Since $u_* = 0$ on $\partial\Omega$, we get $\mathcal{R}u_* = \frac{\partial u_*}{\partial \nu} \leq 0$ on $\partial\Omega$. It is easy to see that $u_*$ is a sub-solution of (3.37) and $u_* \leq \ell u^*$ in $\overline{\Omega} \setminus \Omega_0$ for $\ell$ large enough. The conclusion of Lemma 3.2.3 follows now by the sub and super-solutions method. □

**Corollary 3.2.4.** *If $\Omega_0$ is replaced by $\Omega_m := \{x \in \Omega :\ d(x) < 1/m\}$ ($m \geq 1$ is large), then the statement of Lemma 3.2.3 holds.*

*Proof.* The construction of the sub-solution is made as before, while the super-solution can be chosen any number $\ell \geq 1$ large enough. □

### 3.2.5 Proof of the Main Result

The proof of Theorem 3.2.1 will be divided into two steps:

*Step* 3.2.1. Existence of the minimal large solution of (3.32).

For any $n \geq 1$, let $u_n$ be the unique positive solution of problem (3.37) with $\Phi \equiv n$. By Lemma 3.2.2, $u_n(x)$ increases with $n$ for all $x \in \overline{\Omega} \setminus \overline{\Omega}_0$.

Next, we prove that the pointwise limit of $(u_n(x))_n$ exists, for each $x \in \overline{\Omega} \setminus \overline{\Omega}_0$ (see Lemma 5.2 in Cîrstea and Rădulescu (2002*c*)).

**Lemma 3.2.5.** *The sequence $(u_n(x))_n$ is bounded from above by some function $V(x)$ that is uniformly bounded on all compact subsets of $\overline{\Omega} \setminus \overline{\Omega}_0$.*

*Proof.* Let $b^*$ be a $C^2$-function on $\overline{\Omega} \setminus \Omega_0$ such that

$$0 < b^*(x) \leq b(x) \quad \forall x \in \overline{\Omega} \setminus \overline{\Omega}_0.$$

For $x$ bounded away from $\partial\Omega_0$ is not a problem to find such a function $b^*$. For $x$ satisfying $0 < d(x) < \delta$ with $\delta > 0$ small such that $x \to d(x)$ is a $C^2$-function, we can take

$$b^*(x) = \int_0^{d(x)} \int_0^t [\min_{d(z)\geq s} b(z)]\, ds\, dt.$$

Let $g \in \mathcal{G}$ be a function such that (3.22) holds. The existence of $g$ is guaranteed by Lemma 3.1.8. Since $b^*(x) \to 0$ as $d(x) \searrow 0$, we deduce, by Remark 3.1.4 and (3.4), the existence of some $\delta > 0$ such that

$$\frac{b^*(x) f(g(b^*(x))\xi)}{g''(b^*(x))\,\xi} > \sup_{\overline{\Omega}\setminus\Omega_0} |\nabla b^*|^2 + \frac{g'(b^*(x))}{g''(b^*(x))} \inf_{\overline{\Omega}\setminus\Omega_0} (\Delta b^*) + a\,\frac{g(b^*(x))}{g''(b^*(x))}$$

for all $x \in \Omega$ with $0 < d(x) < \delta$ and $\xi > 1$. Here, $\delta > 0$ is taken sufficiently small so that $g'(b^*(x)) < 0$ and $g''(b^*(x)) > 0$ for all $x$ with $0 < d(x) < \delta$.

For $n_0 \geq 1$ fixed, define $V^*$ as follows

(i) $V^*(x) = u_{n_0}(x) + 1$ for $x \in \overline{\Omega}$ and near $\partial\Omega$;

(ii) $V^*(x) = g(b^*(x))$ for $x$ satisfying $0 < d(x) < \delta$;

(iii) $V^* \in C^2(\overline{\Omega} \setminus \overline{\Omega}_0)$ is positive on $\overline{\Omega} \setminus \overline{\Omega}_0$.

We show that for $\xi > 1$ large enough the upper bound of the sequence $(u_n(x))_n$ can be taken as $V(x) = \xi V^*(x)$. We see that

$$\mathcal{B}V(x) = \xi\,\mathcal{B}V^*(x) \geq \xi \min\{1, \beta(x)\} \geq 0, \quad \forall x \in \partial\Omega$$

and

$$\lim_{d(x)\searrow 0} [u_n(x) - V(x)] = -\infty < 0.$$

Thus, to conclude that $u_n(x) \leq V(x)$ for all $x \in \overline{\Omega} \setminus \overline{\Omega}_0$ it is sufficient to show, by virtue of Lemma 3.2.2, that

$$-\Delta V(x) \geq aV(x) - b(x) f(V(x)), \qquad \forall x \in \Omega \setminus \overline{\Omega}_0. \tag{3.41}$$

For $x \in \Omega$ satisfying $0 < d(x) < \delta$ and $\xi > 1$, we have

$$\begin{aligned} -\Delta V - aV + b(x) f(V) &= -\xi \Delta g(b^*(x)) - a\,\xi g(b^*(x)) + b(x) f(g(b^*(x))\xi) \\ &\geq \xi g''(b^*(x)) \left( -\frac{g'(b^*(x))}{g''(b^*(x))}\,\Delta b^*(x) - |\nabla b^*(x)|^2 \right. \\ &\qquad \left. -a\,\frac{g(b^*(x))}{g''(b^*(x))} + b^*(x)\,\frac{f(g(b^*(x))\xi)}{g''(b^*(x))\,\xi} \right) > 0. \end{aligned}$$

For $x \in \Omega$ satisfying $d(x) \geq \delta$, we get

$$-\Delta V - aV + b(x) f(V) = \xi \left( -\Delta V^*(x) - aV^*(x) + b(x)\,\frac{f(\xi V^*(x))}{\xi} \right) \geq 0$$

for $\xi$ sufficiently large. In the last inequality, we have used (iii) and Remark 3.1.3. It follows that (3.41) is fulfilled provided that $\xi$ is large enough. This finishes the proof of the lemma. □

By Lemma 3.2.5, $\underline{U}_a(x) \equiv \lim_{n\to\infty} u_n(x)$ exists, for any $x \in \overline{\Omega}\setminus\overline{\Omega}_0$. Moreover, $\underline{U}_a$ is a positive large solution of (3.32). Using Lemma 3.2.2 once more, we infer that any large solution $u$ of (3.32) satisfies $u \geq u_n$ on $\overline{\Omega}\setminus\overline{\Omega}_0$, for all $n \geq 1$. Hence, $\underline{U}_a$ is the minimal large solution of (3.32).

*Step* 3.2.2. Existence of the maximal large solution of (3.32).

We show that if in (3.32) we replace $\Omega_0$ by $\Omega_m$, defined in Corollary 3.2.4, then the new problem has a minimal large solution (cf. Lemma 5.3 in Cîrstea and Rădulescu (2002*c*)).

**Lemma 3.2.6.** *Problem* (3.32) *with* $\Omega_0$ *replaced by* $\Omega_m$ *has a minimal large solution provided that* (3.4) *and* (3.6) *are fulfilled.*

*Proof.* The argument used here (much easier, since $b > 0$ on $\overline{\Omega} \setminus \Omega_m$) is similar to that in Step 3.2.1. The only difference which appears in the proof (except the replacement of $\Omega_0$ by $\Omega_m$) is related to the construction of $V^*(x)$ for $x$ near $\partial\Omega_m$. Here, instead of Lemma 3.1.8 we use Theorem 3.1.1 which says that, for any $a \in \mathbb{R}$, there exists a positive large solution $u_{a,\infty}$ of problem (3.2) in the domain $\Omega \setminus \overline{\Omega}_m$. We define $V^*(x) = u_{a,\infty}(x)$ for $x \in \Omega \setminus \overline{\Omega}_m$ and near $\partial\Omega_m$.

For $\xi > 1$ and $x \in \Omega \setminus \overline{\Omega}_m$ near $\partial\Omega_m$, we have

$$\begin{aligned} -\Delta V(x) - aV(x) + b(x)f(V(x)) &= -\xi\Delta V^*(x) - a\xi V^*(x) + b(x)f(\xi V^*(x)) \\ &= b(x)[f(\xi V^*(x)) - \xi f(V^*(x)] \geq 0. \end{aligned}$$

This completes the proof. □

Let $v_m$ be the minimal large solution for the problem considered in the statement of Lemma 3.2.6. By Lemma 3.2.2, $v_m \geq v_{m+1} \geq u$ on $\overline{\Omega} \setminus \overline{\Omega}_m$, where $u$ is any large solution of (3.32). Hence, $\overline{U}_a(x) := \lim_{m\to\infty} v_m(x) \geq u(x)$. A regularity and compactness argument shows that $\overline{U}_a$ is a positive large solution of (3.32). Consequently, $\overline{U}_a$ is the maximal large solution. This concludes the proof of Theorem 3.2.1. □

## 3.3 Degenerate Mixed Boundary Value Problems

### 3.3.1 Existence of Large Solutions

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a smooth bounded domain. As in §3.2, $\mathcal{B}$ denotes either the Dirichlet boundary operator $\mathcal{D}u := u$ or the Neumann/Robin boundary operator $\mathcal{R}u = \frac{\partial u}{\partial \nu} + \beta(x)u$, where $\nu$ is the outward unit normal to $\partial\Omega$ and $\beta \geq 0$ is in $C^{1,\mu}(\partial\Omega)$, $\mu \in (0,1)$.

Let $b \in C^{0,\mu}(\overline{\Omega})$ satisfy $b \geq 0$, $b \not\equiv 0$ in $\Omega$. Set $\Omega_{0,b} := \{x \in \Omega : \; b(x) = 0\}$.

We assume that $\Omega_{0,b} = D_0 \cup \overline{D}_1$, where $D_0 \neq \emptyset$ is a closed set such that $\Omega \setminus D_0$ is connected with the smooth boundary, and $D_1 \subset\subset \Omega \setminus D_0$ is a connected set.

Suppose that $b > 0$ on $\partial\Omega$ if $\mathcal{B} = \mathcal{R}$ and $\partial D_1$ satisfies the exterior cone condition (possibly, $D_1 = \emptyset$). Let $\lambda_{\infty,1}(D_1)$ be the first Dirichlet eigenvalue of $(-\Delta)$ in $H_0^1(D_1)$. Set $\lambda_{\infty,1}(D_1) = \infty$ if $D_1 = \emptyset$.

The purpose of section §3.3 is to prove the existence of *large solutions* for the degenerate boundary value problem

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega \setminus D_0, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.42}$$

that is, non-negative $C^2(\overline{\Omega} \setminus D_0)$-solutions of (3.42) that satisfy

$$u(x) \to \infty \quad \text{as } x \in \Omega \setminus D_0 \quad \text{and } d(x) := \operatorname{dist}(x, D_0) \to 0.$$

The degenerate character of (3.42) refers to the fact that $b$ may vanish in $\Omega \setminus D_0$.

We state below the main result of §3.3 (see Theorem 1.1 in Cîrstea and Rădulescu (2004)).

**Theorem 3.3.1.** *Let (3.4) and (3.6) hold. If (3.42) has a large solution, then $a < \lambda_{\infty,1}(D_1)$. Furthermore, for any $a < \lambda_{\infty,1}(D_1)$, there exists a minimal (resp., maximal) large solution of (3.42).*

*Remark* 3.3.1. Theorem 3.3.1 improves Theorem 3.2.1 where we assume that $b > 0$ on $\overline{\Omega} \setminus D_0$ and the additional hypothesis $\lim_{u\to\infty}(F/f)'(u) = \gamma$ is required. Moreover, in the case $\mathcal{B} = \mathcal{D}$, we remove the assumption $b > 0$ on $\partial\Omega$, which is made in Theorem 3.2.1 and in Du and Huang (1999).

In §3.3.2 we extend the comparison principles given by Lemmas 3.1.6 and 3.2.2 by treating the degenerate case for $b$. We conclude §3.3 by proving Theorem 3.3.1.

### 3.3.2 Comparison Principles

In Lemmas 3.3.2 and 3.3.3 (cf. (Cîrstea and Rădulescu, 2004, Lemmas 2.1 and 2.2)) we assume that $f$ is continuous on $(0,\infty)$ and $\frac{f(u)}{u}$ is increasing for $u>0$.

**Lemma 3.3.2.** *Let $D\subset\mathbb{R}^N$ be a bounded domain and $0\not\equiv p\in C(D)$ be a non-negative function. If $u_1,u_2\in C^2(D)$ are positive such that*

$$\Delta u_1+au_1-p(x)f(u_1)\le 0\le \Delta u_2+au_2-p(x)f(u_2)\quad \text{in } D, \tag{3.43}$$

$$\limsup_{\mathrm{dist}(x,\partial D)\to 0}(u_2-u_1)(x)\le 0$$

*then $u_1\ge u_2$ on $D$.*

*Proof.* We use here some ideas and approximation techniques introduced by Marcus and Véron (1997). Set

$$\mathcal{O}=\{x\in D:\ u_1(x)<u_2(x)\}.$$

Of course, $u_1\ge u_2$ on $D$ is equivalent to $\mathcal{O}=\emptyset$.

Let $\phi_1$, $\phi_2$ be two non-negative $C^2$-functions on $\overline{D}$ vanishing near $\partial D$.

Using (3.43), we have

$$\begin{aligned} a\int_D(u_2\phi_2-u_1\phi_1)\,dx\ge&\int_D(\nabla u_2\cdot\nabla\phi_2-\nabla u_1\cdot\nabla\phi_1)\,dx\\ &+\int_D p(x)(f(u_2)\phi_2-f(u_1)\phi_1)\,dx.\end{aligned}\tag{3.44}$$

Fix $\varepsilon>0$. Set

$$\begin{aligned} &D_\varepsilon=\{x\in D:\ u_2(x)>u_1(x)+\varepsilon\}\\ &v_i=(u_i+2\varepsilon/i)^{-1}((u_2+\varepsilon)^2-(u_1+2\varepsilon)^2)^+\quad \text{for } i=1,2.\end{aligned}$$

We see that $v_i\in H^1(D)$ and it vanishes outside the set $D_\varepsilon$. We have $D_\varepsilon\subset\subset D$ since $\limsup_{\mathrm{dist}(x,\partial D)\to 0}(u_2-u_1)(x)\le 0$. Hence, $v_i$ can be approximated closely in the $H^1 \cap L^\infty$ topology on $\overline{D}$ by non-negative $C^2$ functions vanishing near $\partial D$. It follows that (3.44) holds with $v_i$ instead of $\phi_i$. Precisely, (3.44) becomes

$$\begin{aligned} a \int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx \geq \int_{D_\varepsilon} (\nabla u_2 \cdot \nabla v_2 - \nabla u_1 \cdot \nabla v_1)\, dx \\ + \int_{D_\varepsilon} p(x)(f(u_2)v_2 - f(u_1)v_1)\, dx. \end{aligned} \tag{3.45}$$

Let $\tau \in (0,1)$ be arbitrary. For any $\varepsilon \in (0,\tau)$, we have

$$\begin{aligned} 0 \leq \int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx = \int_{D_\tau} (u_2 v_2 - u_1 v_1)\, dx \\ + \int_{D_\varepsilon \setminus D_\tau} (u_2 v_2 - u_1 v_1)\, dx. \end{aligned} \tag{3.46}$$

But $\overline{D}_\tau \subset D$ yields $\max_{\overline{D}_\tau} u_2 = M_d < \infty$ and $\min_{\overline{D}_\tau} u_1 = m_d > 0$. Thus, for any $x \in D_\tau$, we obtain

$$0 < \frac{u_2}{u_2+\varepsilon} - \frac{u_1}{u_1+2\varepsilon} \leq 1 - \frac{m_d}{m_d+2\varepsilon} = \frac{2\varepsilon}{m_d+2\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0.$$

Consequently,

$$\frac{u_2}{u_2+\varepsilon} - \frac{u_1}{u_1+2\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0 \text{ uniformly on } D_\tau.$$

It follows that

$$\begin{aligned} 0 &\leq \int_{D_\tau} (u_2 v_2 - u_1 v_1)\, dx \\ &\leq (M_d+1)^2 \int_{D_\tau} \left( \frac{u_2}{u_2+\varepsilon} - \frac{u_1}{u_1+2\varepsilon} \right) dx \to 0 \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{3.47}$$

We see that $u_2 \in (u_1+\varepsilon, u_1+\tau]$ on $D_\varepsilon \setminus D_\tau$. Thus, for each $x \in D_\varepsilon \setminus D_\tau$, we have

$$\begin{aligned} 0 < u_2 v_2 - u_1 v_1 &= \left( \frac{2\varepsilon}{u_1+2\varepsilon} - \frac{\varepsilon}{u_2+\varepsilon} \right) [(u_2+\varepsilon)^2 - (u_1+2\varepsilon)^2] \\ &\leq \frac{2\varepsilon}{u_1+2\varepsilon} [2(u_1+\varepsilon)(\tau-\varepsilon) + \tau^2 - \varepsilon^2] \\ &\leq 2\varepsilon \left[ 2(\tau-\varepsilon) + \frac{\tau^2-\varepsilon^2}{2\varepsilon} \right] < 5\tau^2. \end{aligned}$$

From this we deduce

$$\limsup_{\varepsilon \to 0} \int_{D_\varepsilon \setminus D_\tau} (u_2 v_2 - u_1 v_1)\, dx \leq 5\tau^2 |D|.$$

Using (3.46) and (3.47), we infer that

$$0 \le \liminf_{\varepsilon\to 0} \int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx \le \limsup_{\varepsilon\to 0} \int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx \le 5\tau^2 |D|.$$

Since $\tau > 0$ is arbitrary, we conclude that

$$\lim_{\varepsilon\to 0} \int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx = 0.$$

Assume by contradiction that $\mathcal{O} \neq \emptyset$. Let $x_0 \in \mathcal{O}$ be arbitrary. Since $\mathcal{O}$ is open, there exists a small closed ball $B = B(x_0)$ centred at $x_0$ such that $B \subset \mathcal{O}$. Thus, for each $y \in B$, $u_1(y) < u_2(y)$. By the continuity of $u_i$, $\min_B(u_2 - u_1) = m_B > 0$. It follows that $B \subset D_\varepsilon$, for each $\varepsilon \in (0, m_B)$. It is easy to check that

$$\begin{aligned}\nabla u_2 \cdot \nabla v_2 - \nabla u_1 \cdot \nabla v_1 = & \left| \frac{1}{u_2+\varepsilon} \nabla u_2 - \frac{1}{u_1 + 2\varepsilon} \nabla u_1 \right|^2 \\ & \times \left[(u_2+\varepsilon)^2 + (u_1+2\varepsilon)^2\right] \ge 0 \quad \text{on } D_\varepsilon.\end{aligned}$$

On the other hand, $f(t)/(t+\varepsilon)$ is increasing on $(0,\infty)$. Hence,

$$\frac{f(u_1)}{u_1+2\varepsilon} < \frac{f(u_1+\varepsilon)}{u_1+2\varepsilon} < \frac{f(u_2)}{u_2+\varepsilon} \quad \text{on } D_\varepsilon,$$

which yields $f(u_2)v_2 - f(u_1)v_1 > 0$ on $D_\varepsilon$.

Thus, all the integrands in the right-hand side of (3.45) are non-negative. So, for each $\varepsilon \in (0, m_B)$, we have

$$\begin{aligned} a\int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx \ge & \int_B (\nabla u_2 \cdot \nabla v_2 - \nabla u_1 \cdot \nabla v_1)\, dx \\ & + \int_B p(x)(f(u_2)v_2 - f(u_1)v_1)\, dx \ge 0.\end{aligned}$$

Letting $\varepsilon \searrow 0$, we obtain

$$\frac{\nabla u_2(x)}{u_2(x)} = \frac{\nabla u_1(x)}{u_1(x)} \quad \text{and} \quad p(x) = 0, \text{ for each } x \in B \ni x_0.$$

Since $x_0 \in \mathcal{O}$ is arbitrary, we infer that

$$\nabla(\ln u_2 - \ln u_1) = 0 \quad \text{and} \quad p \equiv 0 \quad \text{on } \mathcal{O}.$$

But $p \not\equiv 0$ in $D$ so that $\mathcal{O} \neq D$. In other words, $\partial\mathcal{O} \cap D \neq \emptyset$. Let $z \in \partial\mathcal{O} \cap D$ and $\mathcal{C}$ be a domain included in $\mathcal{O}$ such that $z \in \partial\mathcal{C}$. Hence $u_1(z) = u_2(z)$ and $\nabla(\ln u_2 - \ln u_1) \equiv 0$ on $\mathcal{C}$, that is, $u_2/u_1 = \text{Const.} > 0$ on $\mathcal{C}$. By the continuity of $u_i$, we conclude that $u_1 = u_2$ on $\mathcal{C}$. This contradicts $\mathcal{C} \subseteq \mathcal{O}$. □

**Lemma 3.3.3.** *Let $\omega \subset\subset \Omega$ and $0 \not\equiv p \in C(\overline{\Omega} \setminus \omega)$ be a non-negative function. If $u_1$, $u_2 \in C^2(\overline{\Omega} \setminus \overline{\omega})$ are positive functions in $\Omega \setminus \overline{\omega}$ such that*

$$\limsup_{\text{dist}\,(x,\partial\omega)\to 0} (u_2 - u_1)(x) \leq 0$$

$$\Delta u_1 + a u_1 - p(x) f(u_1) \leq 0 \leq \Delta u_2 + a u_2 - p(x) f(u_2) \quad \text{in } \Omega \setminus \overline{\omega} \tag{3.48}$$

$$\begin{cases} \text{either } \mathcal{B}u_1 \geq \mathcal{B}u_2 \text{ on } \partial\Omega & \text{if } \mathcal{B} = \mathcal{D}, \\ \text{or } \mathcal{B}u_1 \geq 0 \geq \mathcal{B}u_2 \text{ on } \partial\Omega & \text{if } \mathcal{B} = \mathcal{R}, \end{cases} \tag{3.49}$$

*then $u_1 \geq u_2$ on $\overline{\Omega} \setminus \overline{\omega}$.*

*Proof.* If $\mathcal{B} = \mathcal{D}$, then the assertion follows by Lemma 3.3.2.

Suppose that $\mathcal{B} = \mathcal{R}$. Set $D := \Omega \setminus \overline{\omega}$ and define $\mathcal{O}$ as in the proof of Lemma 3.3.2. Assume by contradiction that $\mathcal{O} = \emptyset$.

Let $\phi_1$, $\phi_2$ be two non-negative $C^2$-functions on $\overline{\Omega} \setminus \omega$ vanishing near $\partial\omega$. Using (3.48) and (3.49), we find

$$a \int_D (u_2\phi_2 - u_1\phi_1)\, dx \geq \int_{\partial\Omega} \beta\, (u_2\phi_2 - u_1\phi_1)\, dS + \int_D (\nabla u_2 \cdot \nabla \phi_2 - \nabla u_1 \cdot \nabla \phi_1)\, dx$$
$$+ \int_D p\, (f(u_2)\phi_2 - f(u_1)\phi_1)\, dx.$$

For $\varepsilon > 0$ arbitrary, let $D_\varepsilon$ and $v_i$ be defined as in the proof of Lemma 3.3.2.

It follows that

$$a \int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx \geq \int_{\partial\Omega} \beta\, (u_2 v_2 - u_1 v_1)\, dS + \int_{D_\varepsilon} (\nabla u_2 \cdot \nabla v_2 - \nabla u_1 \cdot \nabla v_1)\, dx$$
$$+ \int_{D_\varepsilon} p\, (f(u_2) v_2 - f(u_1) v_1)\, dx.$$

Let $\tau \in (0, 1)$ be arbitrary. Set

$$G_\tau = \{x \in D_\tau : \ \text{dist}(x, \partial\Omega) \geq \tau\},$$
$$L_\tau = \{x \in \Omega : \ \text{dist}(x, \partial\Omega) < \tau\},$$
$$K_{\varepsilon\tau} = \{x \in D_\varepsilon : \ \text{dist}(x, \partial\Omega) < \tau\}.$$

For any $\varepsilon \in (0, \tau)$, we have

$$0 \leq \int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx \leq \int_{K_{\varepsilon\tau}} (u_2 v_2 - u_1 v_1)\, dx + \int_{G_\tau} (u_2 v_2 - u_1 v_1)\, dx$$
$$+ \int_{D_\varepsilon \setminus D_\tau} (u_2 v_2 - u_1 v_1)\, dx. \tag{3.50}$$

As in Lemma 3.3.2, we obtain

$$\frac{u_2}{u_2+\varepsilon} - \frac{u_1}{u_1+2\varepsilon} \to 0 \ \text{ as } \varepsilon \to 0 \ \text{ uniformly on } G_\tau.$$

We also deduce

$$\lim_{\varepsilon\to 0} \int_{G_\tau} (u_2 v_2 - u_1 v_1)\, dx = 0 \text{ (see (3.47))}$$

and

$$\limsup_{\varepsilon\to 0} \int_{D_\varepsilon \setminus D_\tau} (u_2 v_2 - u_1 v_1)\, dx \le 5\tau^2 |D|.$$

Note that

$$\int_{K_{\varepsilon\tau}} (u_2 v_2 - u_1 v_1)\, dx \le 2 \max_{x\in \overline{L}_\tau} (u_2(x)+1)^2 |L_\tau|.$$

By (3.50), we find

$$\begin{aligned} 0 &\le \liminf_{\varepsilon\to 0} \int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx \le \limsup_{\varepsilon\to 0} \int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx \\ &\le 2 \max_{x\in \overline{L}_\tau} (u_2+1)^2 |L_\tau| + 5\tau^2 |D|. \end{aligned}$$

Since $|D| < \infty$ and $|L_\tau| \to 0$ as $\tau \to 0$, we regain

$$\lim_{\varepsilon\to 0} \int_{D_\varepsilon} (u_2 v_2 - u_1 v_1)\, dx = 0.$$

The same argument used before leads to a contradiction. □

### 3.3.3 Proof of the Main Result

**Lemma 3.3.4.** *Assume that (3.4) and (3.6) hold. If $0 \not\equiv \Phi \in C^{2,\mu}(\partial D_0)$ is a non-negative function, then*

$$\begin{cases} \Delta u + au = b(x) f(u) & \text{in } \Omega \setminus D_0, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \\ u = \Phi & \text{on } \partial D_0 \end{cases} \tag{3.51}$$

*has a positive solution if and only if $a < \lambda_{\infty,1}(D_1)$; in this case, the solution is unique.*

*Proof.* Let $\widetilde{\Omega}$ be a smooth sub-domain of $\Omega \setminus D_0$ such that $b > 0$ on $\partial\widetilde{\Omega}$ and $\overline{D}_1 \subset \widetilde{\Omega}$. If $u_{\mathcal{B}}$ is a positive solution of (3.51), then it satisfies

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \widetilde{\Omega}, \\ u = u_{\mathcal{B}} & \text{on } \partial\widetilde{\Omega}. \end{cases}$$

By Lemma 3.1.11, we conclude that $a \in (-\infty, \lambda_{\infty,1}(D_1))$.

Fix $a \in (-\infty, \lambda_{\infty,1}(D_1))$. Let $v_\infty$ be a large solution of $\Delta u + au = b(x)f(u)$ in $\Omega \setminus D_0$ (see Theorem 3.1.1). Let $\delta > 0$ be small such that

$$b > 0 \quad \text{on } T_{2\delta} := \{x \in \Omega : \ \text{dist}\,(x, \partial\Omega) < 2\delta\}.$$

Set

$$C_\delta = \{y \in \mathbb{R}^N : \ \text{dist}\,(y, \partial\Omega) < \delta\}.$$

Let $p \in C^{0,\mu}(\overline{C}_\delta)$ be such that $p > 0$ on $\overline{C}_\delta \setminus \overline{\Omega}$, $p = 0$ on $\overline{T}_\tau$ and $0 < p \le b$ on $\overline{T}_\delta \setminus \overline{T}_\tau$. We choose $\tau \in (0, \delta)$ such that $a$ is less than the first Dirichlet eigenvalue of $(-\Delta)$ in $T_\tau$.

Let $u^*$ be the unique positive solution of

$$\begin{cases} \Delta u + au = p(x)f(u) & \text{in } C_\delta \\ u = 1 & \text{on } \partial C_\delta. \end{cases}$$

We define $0 < u^+ \in C^2(\overline{\Omega} \setminus D_0)$ such that

$$\begin{cases} u^+ = v_\infty & \text{on } \Omega \setminus (T_\delta \cup D_0) \\ u^+ = 1 & \text{on } \overline{T}_{\delta/2} \text{ if } \mathcal{B} = \mathcal{R} \\ u^+ = u^* & \text{on } \overline{T}_{\delta/2} \text{ if } \mathcal{B} = \mathcal{D}. \end{cases}$$

For $\xi > 1$ large, we show that $\tilde{u} = \xi u^+$ is a positive super-solution of (3.51).

Clearly, $\tilde{u} = \infty$ on $\partial D_0$ and $\mathcal{B}\tilde{u} \ge 0$ on $\partial\Omega$. By (3.4), we find

$$\Delta\tilde{u} + a\tilde{u} - b(x)f(\tilde{u}) \le 0 \quad \text{on } \Omega \setminus (T_\delta \cup D_0), \ \forall \xi > 1.$$

If $\mathcal{B} = \mathcal{D}$, then

$$\Delta\tilde{u} + a\tilde{u} - b(x)f(\tilde{u}) \le \xi\Delta u^* + a\xi u^* - p(x)f(\xi u^*) \le 0 \text{ on } T_{\delta/2}, \ \forall \xi > 1.$$

If $\mathcal{B} = \mathcal{R}$ (resp., $\mathcal{B} = \mathcal{D}$) then $\min_{\overline{T}_\delta} b > 0$ (resp., $\inf_{T_\delta \setminus T_{\delta/2}} b > 0$). Since $\lim_{t\to\infty} f(t)/t = \infty$, for $\xi > 1$ large we have

$$\begin{aligned}\Delta\tilde{u} + a\tilde{u} - b(x)f(\tilde{u}) &= \xi(\Delta u^+ + au^+ - b(x)f(\xi u^+)/\xi) \\ &\leq 0 \quad \text{on } T_\delta \text{ (resp., } T_\delta \setminus T_{\delta/2}) \text{ when } \mathcal{B} = \mathcal{R} \text{ (resp., } \mathcal{B} = \mathcal{D}).\end{aligned}$$

The sub-supersolutions method and the strong maximum principle (see Corollary 3.1.4) yield the existence of a positive solution of (3.51). The uniqueness follows by Lemma 3.3.3. □

*Proof of Theorem 3.3.1 concluded.* If (3.42) has a large solution then, by the strong maximum principle, it is positive. By the assumption $\overline{D}_1 \subset \Omega \setminus D_0$ and Lemma 3.1.11, we get $a < \lambda_{\infty,1}(D_1)$.

Fix $a < \lambda_{\infty,1}(D_1)$ and let $u_n$ $(n \geq 1)$ be the unique positive solution of (3.51) with $\Phi \equiv n$. By Lemma 3.3.3, $u_n \leq u_{n+1} \leq \tilde{u}$ on $\overline{\Omega} \setminus D_0$. Thus $(u_n)$ converges to the *minimal* large solution of (3.42).

Define $\Omega_m = \{x \in \Omega : \ d(x) \leq 1/m\}$ for $m \geq m_1$, where $m_1 > 0$ is large so that $b > 0$ on $\Omega_{m_1} \setminus D_0$. Let $v_m$ be the minimal large solution of (3.42) with $D_0$ replaced by $\Omega_m$. By Lemma 3.3.3, $v_m \geq v_{m+1} \geq u$ on $\overline{\Omega} \setminus \Omega_m$, where $u$ is any large solution of (3.42). This, together with a regularity and compactness argument, shows that the pointwise limit of $(v_m)$ is the maximal large solution of (3.42). □

# Chapter 4

# Large Solutions for Logistic-type Equations: Uniqueness

**"Proof is the idol before whom the pure mathematician tortures himself."**
**(Sir Arthur Eddington)**

Chapter 3 establishes that the existence of large solutions of (3.1) holds if and only if the parameter $a$ is suitably connected to the zero set of $b(x)$ in $\Omega$. The growth rate of $f$ at infinity of Keller–Osserman type covers a large array of nonlinearities.

The delicate issues of uniqueness and asymptotic behavior of the large solution near the boundary are investigated in Chapters 4 and 5. To resolve them, Chapter 4 concentrates on the case that $f$ varies regularly at $\infty$ like a super-linear power. The originality of this chapter is to bring together regular variation theory in applied probability (see §4.1) and the blow-up theory in elliptic equations. Using this interplay, we succeed in getting a computationally convenient estimate of the blow-up rate when competition near the boundary is involved through a weight $b(x)$ vanishing on $\partial\Omega$. The decay rate of $b(x)$ is controlled via a ratio, whose limit is finite when the distance to the boundary approaches zero. This condition allows for various vanishing rates at the boundary, which will be characterized in terms of regular variation theory. In addition, a new phenomenon will be revealed in regard to the two-term asymptotic expansion of the large solution. Namely, its dependence on the curvature of the boundary occurring in the non-competing case (and also in a specific competing case) is destroyed by a critical combination between the decay rate of $b(x)$ and the variation of $f$ at $\infty$.

## 4.1 Preliminaries: Regular Variation Theory

Regular variation theory was instituted by Karamata (1930, 1933) and subsequently developed by himself and many others. Although Karamata originally introduced his theory in order to use it in Tauberian theorems, regularly varying functions have been subsequently applied in several branches of Analysis: Abelian theorems (asymptotics of series and integrals—Fourier ones in particular), analytic (entire) functions, analytic number theory, etc. The great potential of regular variation for probability theory and its applications was realised by Feller (1971) and also stimulated by de Haan (1970). The first monograph on regularly varying functions is that of Seneta (1976), while the theory and various applications of the subject are presented in the comprehensive treatise of Bingham et al. (1987).

We give a brief account of the definitions and properties of regularly varying functions involved in this thesis (see Bingham et al. (1987) or Seneta (1976)).

**Definition 4.1.1.** A positive measurable function $Z$ defined on $[A, \infty)$, for some $A > 0$, is called *regularly varying (at infinity) with index* $q \in \mathbb{R}$, written $Z \in RV_q$, provided that

$$\lim_{u \to \infty} \frac{Z(\xi u)}{Z(u)} = \xi^q, \qquad \text{for all } \xi > 0.$$

When the index $q$ is zero, we say that the function is *slowly varying*.

*Remark* 4.1.1. Let $Z : [A, \infty) \to (0, \infty)$ be a measurable function, then

(i) $Z$ is regularly varying if and only if $\lim_{u \to \infty} Z(\xi u)/Z(u)$ is finite and positive for each $\xi$ in a set $S \subset (0, \infty)$ of positive measure (cf. Lemma 1.6 and Theorem 1.3 in Seneta (1976)).

(ii) The transformation $Z(u) = u^q L(u)$ reduces regular variation to slow variation. Indeed, $\lim_{u \to \infty} Z(\xi u)/Z(u) = u^q$ if and only if $\lim_{u \to \infty} L(\xi u)/L(u) = 1$, for every $\xi > 0$.

**Example 4.1.1.** Any measurable function on $[A, \infty)$ which has a positive limit at infinity is slowly varying. The logarithm $\log u$, its iterates $\log \log u$ $(= \log_2 u)$, $\log_m u$ $(= \log \log_{m-1} u)$ and powers of $\log_m u$ are non-trivial examples of slowly varying functions. Non-logarithmic examples are given by $\exp\{(\log u)/\log\log u\}$ and $\exp\{(\log u)^{\alpha_1}(\log_2 u)^{\alpha_2}\ldots(\log_m u)^{\alpha_m}\}$, where $\alpha_i \in (0,1)$.

For details on Propositions 4.1.1–4.1.5, we refer to Bingham et al. (1987) (p. 6, 12, 14, 16, 28, 30).

**Proposition 4.1.1 (Uniform Convergence Theorem).** *If $L$ is a slowly varying function, then the convergence $L(\xi u)/L(u) \to 1$ as $u \to \infty$ holds uniformly on each compact $\xi$-set in $(0,\infty)$.*

**Proposition 4.1.2 (Representation Theorem).** *The function $L(u)$ is slowly varying if and only if it can be written in the form*

$$L(u) = M(u)\exp\left\{\int_B^u \frac{y(t)}{t}\,dt\right\} \quad (u \ge B) \tag{4.1}$$

*for some $B > A$, where $y \in C[B,\infty)$ satisfies $\lim_{u\to\infty} y(u) = 0$ and $M(u)$ is measurable on $[B,\infty)$ such that $\lim_{u\to\infty} M(u) := \widehat{M} \in (0,\infty)$.*

The Karamata representation (4.1) is non-unique because we can adjust one of $M(u)$, $y(u)$ and suitably modify the other. Thus, the function $y$ may be assumed arbitrarily smooth, but the smoothness properties of $M(u)$ can ultimately reach those of $L(u)$. If $M(u)$ is replaced by its limit at infinity $\widehat{M} > 0$, then the new function, say $\widehat{L}(u)$, is referred to as a *normalized* slowly varying function. Notice that $\widehat{L} \in C^1[B,\infty)$ and $y(u) = u\widehat{L}'(u)/\widehat{L}(u)$, for each $u \ge B$.

In general, a function $\widehat{Z}(u)$ defined for $u > B$ is called a *normalized regularly varying function of index $q$* if it is $C^1$ and satisfies

$$\lim_{u\to\infty} \frac{u\widehat{Z}'(u)}{Z(u)} = q.$$

We use $NRV_q$ to denote the set of all normalized regularly varying functions of index $q$, that is

$$NRV_q = \left\{Z \in RV_q :\ Z(u)u^{-q} \text{ is a } \textit{normalized} \text{ slowly varying function}\right\}. \tag{4.2}$$

*Remark* 4.1.2. For any $Z \in RV_q$ there exists $\widehat{Z} \in NRV_q$ such that $\widehat{Z}(u)/Z(u) \to 1$ as $u \to \infty$. Indeed, $L(u) := Z(u)/u^q$ is slowly varying so that, by Proposition 4.1.2, it can be represented as in (4.1). We define $\widehat{L}(u)$ as above with $\widehat{M}$ instead of $M$(u). Then $\widehat{Z}(u) = u^q \widehat{L}(u)$ satisfies

$$\widehat{Z} \in C^1, \lim_{u\to\infty} \frac{\widehat{Z}(u)}{Z(u)} = 1, \lim_{u\to\infty} \frac{u\widehat{Z}'(u)}{\widehat{Z}(u)} = q + \lim_{u\to\infty} \frac{u\widehat{L}'(u)}{\widehat{L}(u)} = q.$$

**Proposition 4.1.3 (Elementary properties of slowly varying functions).** *If $L$ is slowly varying, then we have:*

(i) *For any $\alpha > 0$, $u^\alpha L(u) \to \infty$, $u^{-\alpha} L(u) \to 0$ as $u \to \infty$.*

(ii) *$(L(u))^\alpha$ varies slowly for every $\alpha \in \mathbb{R}$.*

(iii) *If $L_1$ varies slowly, so do $L(u)L_1(u)$ and $L(u) + L_1(u)$.*

From Proposition 4.1.3 (i) and Remark 4.1.1 (ii), $\lim_{u\to\infty} Z(u) = \infty$ (resp., 0) for any function $Z \in RV_q$ with $q > 0$ (resp., $q < 0$).

*Remark* 4.1.3. Note that the behavior at infinity of a slowly varying function cannot be predicted. For instance,

$$L(u) = \exp\left\{(\log u)^{1/2} \cos((\log u)^{1/2})\right\}$$

exhibits infinite oscillation in the sense that

$$\liminf_{u\to\infty} L(u) = 0 \quad \text{and} \quad \limsup_{u\to\infty} L(u) = \infty.$$

**Proposition 4.1.4 (Karamata's Theorem; direct half).** *Let $Z \in RV_q$ be locally bounded in $[A, \infty)$, then*

(i) *for any $j \geq -(q+1)$,*

$$\lim_{u\to\infty} \frac{u^{j+1} Z(u)}{\int_A^u x^j Z(x)\, dx} = j + q + 1. \tag{4.3}$$

(ii) *for any $j < -(q+1)$ (and for $j = -(q+1)$ if $\int^\infty x^{-(q+1)} Z(x)\, dx < \infty$)*

$$\lim_{u\to\infty} \frac{u^{j+1} Z(u)}{\int_u^\infty x^j Z(x)\, dx} = -(j + q + 1). \tag{4.4}$$

**Proposition 4.1.5 (Karamata's Theorem; converse half).** *Let $Z$ be positive and locally integrable in $[A, \infty)$.*

(i) *If (4.3) holds for some $j > -(q+1)$, then $Z \in RV_q$.*

(ii) *If (4.4) is satisfied for some $j < -(q+1)$, then $Z \in RV_q$.*

If $H$ is a non-decreasing function on $\mathbb{R}$, then we define (as in Resnick (1987)) the (left continuous) inverse of $H$ by

$$H^{\leftarrow}(y) = \inf\{s : \ H(s) \geq y\}.$$

**Proposition 4.1.6 (Proposition 0.8 in Resnick (1987)).** *We have*

(i) *If $Z \in RV_q$, then $\lim_{u \to \infty} \log Z(u) / \log u = q$.*

(ii) *If $Z_1 \in RV_{q_1}$ and $Z_2 \in RV_{q_2}$ with $\lim_{u\to\infty} Z_2(u) = \infty$, then*

$$Z_1 \circ Z_2 \in RV_{q_1 q_2}.$$

(iii) *Suppose $Z$ is non-decreasing, $Z(\infty) = \infty$, and $Z \in RV_q$, $0 < q < \infty$, then*

$$Z^{\leftarrow} \in RV_{1/q}.$$

(iv) *Suppose $Z_1$, $Z_2$ are non-decreasing and $q$-varying, $0 < q < \infty$, then, for $c \in (0, \infty)$ we have*

$$\lim_{u\to\infty} \frac{Z_1(u)}{Z_2(u)} = c \quad \textit{if and only if} \quad \lim_{u\to\infty} \frac{Z_1^{\leftarrow}(u)}{Z_2^{\leftarrow}(u)} = c^{-1/q}.$$

The next result shows that any function $Z$ varying regularly with non-zero index is asymptotic to a monotone function.

**Proposition 4.1.7 (Theorem 1.5.3 in Bingham et al. (1987)).** *Let $Z \in RV_q$ and choose $B \geq 0$ so that $Z$ is locally bounded on $[B, \infty)$. If $q > 0$, then*

(i) *$\overline{Z}(u) := \sup\{Z(y) : \ B \leq y \leq u\} \sim Z(u)$ as $u \to \infty$,*

(ii) *$\underline{Z}(u) := \inf\{Z(y) : \ y \geq u\} \sim Z(u)$ as $u \to \infty$.*

*If $q < 0$, then*

$$\sup\{Z(y): \ y \geq u\} \sim Z(u) \quad \text{as } u \to \infty,$$
$$\inf\{Z(y): \ B \leq y \leq u\} \sim Z(u) \quad \text{as } u \to \infty.$$

We give the definitions of $\Gamma$ and $\Pi$-varying functions, which are extensions of regular variation due to de Haan (1970); for more details see §5.3.2 of Chapter 5.

**Definition 4.1.2 (p. 26 in Resnick (1987)).** A non-decreasing function $U$ is $\Gamma$-varying at $\infty$ (written $U \in \Gamma$) if $U$ is defined on an interval $(A, \infty)$, $\lim_{x\to\infty} U(x) = \infty$ and there exists $g : (A, \infty) \to (0, \infty)$ such that

$$\lim_{y\to\infty} \frac{U(y + \lambda g(y))}{U(y)} = e^{\lambda}, \ \forall \lambda \in \mathbb{R}.$$

**Definition 4.1.3 (p. 27 in Resnick (1987)).** A non-negative, non-decreasing function $V$ defined on a semi-infinite interval $(z, \infty)$ is $\Pi$-varying (written $V \in \Pi$) if there exists a function $\alpha(u) > 0$ such that for $\lambda > 0$

$$\lim_{u\to\infty} \frac{V(\lambda u) - V(u)}{\alpha(u)} = \log \lambda. \tag{4.5}$$

The functions $g(\cdot)$ and $\alpha(\cdot)$ are called *auxiliary functions*; they are unique up to asymptotic equivalence.

A convenient relationship between $\Pi$ and $\Gamma$ is provided below.

**Proposition 4.1.8 (Proposition 0.9 in Resnick (1987)).** *The following hold:*

(a) *If $U \in \Gamma$ with auxiliary function $\chi$, then $U^{\leftarrow} \in \Pi$ with auxiliary function $\alpha(u) = \chi \circ U^{\leftarrow}(u)$.*

(b) *If $V \in \Pi$ with auxiliary function $\alpha(\cdot)$, then $V^{\leftarrow} \in \Gamma$ with auxiliary function $\chi(u) = \alpha \circ V^{\leftarrow}(u)$.*

## 4.2 Pure Boundary Blow-up Problems

### 4.2.1 Introduction

Our major goal here is to advance innovative methods to study the uniqueness and asymptotic behavior of large solutions of the problem (3.2) considered in §3.1.1.

We open up a new line of research to obtain the blow-up rate of the large solution and to gain insight into the two-term asymptotic expansion of the large solution near $\partial\Omega$. Our approach relies essentially on *regular variation theory* not only in the statement but in the proof as well. This enables us to obtain significant information about the qualitative behavior of the large solution to (3.2) in a general setting that removes previous restrictions imposed in the literature.

We point out that, despite a long history and intense research on the topic of large solutions, regular variation theory arising in probability theory has not been exploited before in this context.

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a smooth bounded domain. Consider the semilinear elliptic equation

$$\Delta u + au = b(x)f(u) \qquad \text{in } \Omega, \tag{4.6}$$

where $a$ is a real parameter and $b \in C^{0,\mu}(\overline{\Omega})$, for some $\mu \in (0,1)$, such that $b \geq 0$, $b \not\equiv 0$ in $\Omega$. Suppose that $f \in C^1[0,\infty)$ satisfies (3.4), that is

$$f \geq 0 \text{ and } f(u)/u \text{ is increasing on } (0,\infty). \tag{4.7}$$

Let $\Omega_0$ denote the interior of the zero set of $b$ in $\Omega$, namely:

$$\Omega_0 := \text{int}\,\{x \in \Omega : \; b(x) = 0\}.$$

We assume, throughout, that $\partial\Omega_0$ satisfies the exterior cone condition (possibly, $\Omega_0 = \emptyset$), $\Omega_0$ is connected, $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ on $\Omega \setminus \overline{\Omega}_0$. Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of $(-\Delta)$ in $H_0^1(\Omega_0)$ (with $\lambda_{\infty,1} = +\infty$ if $\Omega_0 = \emptyset$).

We recall that by a *large* (or *blow-up*) solution of (4.6) we mean a $C^2(\Omega)$-solution $u$ of (4.6) such that $u \geq 0$ in $\Omega$ and $u(x) \to \infty$ as $d(x) := \text{dist}\,(x, \partial\Omega) \to 0$.

As in Cîrstea (2002), we denote by $\mathcal{K}$ the set of all positive, non-decreasing $C^1$-functions $k$ defined on $(0,\nu)$, for some $\nu > 0$, which satisfy

$$\lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{k(t)} = \ell_0 \quad \text{and} \quad \lim_{t\to 0^+} \left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)' = \ell_1. \tag{4.8}$$

We have $\ell_0 = 0$ and $\ell_1 \in [0,1]$, for every $k \in \mathcal{K}$. Thus, $\mathcal{K} = \mathcal{K}_{(01]} \cup \mathcal{K}_0$, where

$$\mathcal{K}_{(01]} = \{k \in \mathcal{K} : \quad 0 < \ell_1 \le 1\} \quad \text{and} \quad \mathcal{K}_0 = \{k \in \mathcal{K} : \quad \ell_1 = 0\}.$$

The exact characterization of $\mathcal{K}_{(01]}$ and $\mathcal{K}_0$ will be provided in §4.2.4.

Some simple examples of $k \in \mathcal{K}$ are:

(i) $k(t) = t^\alpha$ with $\alpha > 0$, where $\ell_1 = 1/(1+\alpha)$.

(ii) $k(t) = \exp(-1/t^\alpha)$ with $\alpha > 0$, where $\ell_1 = 0$.

(iii) $k(t) = 1/\ln(1/t^\alpha)$ with $\alpha > 0$, where $\ell_1 = 1$.

### 4.2.2 Main Results

We first establish the uniqueness and blow-up rate of the large solution of (4.6) (see Theorem 1.1 in Cîrstea and Rădulescu (2005)).

**Theorem 4.2.1.** *Let (4.7) hold and $f \in RV_{\rho+1}$ with $\rho > 0$. Suppose there exists $k \in \mathcal{K}$ such that*

$$b(x) = k^2(d) + o(k^2(d)) \text{ as } d(x) \to 0, \tag{4.9}$$

*then, for any $a \in (-\infty, \lambda_{\infty,1})$, (4.6) admits a unique large solution $u_a$. Moreover, the asymptotic behavior is given by*

$$u_a(x) = [2(2+\ell_1\rho)/\rho^2]^{1/\rho}\,\varphi(d) + o(\varphi(d)) \quad \text{as } d(x) \to 0, \tag{4.10}$$

*where $\varphi$ is defined by*

$$\frac{f(\varphi(t))}{\varphi(t)} = \frac{1}{\left(\int_0^t k(s)\,ds\right)^2}, \quad \text{for } t > 0 \text{ small.} \tag{4.11}$$

*Remark* 4.2.1. This result generalizes Theorem 1 in Cîrstea (2002), where the case $f(u) = u^{\rho+1}$ has been treated.

In the setting of Theorem 4.2.1, let $r(t)$ satisfy $\lim_{t\searrow 0}\left(\int_0^t k(s)\,ds\right)^2 r(t)=1$ and $\widehat{f}(u)$ be chosen such that $\lim_{u\to\infty}\widehat{f}(u)/f(u)=1$ and $j(u)=\widehat{f}(u)/u$ is non-decreasing for $u>0$ large. Then, $\lim_{t\searrow 0}\varphi(t)/\widehat{\varphi}(t)=1$, where $\varphi$ is defined by (4.11) and $\widehat{\varphi}(t)=j^{\leftarrow}(r(t))$ for $t>0$ small.

The behavior of $\varphi(t)$ for small $t>0$ will be described in §4.2.4. In particular, if $k\in\mathcal{K}$ with $\ell_1\neq 0$, then $\varphi(1/u)\in RV_{2/(\rho\ell_1)}$. In contrast, if $k\in\mathcal{K}$ with $\ell_1=0$, then $\varphi(1/u)\notin RV_q$, for all $q\in\mathbb{R}$ (see Remark 4.2.8).

*Remark* 4.2.2. The asymptotic behavior of the unique large solution $u_a$ can also be expressed as follows (cf. Theorem 1 in Cîrstea and Rădulescu (2002*d*))

$$u_a(x)=\xi_0 h(d)+o(h(d))\quad \text{as } d(x)\to 0, \tag{4.12}$$

where $\xi_0=\left(\frac{2+\ell_1\rho}{2+\rho}\right)^{1/\rho}$ and $h$ is given by

$$\int_{h(t)}^{\infty}\frac{ds}{\sqrt{2F(s)}}=\int_0^t k(s)\,ds,\quad \text{for } t>0 \text{ small.} \tag{4.13}$$

The next objective is to find the two-term blow-up rate of $u_a$ when (4.9) is replaced by

$$b(x)=k^2(d)(1+\widetilde{c}d^\theta+o(d^\theta)) \text{ as } d(x)\to 0, \tag{4.14}$$

where $\theta>0$, $\widetilde{c}\in\mathbb{R}$ are constants. To simplify the exposition, we assume that $f'\in RV_\rho$ $(\rho>0)$, which is equivalent to $f(u)$ being of the form

$$f(u)=Cu^{\rho+1}\exp\left\{\int_B^u\frac{\phi(t)}{t}\,dt\right\},\quad \forall u\geq B, \tag{4.15}$$

for some constants $B$, $C>0$, where $\phi\in C[B,\infty)$ satisfies $\lim_{u\to\infty}\phi(u)=0$. In this case, $f(u)/u$ is increasing on $[B,\infty)$ provided that $B$ is large enough.

We prove that the two-term asymptotic expansion of $u_a$ near $\partial\Omega$ depends on the chosen subclass for $k\in\mathcal{K}$ and the additional hypotheses on $f$ (by means of $\phi$ in (4.15)).

Let $-\rho-2<\eta\leq 0$ and $\tau,\zeta>0$. We define

$$
\begin{aligned}
\mathcal{F}_{\rho\eta} &= \{f' \in RV_\rho\ (\rho>0):\ \text{either } \phi\in RV_\eta \text{ or } -\phi\in RV_\eta\},\\
\mathcal{F}_{\rho 0,\tau} &= \{f\in\mathcal{F}_{\rho 0}:\ \lim_{u\to\infty}(\ln u)^\tau \phi(u)=\ell^\star\in\mathbb{R}\},\\
\mathcal{K}_{(01],\tau} &= \left\{k\in\mathcal{K}_{(01]}:\ \lim_{t\searrow 0}(-\ln t)^\tau\left[\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)'-\ell_1\right]:=L_\sharp\in\mathbb{R}\right\},\\
\mathcal{K}_{0,\zeta} &= \left\{k\in\mathcal{K}_0:\ \lim_{t\searrow 0}\frac{1}{t^\zeta}\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)':=L_\star\in\mathbb{R}\right\}.
\end{aligned}
$$

Further on in section 4.2, $\eta$, $\tau$ and $\zeta$ are understood to be in the above range.

The next result is a consequence of Theorem 1 in Cîrstea and Rădulescu (2003*a*) and Proposition 4.2.11.

**Theorem 4.2.2.** *Suppose* (4.7), (4.14) *with* $k\in\mathcal{K}_{0,\zeta}$, *and one of the following growth conditions at infinity:*

(i) $f(u)=Cu^{\rho+1}$ *in a neighborhood of infinity (i.e.,* $\phi\equiv 0$ *in* (4.15)*);*

(ii) $f\in\mathcal{F}_{\rho\eta}$ *with* $\eta\neq 0$*;*

(iii) $f\in\mathcal{F}_{\rho 0,\tau_1}$ *with* $\tau_1=\varpi/\zeta$, *where* $\varpi=\min\{\theta,\zeta\}$.

*For any* $a\in(-\infty,\lambda_{\infty,1})$, *the two-term blow-up rate of* $u_a$ *is then*

$$u_a(x)=\xi_0 h(d)(1+\chi d^\varpi+o(d^\varpi))\quad \text{as } d(x)\searrow 0 \tag{4.16}$$

*where* $h$ *is given by* (4.13), $\xi_0=[2/(2+\rho)]^{1/\rho}$ *and*

$$
\chi=\begin{cases}
\dfrac{L_\star}{2}\text{Heaviside}(\theta-\zeta)-\dfrac{\widetilde{c}}{\rho}\text{Heaviside}(\zeta-\theta):=\chi_1 & \text{if (i) or (ii) holds,}\\[2ex]
\chi_1-\dfrac{\ell^\star}{\rho}\left[\dfrac{\rho\zeta L_\star}{2(1+\zeta)}\right]^{\tau_1}\left(\dfrac{1}{\rho+2}+\ln\xi_0\right) & \text{if } f \text{ obeys (iii).}
\end{cases}
$$

The situation corresponding to $k\in\mathcal{K}_{(01]}$ is treated below (see Theorem 1.3 in Cîrstea and Rădulescu (2005)).

**Theorem 4.2.3.** *Suppose* (4.7), (4.14) *with* $k \in \mathcal{K}_{(01],\tau}$, *and one of the following conditions:*

(i) $f \in \mathcal{F}_{\rho\eta}$ *with* $\eta L_\sharp \neq 0$;

(ii) $f \in \mathcal{F}_{\rho 0,\tau}$ *with* $[\ell^\star(\ell_1 - 1)]^2 + L_\sharp^2 \neq 0$.

*For any* $a \in (-\infty, \lambda_{\infty,1})$, *the two-term blow-up rate of* $u_a$ *is then*

$$u_a(x) = \xi_0 h(d)[1 + \widetilde{\chi}\,(-\ln d)^{-\tau} + o((-\ln d)^{-\tau})] \quad \textit{as } d(x) \searrow 0, \tag{4.17}$$

*where* $h$ *is given by* (4.13), $\xi_0 = [(2 + \ell_1\rho)/(2 + \rho)]^{1/\rho}$ *and*

$$\widetilde{\chi} = \begin{cases} \dfrac{L_\sharp}{2 + \rho\ell_1} := \chi_2 & \textit{if (i) holds,} \\ \chi_2 - \dfrac{\ell^\star}{\rho}\left(\dfrac{\rho\ell_1}{2}\right)^\tau \left[\dfrac{2(1-\ell_1)}{(\rho+2)(\rho\ell_1+2)} + \ln \xi_0\right] & \textit{if } f \textit{ obeys (ii).} \end{cases} \tag{4.18}$$

*Remark* 4.2.3. Note that Theorems 4.2.2 and 4.2.3 distinguish from Theorem 1 in García-Melián et al. (2001), which treats the particular case $f(u) = u^p$ $(p > 1)$, $\Omega_0 = \emptyset$, $k(t) = \sqrt{C_0 t^\gamma}$ $(C_0, \gamma > 0)$ and $\theta = 1$ in (4.14). The second term in the asymptotic expansion of $u_a$ near $\partial\Omega$ involves, in García-Melián et al. (2001), both the distance function $d(x)$ and the mean curvature of $\partial\Omega$.

Theorem 4.2.2 admits the case $f(u) = u^p$ assuming that $k \in \mathcal{K}_{0,\zeta}$, while the alternative (ii) of Theorem 4.2.3 includes the case $k(t) = \sqrt{C_0 t^\gamma}$ (when $L_\sharp = 0$) provided that $f \in \mathcal{F}_{\rho 0,\tau}$ with $\ell^\star \neq 0$. Relations (4.16) and (4.17) show how dramatically the two-term asymptotic expansion of $u_a$ changes from the result in García-Melián et al. (2001). Our approach is completely different from that in Bandle and Essén (1994); Bandle and Marcus (1992*a*); García-Melián et al. (2001); Lazer and McKenna (1994), as we essentially use Karamata's theory.

We point out that the general asymptotic results stated in the above theorems do not involve the difference or the quotient of $u(x)$ and $\psi(d(x))$, as established in Bandle and Marcus (1992*a*), Bieberbach (1916), Lazer and McKenna (1994), Rademacher (1943) for $a = 0$ and $b = 1$, where $\psi$ is a large solution of

$$\psi''(r) = f(\psi(r)) \qquad \text{on } (0, \infty)\,.$$

The rest of the section 4.2 is organized as follows. In §4.2.3 we prove some auxiliary results which will be repeatedly invoked. In §4.2.4 we characterize the class $\mathcal{K}$ as well as its subclasses $\mathcal{K}_{0,\zeta}$ and $\mathcal{K}_{(01],\tau}$ that appear in Theorems 4.2.2 and 4.2.3. The uniqueness of the large solution of (4.6) will be proved in §4.2.5. We provide two proofs of Theorem 4.2.1: the first one follows the pattern in Cîrstea (2002) (where $f(u) = u^{\rho+1}$) and Cîrstea and Rădulescu (2002*d*) (where $f' \in RV_\rho$), while the second one is useful to our next purpose. The asymptotic expansion of the large solution given by Theorems 4.2.2 and 4.2.3 will be analyzed in §4.2.6.

### 4.2.3 Auxiliary Results

Based on regular variation theory, we first prove two lemmas that have been only stated in Cîrstea and Rădulescu (2003*a*). The results of §4.2.3, with the exception of Lemma 4.2.6 appearing in Cîrstea and Rădulescu (2003*a*), have been incorporated in Cîrstea and Rădulescu (2005).

*Remark* 4.2.4. If $f \in RV_{\rho+1}$ ($\rho > 0$) is continuous, then

$$\Xi(u) := \frac{\sqrt{F(u)}}{f(u)\int_u^\infty [F(s)]^{-1/2}\,ds} \to \frac{\rho}{2(\rho+2)} \quad \text{as } u \to \infty, \tag{4.19}$$

where $F$ stands for an antiderivative of $f$. Indeed, by Proposition 4.1.4, we have

$$\lim_{u\to\infty} \frac{F(u)}{uf(u)} = \frac{1}{\rho+2} \quad \text{and} \quad \lim_{u\to\infty} \frac{u[F(u)]^{-1/2}}{\int_u^\infty [F(s)]^{-1/2}\,ds} = \frac{\rho}{2}. \tag{4.20}$$

**Lemma 4.2.4 (Properties of $h$).** *If $f \in RV_{\rho+1}$ ($\rho > 0$) is continuous and $k \in \mathcal{K}$, then $h$ defined by (4.13) is a $C^2$-function satisfying the following:*

(i) $\displaystyle\lim_{t\searrow 0} \frac{h''(t)}{k^2(t)f(\xi h(t))} = \frac{2+\rho\ell_1}{\xi^{\rho+1}(2+\rho)}$, *for each* $\xi > 0$;

(ii) $\displaystyle\lim_{t\searrow 0} \frac{h(t)h''(t)}{[h'(t)]^2} = \frac{2+\rho\ell_1}{2}$ *and* $\displaystyle\lim_{t\searrow 0} \frac{\ln k(t)}{\ln h(t)} = \frac{\rho(\ell_1-1)}{2}$;

(iii) $\displaystyle\lim_{t\searrow 0} \frac{h'(t)}{th''(t)} = -\frac{\rho\ell_1}{2+\rho\ell_1}$ *and* $\displaystyle\lim_{t\searrow 0} \frac{h(t)}{t^2h''(t)} = \frac{\rho^2\ell_1^2}{2(2+\rho\ell_1)}$;

(iv) $\displaystyle\lim_{t\searrow 0} \frac{h(t)}{th'(t)} = \lim_{t\searrow 0} \frac{\ln t}{\ln h(t)} = -\frac{\rho\ell_1}{2}$;

(v) $\lim_{t\searrow 0} t^j h(t) = \infty$, *for all* $j > 0$, *provided that* $k \in \mathcal{K}_0$. *If, in addition,* $k \in \mathcal{K}_{0,\zeta}$ *then* $\lim_{t\searrow 0} \frac{1}{-\zeta t^\zeta \ln h(t)} = \lim_{t\searrow 0} \frac{h'(t)}{t^{\zeta+1} h''(t)} = \frac{-\rho L_\star}{2(\zeta+1)}$.

*Proof.* By (4.13), the function $h \in C^2(0,\nu)$, for some $\nu > 0$, and $\lim_{t\searrow 0} h(t) = \infty$.
For any $t \in (0,\nu)$, we have $h'(t) = -k(t)\sqrt{2F(h(t))}$ and

$$h''(t) = k^2(t) f(h(t)) \left\{ 1 + 2\Xi(h(t)) \left[ \left( \frac{\int_0^t k(s)\,ds}{k(t)} \right)' - 1 \right] \right\}. \tag{4.21}$$

Using Remark 4.2.4 and $f \in RV_{\rho+1}$, we reach (i).

(ii). By (i) and (4.20), we get

$$\lim_{t\searrow 0} \frac{h(t)h''(t)}{[h'(t)]^2} = \lim_{t\searrow 0} \frac{h''(t)}{k^2(t) f(h(t))} \frac{h(t) f(h(t))}{2F(h(t))} = \frac{2+\rho\ell_1}{2}, \tag{4.22}$$

respectively

$$\lim_{t\searrow 0} \frac{k'(t)}{k(t)} \frac{h(t)}{h'(t)} = \lim_{t\searrow 0} \frac{h(t) f(h(t))}{F(h(t))} \frac{-\left(\int_0^t k(s)\,ds\right)}{k^2(t)/k'(t)} \Xi(h(t)) = \frac{\rho(\ell_1 - 1)}{2}. \tag{4.23}$$

(iii). Using (i) and Remark 4.2.4, we find

$$\lim_{t\searrow 0} \frac{h'(t)}{th''(t)} = \frac{-2(2+\rho)}{2+\rho\ell_1} \lim_{t\searrow 0} \frac{\int_0^t k(s)\,ds}{tk(t)} \Xi(h(t)) = \frac{-\rho\ell_1}{2+\rho\ell_1},$$

which, together with (4.22), implies that

$$\lim_{t\searrow 0} \frac{h(t)}{t^2 h''(t)} = \lim_{t\searrow 0} \frac{h(t)h''(t)}{[h'(t)]^2} \left[ \frac{h'(t)}{th''(t)} \right]^2 = \frac{\rho^2 \ell_1^2}{2(2+\rho\ell_1)}.$$

(iv). If $\ell_1 \neq 0$, then by (iii), we have

$$\lim_{t\searrow 0} \frac{h(t)}{th'(t)} = \lim_{t\searrow 0} \frac{h(t)}{t^2 h''(t)} \frac{th''(t)}{h'(t)} = \frac{-\rho\ell_1}{2}.$$

If $\ell_1 = 0$, then we derive

$$\lim_{t\searrow 0} \frac{k(t)}{tk'(t)} = \lim_{t\searrow 0} \frac{k^2(t)}{k'(t)\left(\int_0^t k(s)\,ds\right)} \frac{\int_0^t k(s)\,ds}{tk(t)} = 0. \tag{4.24}$$

This and (4.23) yield $\lim_{t\searrow 0} \frac{h(t)}{th'(t)} = 0$, which concludes (iv).

(v). If $k \in \mathcal{K}_0$, then using (iv), we obtain $\lim_{t\searrow 0} \ln[t^j h(t)] = \infty$, for all $j > 0$. Suppose $k \in \mathcal{K}_{0,\zeta}$, for some $\zeta > 0$. Then, $\lim_{t\searrow 0} \frac{\int_0^t k(s)\,ds}{t^{\zeta+1}k(t)} = \frac{L_\star}{\zeta+1}$ and

$$\frac{L_\star}{\zeta+1} = \lim_{t\searrow 0} \frac{\int_0^t k(s)\,ds}{t^{\zeta+1}k(t)} \frac{k^2(t)/k'(t)}{\left(\int_0^t k(s)\,ds\right)} = \lim_{t\searrow 0} \frac{k(t)}{t^{\zeta+1}k'(t)} = \frac{-1}{\zeta} \lim_{t\searrow 0} \frac{1}{t^\zeta \ln k(t)}. \quad (4.25)$$

By (4.22), (4.23) and (4.25), we deduce

$$\lim_{t\searrow 0} \frac{h'(t)}{t^{\zeta+1}h''(t)} = \lim_{t\searrow 0} \frac{h(t)}{h'(t)t^{\zeta+1}} = \lim_{t\searrow 0} \frac{k'(t)h(t)}{k(t)h'(t)} \frac{k(t)}{t^{\zeta+1}k'(t)} = \frac{-\rho L_\star}{2(\zeta+1)}.$$

This completes the proof of the lemma. □

Let $\tau > 0$ be arbitrary and let $f$ be as in Remark 4.2.4. For $u > 0$ sufficiently large, we define

$$T_{1,\tau}(u) = \left[\frac{\rho}{2(\rho+2)} - \Xi(u)\right] (\ln u)^\tau \text{ and } T_{2,\tau}(u) = \left[\frac{f(\xi_0 u)}{\xi_0 f(u)} - \xi_0^\rho\right] (\ln u)^\tau. \quad (4.26)$$

*Remark* 4.2.5. When $f(u) = Cu^{\rho+1}$, we have $T_{1,\tau}(u) = T_{2,\tau}(u) = 0$.

**Lemma 4.2.5.** *Let $f \in \mathcal{F}_{\rho\eta}$ (where $-\rho-2 < \eta \leq 0$). The following hold:*

(i) *If $f \in \mathcal{F}_{\rho 0,\tau}$, then*

$$\lim_{u\to\infty} T_{1,\tau}(u) = \frac{-\ell^\star}{(\rho+2)^2} \quad \textit{and} \quad \lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \ell^\star \ln \xi_0.$$

(ii) *If $f \in \mathcal{F}_{\rho\eta}$ with $\eta \neq 0$, then*

$$\lim_{u\to\infty} T_{1,\tau}(u) = \lim_{u\to\infty} T_{2,\tau}(u) = 0.$$

*Proof.* Using the second limit in (4.20), we obtain

$$\lim_{u\to\infty} T_{1,\tau}(u) = \frac{\rho}{2} \lim_{u\to\infty} \frac{\frac{\rho}{2(\rho+2)} \int_u^\infty [F(s)]^{-1/2}\,ds - \sqrt{F(u)}/f(u)}{u[F(u)]^{-1/2}\,(\ln u)^{-\tau}}.$$

By L'Hospital's rule, we arrive at

$$\lim_{u\to\infty} T_{1,\tau}(u) = \lim_{u\to\infty} \left[\frac{\rho+1}{\rho+2} - \frac{F(u)f'(u)}{f^2(u)}\right] (\ln u)^\tau := \lim_{u\to\infty} Q_{1,\tau}(u).$$

A simple calculation shows that, for $u > 0$ large,

$$Q_{1,\tau}(u) = \frac{(\ln u)^\tau}{\rho+2}\left[\rho + 1 - \frac{uf'(u)}{f(u)}\right] + \frac{uf'(u)}{f(u)}\left[\frac{1}{\rho+2} - \frac{F(u)}{uf(u)}\right](\ln u)^\tau$$
$$=: \frac{1}{\rho+2}Q_{2,\tau}(u) + \frac{uf'(u)}{f(u)}Q_{3,\tau}(u).$$

Since (4.15) holds with $\phi \in RV_\eta$ or $-\phi \in RV_\eta$, we can assume $B > 0$ such that $\phi \neq 0$ on $[B, \infty)$. For any $u > B$, we have $Q_{2,\tau}(u) = -\phi(u)(\ln u)^\tau$ and

$$Q_{3,\tau}(u) = \widetilde{C}\,\frac{(\ln u)^\tau}{uf(u)} + \frac{\int_B^u f(s)\phi(s)\,ds}{(\rho+2)uf(u)\phi(u)}\,\phi(u)(\ln u)^\tau,$$

where $\widetilde{C} \in \mathbb{R}$ is a constant. Since either $f\phi \in RV_{\rho+\eta+1}$ or $-f\phi \in RV_{\rho+\eta+1}$, by Proposition 4.1.4,

$$\lim_{u\to\infty} \frac{uf(u)\phi(u)}{\int_B^u f(x)\phi(x)\,dx} = \rho + \eta + 2.$$

If (i) holds, then $\lim_{u\to\infty} Q_{2,\tau}(u) = -\ell^\star$ and $\lim_{u\to\infty} Q_{3,\tau}(u) = \ell^\star(\rho+2)^{-2}$. Thus,

$$\lim_{u\to\infty} T_{1,\tau}(u) = \lim_{u\to\infty} Q_{1,\tau}(u) = -\ell^\star/(\rho+2)^2.$$

If (ii) holds, then by Proposition 4.1.3, $\lim_{u\to\infty}(\ln u)^\tau\phi(u) = 0$. It follows that

$$\lim_{u\to\infty} Q_{2,\tau}(u) = \lim_{u\to\infty} Q_{3,\tau}(u) = 0$$

which yields $\lim_{u\to\infty} T_{1,\tau}(u) = 0$. Note that the proof is finished if $\xi_0 = 1$, since $T_{2,\tau}(u) = 0$ for each $u > 0$.

Arguing by contradiction, let us suppose that $\xi_0 \neq 1$, then, by (4.15),

$$T_{2,\tau}(u) = \xi_0^\rho\left[\exp\left\{\int_u^{\xi_0 u} \frac{\phi(t)}{t}\,dt\right\} - 1\right](\ln u)^\tau, \quad \forall u > B/\xi_0.$$

But, $\lim_{u\to\infty} \phi(us)/s = 0$, uniformly with respect to $s \in [\xi_0, 1]$. So

$$\lim_{u\to\infty} \int_u^{\xi_0 u} \frac{\phi(t)}{t}\,dt = \lim_{u\to\infty} \int_1^{\xi_0} \frac{\phi(su)}{s}\,ds = 0$$

which leads to

$$\lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \lim_{u\to\infty}\left(\int_u^{\xi_0 u} \frac{\phi(t)}{t}\,dt\right)(\ln u)^\tau.$$

If (i) occurs, then by Proposition 4.1.1, we have

$$\lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \lim_{u\to\infty} (\ln u)^\tau \phi(u) \int_1^{\xi_0} \frac{\phi(tu)}{\phi(u)} \frac{dt}{t} = \xi_0^\rho \ell^\star \ln \xi_0.$$

If (ii) occurs, then by Proposition 4.1.3, we infer that

$$\lim_{u\to\infty} T_{2,\tau}(u) = \frac{-\xi_0^\rho}{\tau} \lim_{u\to\infty} \left[\phi(\xi_0 u) - \phi(u)\right] (\ln u)^{\tau+1} = 0.$$

The proof of Lemma 4.2.5 is now complete. □

**Lemma 4.2.6.** *Under the assumptions of Theorem 4.2.2, we have*

$$\mathfrak{I}(t) := t^{-\varpi} \left(1 - \frac{k^2(t) f(\xi_0 h(t))}{\xi_0 h''(t)}\right) \to \rho\chi + \widetilde{c}\,\text{Heaviside}\,(\zeta - \theta) \quad \text{as } t \searrow 0. \quad (4.27)$$

*Proof.* From (4.21), $\mathfrak{I}(t) = \frac{k^2(t) f(h(t))}{h''(t)} \sum_{j=1}^3 \mathfrak{I}_j(t)$, for $t > 0$ small, where

$$\begin{cases} \mathfrak{I}_1(t) := 2\dfrac{\Xi(h(t))}{t^\varpi} \left(\dfrac{\int_0^t k(s)\,ds}{k(t)}\right)' \to \dfrac{\rho L_\star}{\rho+2} \text{Heaviside}\,(\theta - \zeta) \quad \text{as } t \searrow 0, \\ \mathfrak{I}_2(t) = 2\dfrac{T_{1,\tau_1}(h(t))}{[t^\zeta \ln h(t)]^{\tau_1}} \quad \text{and} \quad \mathfrak{I}_3(t) = -\dfrac{T_{2,\tau_1}(h(t))}{[t^\zeta \ln h(t)]^{\tau_1}}. \end{cases}$$

*Case* (i) *or* (ii) *of Theorem* 4.2.2. By Lemmas 4.2.4 and 4.2.5, we have $\lim_{t\searrow 0} \mathfrak{I}_2(t) = \lim_{t\searrow 0} \mathfrak{I}_3(t) = 0$. Thus, we arrive at

$$\lim_{t\searrow 0} \mathfrak{I}(t) = \frac{\rho L_\star}{2} \text{Heaviside}\,(\theta - \zeta) =: \rho\chi + \widetilde{c}\,\text{Heaviside}\,(\zeta - \theta).$$

*Case* (iii) *of Theorem* 4.2.2. Using again Lemmas 4.2.4 and 4.2.5, we find

$$\lim_{t\searrow 0} \mathfrak{I}_2(t) = -\frac{2\ell^\star}{(\rho+2)^2} \left[\frac{\rho\zeta L_\star}{2(\zeta+1)}\right]^{\tau_1}, \quad \lim_{t\searrow 0} \mathfrak{I}_3(t) = -\xi_0^\rho \ell^\star \ln \xi_0 \left[\frac{\rho\zeta L_\star}{2(\zeta+1)}\right]^{\tau_1}.$$

It follows that

$$\begin{aligned} \lim_{t\searrow 0} \mathfrak{I}(t) &= \frac{\rho L_\star}{2} \text{Heaviside}\,(\theta - \zeta) - \ell^\star \left[\frac{\rho\zeta L_\star}{2(\zeta+1)}\right]^{\tau_1} \left(\frac{1}{\rho+2} + \ln \xi_0\right) \\ &=: \rho\chi + \widetilde{c}\,\text{Heaviside}\,(\zeta - \theta). \end{aligned}$$

This concludes the proof. □

**Lemma 4.2.7.** *Under the assumptions of Theorem 4.2.3, we have*

$$\mathcal{H}(t) := (-\ln t)^{\tau} \left(1 - \frac{k^2(t) f(\xi_0 h(t))}{\xi_0 h''(t)}\right) \to \rho \widetilde{\chi} \quad \text{as } t \searrow 0, \tag{4.28}$$

*where* $\widetilde{\chi}$ *is defined by* (4.18).

*Proof.* Using (4.21), we write $\mathcal{H}(t) = \frac{k^2(t) f(h(t))}{h''(t)} \sum_{i=1}^{3} \mathcal{H}_i(t)$, for $t > 0$ small, where

$$\begin{cases} \mathcal{H}_1(t) := 2\Xi(h(t))(-\ln t)^{\tau} \left[\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)' - \ell_1\right], \\ \mathcal{H}_2(t) := 2(1-\ell_1)\left(\frac{-\ln t}{\ln h(t)}\right)^{\tau} T_{1,\tau}(h(t)) \\ \mathcal{H}_3(t) := -\left(\frac{-\ln t}{\ln h(t)}\right)^{\tau} T_{2,\tau}(h(t)). \end{cases}$$

By Remark 4.2.4, we find $\lim_{t\searrow 0} \mathcal{H}_1(t) = \rho L_\sharp/(\rho+2)$.

*Case* (i) *of Theorem* 4.2.3. By Lemmas 4.2.4 and 4.2.5, it turns out that

$$\lim_{t\searrow 0} \mathcal{H}_2(t) = \lim_{t\searrow 0} \mathcal{H}_3(t) = 0 \quad \text{and} \quad \lim_{t\searrow 0} \mathcal{H}(t) = \frac{\rho L_\sharp}{2+\rho\ell_1} =: \rho\widetilde{\chi}.$$

*Case* (ii) *of Theorem* 4.2.3. By Lemmas 4.2.4 and 4.2.5, we get

$$\lim_{t\searrow 0} \mathcal{H}_2(t) = \frac{-2(1-\ell_1)\ell^\star}{(\rho+2)^2}\left(\frac{\rho\ell_1}{2}\right)^{\tau} \text{ and } \lim_{t\searrow 0} \mathcal{H}_3(t) = \frac{-\ell^\star(2+\rho\ell_1)}{(2+\rho)}\left(\frac{\rho\ell_1}{2}\right)^{\tau} \ln \xi_0.$$

Thus, we arrive at

$$\lim_{t\searrow 0} \mathcal{H}(t) = \frac{\rho L_\sharp}{2+\rho\ell_1} - \ell^\star \left(\frac{\rho\ell_1}{2}\right)^{\tau} \left[\frac{2(1-\ell_1)}{(\rho+2)(2+\rho\ell_1)} + \ln \xi_0\right] =: \rho\widetilde{\chi}.$$

This finishes the proof. □

### 4.2.4 Characterization of $\mathcal{K}$ and its Subclasses

The results of §4.2.4 have been included in Cîrstea and Rădulescu (2005).

Definition 4.1.1 extends to *regular variation at the origin*. We say that $Z$ is regularly varying (on the right) at the origin with index $q$ (and write, $Z \in RV_q(0+)$) if $Z(1/u) \in RV_{-q}$. Moreover, by $Z \in NRV_q(0+)$ we mean that $Z(1/u) \in NRV_{-q}$. The meaning of $NRV_q$ is given by (4.2).

**Proposition 4.2.8.** *We have $k \in \mathcal{K}_{(01]}$ if and only if $k$ is non-decreasing near the origin and $k$ belongs to $NRV_\alpha(0+)$ for some $\alpha \geq 0$ (where $\alpha = 1/\ell_1 - 1$).*

*Proof.* If $k \in \mathcal{K}_{(01]}$, then from the definition

$$\lim_{t\to 0^+} \frac{\int_0^t k(s)ds}{k(t)} \Big/ t = \lim_{t\to 0^+} \left(\frac{\int_0^t k(s)ds}{k(t)}\right)' = \ell_1,$$

which implies that

$$\lim_{u\to\infty} \frac{u\,\frac{d}{du}k(1/u)}{k(1/u)} = \lim_{t\to 0^+} \frac{-tk'(t)}{k(t)} = \frac{\ell_1 - 1}{\ell_1}.$$

Thus $k(1/u)$ belongs to $NRV_{1-1/\ell_1}$. Conversely, if $k$ belongs to $NRV_\alpha(0+)$ with $\alpha \geq 0$, then $k$ is a positive $C^1$-function on some interval $(0, \nu)$ and

$$\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \alpha. \tag{4.29}$$

By Proposition 4.1.4, we deduce

$$\lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{tk(t)} = \lim_{u\to\infty} \frac{\int_u^\infty x^{-2}k(1/x)dx}{u^{-1}k(1/u)} = \frac{1}{1+\alpha}. \tag{4.30}$$

Combining (4.29) and (4.30), we get $\lim_{t\to 0^+}\left(\int_0^t k(s)\,ds/k(t)\right)' = 1/(1+\alpha)$. If, in addition, $k$ is non-decreasing near 0, then $k \in \mathcal{K}$ with $\ell_1 = 1/(1+\alpha)$. Note that by (4.29), $k$ is increasing near the origin if $\alpha > 0$; however, when $k$ is slowly varying at 0, then we cannot draw any conclusion about the monotonicity of $k$ near the origin (see Remark 4.1.3). □

*Remark* 4.2.6. By Propositions 4.2.8 and 4.1.2, we deduce $k \in \mathcal{K}_{(01]}$ if and only $k$ is of the form

$$k(t) = c_0 t^\alpha \exp\left\{\int_t^{c_1} \frac{E(y)}{y}\,dy\right\} \quad (0 < t < c_1), \text{ for some } 0 \leq \alpha (= 1/\ell_1 - 1) \tag{4.31}$$

where $c_0, c_1 > 0$ are constants, $E \in C[0, c_1)$ with $E(0) = 0$ and (only for $\ell_1 = 1$) $E(t) \leq \alpha$.

**Proposition 4.2.9.** *We have $k \in \mathcal{K}_{(01],\tau}$ if and only if $k$ is of the form (4.31) where, in addition,*

$$\lim_{t\searrow 0} (-\ln t)^\tau E(t) = \ell_\sharp \in \mathbb{R} \quad \textit{with } \ell_\sharp = (1+\alpha)^2 L_\sharp. \tag{4.32}$$

*Proof.* Suppose $k$ satisfies (4.31) and (4.32). A simple calculation leads to

$$\lim_{t\searrow 0}(-\ln t)^\tau \left[\frac{1-\ell_1}{\ell_1} - \frac{tk'(t)}{k(t)}\right] = \lim_{t\searrow 0}(-\ln t)^\tau E(t) = \ell_\sharp. \tag{4.33}$$

By L'Hospital's rule, we find

$$\begin{aligned}\lim_{t\searrow 0}(-\ln t)^\tau \left[\ell_1 - \frac{\int_0^t k(s)\,ds}{tk(t)}\right] &= \lim_{t\searrow 0}\frac{(\ell_1-1)+\ell_1 tk'(t)/k(t)}{(-\ln t)^{-\tau}\left[1+\frac{tk'(t)}{k(t)}-\frac{\tau}{\ln t}\right]}\\ &= -\ell_1^2\lim_{t\searrow 0}(-\ln t)^\tau\left[\frac{1-\ell_1}{\ell_1}-\frac{tk'(t)}{k(t)}\right] = \frac{-\ell_\sharp}{(\alpha+1)^2}.\end{aligned} \tag{4.34}$$

We see that, for each $t>0$ small,

$$\left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)' - \ell_1 = \frac{tk'(t)}{k(t)}\left[\ell_1 - \frac{\int_0^t k(s)\,ds}{tk(t)}\right] + \ell_1\left[\frac{1-\ell_1}{\ell_1}-\frac{tk'(t)}{k(t)}\right]. \tag{4.35}$$

By (4.33)–(4.35), we infer that $k\in\mathcal{K}_{(01],\tau}$ with $L_\sharp = \ell_\sharp/(1+\alpha)^2$.

Conversely, if $k\in\mathcal{K}_{(01],\tau}$, then $k$ is of the form (4.31). Moreover, we have

$$\lim_{t\searrow 0}(-\ln t)^\tau\left(\frac{\int_0^t k(s)\,ds}{tk(t)} - \ell_1\right) = \lim_{t\searrow 0}\frac{\left(\int_0^t k(s)\,ds/k(t)\right)' - \ell_1}{(-\ln t)^{-\tau}\left(1-\frac{\tau}{\ln t}\right)} = L_\sharp. \tag{4.36}$$

By (4.35) and (4.36), we deduce

$$L_\sharp = -\alpha L_\sharp + \frac{1}{\alpha+1}\lim_{t\searrow 0}(-\ln t)^\tau E(t).$$

Consequently, $\lim_{t\searrow 0}(-\ln t)^\tau E(t) = (1+\alpha)^2 L_\sharp$. Hence, (4.32) holds. □

**Proposition 4.2.10.** *We have $k\in\mathcal{K}_0$ if and only if $k$ is of the form*

$$k(t) = d_0\left(\exp\left\{-\int_t^{d_1}\frac{dx}{x\mathcal{W}(x)}\right\}\right)' \quad (0<t<d_1), \tag{4.37}$$

*where $d_0, d_1>0$ are constants and $0<\mathcal{W}\in C^1(0,d_1)$ satisfies*

$$\lim_{t\searrow 0}\mathcal{W}(t) = \lim_{t\searrow 0} t\,\mathcal{W}'(t) = 0. \tag{4.38}$$

*Proof.* If $k \in \mathcal{K}_0$, then we set

$$\mathcal{W}(t) = \frac{\int_0^t k(s)\,ds}{tk(t)}, \quad \text{for } t \in (0, d_1). \tag{4.39}$$

Hence, $\lim_{t \searrow 0} \mathcal{W}(t) = 0$ and, for $t > 0$ small,

$$t\mathcal{W}'(t) = \left( \frac{\int_0^t k(s)\,ds}{k(t)} \right)' - \frac{\int_0^t k(s)\,ds}{tk(t)}.$$

It follows that $\lim_{t \searrow 0} t\mathcal{W}'(t) = 0$. By (4.39), we find

$$\int_t^{d_1} \frac{dx}{x\mathcal{W}(x)} = \ln \left( \int_0^{d_1} k(s)\,ds \right) - \ln \left( \int_0^t k(s)\,ds \right), \quad t \in (0, d_1)$$

which proves (4.37). Conversely, if (4.37) holds, then $\lim_{t \searrow 0} \int_t^{d_1} \frac{dx}{x\mathcal{W}(x)} = \infty$ and

$$\int_0^t k(s)\,ds = d_0 \exp\left\{ - \int_t^{d_1} \frac{dx}{x\mathcal{W}(x)} \right\} = tk(t)\mathcal{W}(t), \quad t \in (0, d_1). \tag{4.40}$$

This, together with (4.38), shows that $k \in \mathcal{K}_0$. □

**Proposition 4.2.11.** *We have $k \in \mathcal{K}_{0,\zeta}$ if and only if $k$ is of the form (4.37) where, in addition,*

$$\lim_{t \searrow 0} t^{1-\zeta} \mathcal{W}'(t) = -\ell_\star \quad \textit{with} \quad -\ell_\star = \zeta L_\star/(1+\zeta). \tag{4.41}$$

*Proof.* If $k \in \mathcal{K}_{0,\zeta}$, then (4.37) and (4.40) are fulfilled. Therefore,

$$L_\star = \lim_{t \searrow 0} \frac{(t\mathcal{W}(t))'}{t^\zeta} = \lim_{t \searrow 0} \frac{\mathcal{W}(t) + t\,\mathcal{W}'(t)}{t^\zeta} \text{ and } \frac{L_\star}{\zeta+1} = \lim_{t \searrow 0} \frac{\int_0^t k(s)\,ds}{k(t)t^{\zeta+1}} = \lim_{t \searrow 0} \frac{\mathcal{W}(t)}{t^\zeta},$$

from which (4.41) follows. Conversely, if (4.37) and (4.41) are satisfied, then $\lim_{t \searrow 0} \mathcal{W}(t)/t^\zeta = -\ell_\star/\zeta$. By (4.40), we infer that

$$\frac{1}{t^\zeta} \left( \frac{\int_0^t k(s)\,ds}{k(t)} \right)' = \frac{1}{t^\zeta} (\mathcal{W}(t) + t\mathcal{W}'(t)) \to \frac{-\ell_\star(\zeta+1)}{\zeta} \quad \text{as } t \searrow 0.$$

Thus, $k \in \mathcal{K}_{0,\zeta}$ with $L_\star = -\ell_\star(\zeta+1)/\zeta$. □

*Remark* 4.2.7. If $k \in \mathcal{K}_0$ or $k \in \mathcal{K}_{(01],\tau}$ with $(1-\ell_1)^2 + L_\sharp^2 \neq 0$, then

$$\lim_{t \searrow 0} \frac{k'(t)}{k(t)t^{\theta-1}} = \infty, \quad \text{for every } \theta > 0. \tag{4.42}$$

Indeed, if $k \in \mathcal{K}_0$, then $\lim_{t\searrow 0} \frac{tk'(t)}{k(t)} = \infty$. Assuming that $k \in \mathcal{K}_{(01],\tau}$, we deduce (4.42) from (4.29) when $\ell_1 \neq 1$, otherwise from (4.32) when $L_\sharp \neq 0$ since

$$\lim_{t \searrow 0} \frac{k'(t)}{k(t)t^{\theta-1}} = \lim_{t\searrow 0} -E(t)t^{-\theta} = -L_\sharp \lim_{t\searrow 0} \frac{t^{-\theta}}{(-\ln t)^\tau} = \infty.$$

*Remark* 4.2.8. Under the assumptions of Theorem 4.2.1, we have

(a) Suppose $\lim_{t\searrow 0} \left(\int_0^t k(s)\,ds\right)^2 r(t) = 1$ and let $\widehat{f}(u)$ be chosen such that $\lim_{u\to\infty} \widehat{f}(u)/f(u) = 1$ and $j(u) := \widehat{f}(u)/u$ is non-decreasing for $u > 0$ large. Then $\lim_{t\searrow 0} \widehat{\varphi}(t)/\varphi(t) = 1$, where $\widehat{\varphi}(t) := j^{\leftarrow}(r(t))$, for $t > 0$ small, and $\varphi(t)$ is defined by (4.11).

(b) If $k \in \mathcal{K}$ with $\ell_1 \neq 0$, then $\varphi(1/u) \in RV_{2/(\rho\ell_1)}$.

(c) If $k \in \mathcal{K}_0$, then $\varphi(1/u)$ is $\Gamma$-varying at $u = \infty$ with auxiliary function $\dfrac{\rho u^2 \int_0^{1/u} k(s)\,ds}{2k(1/u)}$.

(d) $\displaystyle\lim_{t\searrow 0} \frac{\varphi(t)}{h(t)} = \left(\frac{2(\rho+2)}{\rho^2}\right)^{-1/\rho}$, where $h(t)$ is given by (4.13).

Indeed, we have $(f(u)/u)^{\leftarrow} \in RV_{1/\rho}$ and $\lim_{u\to\infty} (f(u)/u)^{\leftarrow}/j^{\leftarrow}(u) = 1$ (use Proposition 4.1.6). Then, by Proposition 4.1.1 we deduce (a). We see that (b) follows by Proposition 4.1.6 since $\left(\int_0^{1/u} k(s)\,ds\right)^{-2} \in RV_{2/\ell_1}$ (cf. Proposition 4.2.8) and $f(u)/u \in RV_\rho$. If $k \in \mathcal{K}_0$, then by Proposition 4.2.10 and (Resnick, 1987, p. 106), we get $\left(\int_0^{1/u} k(s)\,ds\right)^{-2}$ is $\Gamma$-varying at $u = \infty$ with auxiliary function $u\mathcal{W}(1/u)/2$. By (Resnick, 1987, p. 36), we conclude (c). Notice that

$$Y(u) := \left(\int_u^\infty [2F(s)]^{-1/2}\,ds\right)^{-2} \in RV_\rho$$

and $Y(h(t)) = \left(\int_0^t k(s)\,ds\right)^{-2}$ for $t > 0$ small. Using Remark 4.2.4, we find $\lim_{u\to\infty} f(u)/[uY(u)] = 2(\rho+2)/\rho^2$. By Proposition 4.1.6, we get (d).

### 4.2.5 Uniqueness of the Large Solution

#### 4.2.5.1 Proof of Theorem 4.2.1: First Approach

Since $f \in RV_{\rho+1}$, by Proposition 4.1.3 we obtain $\lim_{u\to\infty} f(u)/u^q = \infty$, for each $q \in (1, \rho+1)$. Thus, (3.6) is fulfilled. By Theorem 3.1.1, (4.6) possesses large solutions if and only if $a \in (-\infty, \lambda_{\infty,1})$.

Fix $a < \lambda_{\infty,1}$. If we prove that (4.12) holds for an *arbitrary* large solution $u_a$ of (4.6), then we can deduce the uniqueness.

Indeed, if $u_1$ and $u_2$ are two arbitrary large solutions of (4.6), then (4.12) implies that $\lim_{d(x)\to 0^+} u_1(x)/u_2(x) = 1$. Hence, for any $\varepsilon \in (0,1)$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$(1-\varepsilon)u_2(x) \le u_1(x) \le (1+\varepsilon)u_2(x), \quad \forall x \in \Omega \text{ with } 0 < d(x) \le \delta. \tag{4.43}$$

Choosing eventually a smaller $\delta > 0$, we can assume that $\overline{\Omega}_0 \subset C_\delta$, where

$$C_\delta := \{x \in \Omega : \ d(x) > \delta\}.$$

We see that $u_1$ is a positive solution of the boundary value problem

$$\Delta u + au = b(x)f(u) \quad \text{in } C_\delta, \qquad u = u_1 \quad \text{on } \partial C_\delta. \tag{4.44}$$

By (4.7) and (4.43), $\phi^- = (1-\varepsilon)u_2$ (resp., $\phi^+ = (1+\varepsilon)u_2$) is a positive sub-solution (resp., super-solution) of (4.44). By the sub and super-solutions method, (4.44) has a positive solution $\phi_1$ satisfying $\phi^- \le \phi_1 \le \phi^+$ in $C_\delta$. Since $b > 0$ on $\overline{C}_\delta \setminus \overline{\Omega}_0$, by Lemma 3.1.11, we know that (4.44) has a *unique* positive solution, i.e., $u_1 \equiv \phi_1$ in $C_\delta$. This yields

$$(1-\varepsilon)u_2(x) \le u_1(x) \le (1+\varepsilon)u_2(x) \quad \text{in } C_\delta,$$

so that (4.43) holds in $\Omega$. Passing to the limit $\varepsilon \to 0^+$, we conclude that $u_1 \equiv u_2$.

*Proof of* (4.12). Fix $\varepsilon \in (0, 1/2)$. Since (4.9) holds, we take $\delta > 0$ such that

(i) $d(x)$ is a $C^2$-function on the set $\{x \in \Omega : \ d(x) < 2\delta\}$;

(ii) $k$ is non-decreasing on $(0, 2\delta)$;

(iii) $(1-\varepsilon) < b(x)/k^2(d(x)) < (1+\varepsilon)$, for each $x \in \Omega$ with $0 < d(x) < 2\delta$;

(iv) $h''(t) > 0$, for each $t \in (0, 2\delta)$ (cf. Lemma 4.2.4).

Let $\sigma \in (0, \delta)$ be arbitrary. Set $\xi^\pm = \left[\frac{2+\ell_1\rho}{(1\mp 2\varepsilon)(2+\rho)}\right]^{1/\rho}$ and define

$$\begin{cases} v_\sigma^+(x) = h(d(x)-\sigma)\xi^+, & \forall x \in \Omega \text{ with } \sigma < d(x) < 2\delta, \\ v_\sigma^-(x) = h(d(x)+\sigma)\xi^-, & \forall x \in \Omega \text{ with } d(x)+\sigma < 2\delta. \end{cases}$$

Using (i)-(iv), when $\sigma < d(x) < 2\delta$ we obtain (since $|\nabla d(x)| \equiv 1$)

$$\Delta v_\sigma^+ + a v_\sigma^+ - b(x) f(v_\sigma^+) \leq \xi^+ h''(d-\sigma) \left( \frac{h'(d-\sigma)}{h''(d-\sigma)} \Delta d(x) + a \frac{h(d-\sigma)}{h''(d-\sigma)} + 1 \right.$$
$$\left. -(1-\varepsilon) \frac{k^2(d-\sigma) f(h(d-\sigma)\xi^+)}{h''(d-\sigma)\xi^+} \right).$$

Similarly, when $d(x) + \sigma < 2\delta$ we find

$$\Delta v_\sigma^- + a v_\sigma^- - b(x) f(v_\sigma^-) \geq \xi^- h''(d+\sigma) \left( \frac{h'(d+\sigma)}{h''(d+\sigma)} \Delta d(x) + a \frac{h(d+\sigma)}{h''(d+\sigma)} + 1 \right.$$
$$\left. -(1+\varepsilon) \frac{k^2(d+\sigma) f(h(d+\sigma)\xi^-)}{h''(d+\sigma)\xi^-} \right).$$

Thus, by Lemma 4.2.4, we can diminish $\delta > 0$ such that

$$\begin{cases} \Delta v_\sigma^+(x) + a v_\sigma^+(x) - b(x) f(v_\sigma^+(x)) \leq 0 & \forall x \text{ with } \sigma < d(x) < 2\delta; \\ \Delta v_\sigma^-(x) + a v_\sigma^-(x) - b(x) f(v_\sigma^-(x)) \geq 0 & \forall x \text{ with } d(x) + \sigma < 2\delta. \end{cases}$$

Let $\Omega_1$ and $\Omega_2$ be smooth bounded domains such that $\Omega \subset\subset \Omega_1 \subset\subset \Omega_2$ and the first Dirichlet eigenvalue of $(-\Delta)$ in the domain $\Omega_1 \setminus \overline{\Omega}$ is greater than $a$. Let $p \in C^{0,\mu}(\overline{\Omega}_2)$ satisfy $0 < p(x) \leq b(x)$ for $x \in \Omega \setminus C_\delta$, $p = 0$ on $\overline{\Omega}_1 \setminus \Omega$ and $p > 0$ on $\Omega_2 \setminus \overline{\Omega}_1$. By Theorem 3.1.1, we can take $w$ a positive large solution of

$$\Delta w + a w = p(x) f(w) \quad \text{in } \Omega_2 \setminus \overline{C}_\delta.$$

Let $u_a$ be an arbitrary large solution of (4.6). Then $v := u_a + w$ satisfies

$$\Delta v + a v - b(x) f(v) \leq 0 \quad \text{in } \Omega \setminus \overline{C}_\delta.$$

Since $v_{|\partial\Omega} = \infty > v^-_{\sigma|\partial\Omega}$ and $v_{|\partial C_\delta} = \infty > v^-_{\sigma|\partial C_\delta}$, by Lemma 3.1.6 we get

$$u_a + w \geq v_\sigma^- \quad \text{on } \Omega \setminus \overline{C}_\delta. \tag{4.45}$$

Similarly,

$$v_\sigma^+ + w \geq u_a \quad \text{on } C_\sigma \setminus \overline{C}_\delta. \tag{4.46}$$

Letting $\sigma \to 0$ in (4.45) and (4.46), we deduce

$$h(d(x))\xi^+ + 2w \geq u_a + w \geq h(d(x))\xi^-, \quad \forall x \in \Omega \setminus \overline{C}_\delta.$$

Since $w$ is uniformly bounded on $\partial\Omega$, we have

$$\xi^- \leq \liminf_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} \leq \limsup_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} \leq \xi^+.$$

Letting $\varepsilon \to 0^+$ we obtain (4.12). By Remark 4.2.8 (d), we reach (4.10). This concludes the proof of Theorem 4.2.1. □

#### 4.2.5.2 Proof of Theorem 4.2.1: Second Approach

By Theorem 3.1.1, (4.6) admits large solutions if and only if $a \in (-\infty, \lambda_{\infty,1})$.

Cf. Remark 4.2.8 (d), to prove (4.10) it is enough to show (4.12).

Fix $a < \lambda_{\infty,1}$ and let $u_a$ denote an arbitrary large solution of (4.6).

Let $\varepsilon \in (0, 1/2)$ be arbitrary. We choose $\delta > 0$ such that

(i) $d(x)$ is a $C^2$ function on the set $\{x \in \Omega : \ d(x) < \delta\}$;

(ii) $k$ is non-decreasing on $(0, \delta)$;

(iii) $1 - \varepsilon < b(x)/k^2(d(x)) < 1 + \varepsilon$, for each $x \in \Omega$ with $0 < d(x) < \delta$;

(iv) $h'(t) < 0$ and $h''(t) > 0$, for each $t \in (0, \delta)$ (cf. Lemma 4.2.4).

Define $\xi^\pm = \left[\frac{2+\ell_1\rho}{(1\mp 2\varepsilon)(2+\rho)}\right]^{1/\rho}$ and $u^\pm(x) = \xi^\pm h(d(x))$, for any $x$ with $d(x) \in (0, \delta)$.

The proof of (4.12) will be divided into three steps:

*Step* 4.2.1. There exists $\delta_1 \in (0, \delta)$ small such that

$$\begin{cases} \Delta u^+ + au^+ - (1-\varepsilon)k^2(d)f(u^+) \leq 0, & \forall x \text{ with } d(x) \in (0, \delta_1) \\ \Delta u^- + au^- - (1+\varepsilon)k^2(d)f(u^-) \geq 0, & \forall x \text{ with } d(x) \in (0, \delta_1). \end{cases} \tag{4.47}$$

Indeed, for every $x \in \Omega$ with $0 < d(x) < \delta$, we have

$$\begin{aligned} &\Delta u^{\pm} + a u^{\pm} - (1 \mp \varepsilon) k^2(d) f(u^{\pm}) \\ &\quad = \xi^{\pm} h''(d) \left( 1 + a \frac{h(d)}{h''(d)} + \Delta d \, \frac{h'(d)}{h''(d)} - (1 \mp \varepsilon) \frac{k^2(d) f(u^{\pm})}{\xi^{\pm} h''(d)} \right) \\ &\quad =: \xi^{\pm} h''(d) B^{\pm}(d). \end{aligned} \tag{4.48}$$

By Lemma 4.2.4, we deduce $\lim_{d \searrow 0} B^{\pm}(d) = \mp\varepsilon/(1 \mp 2\varepsilon)$, which proves (4.47).

*Step* 4.2.2. There exists $M^+$, $\delta^+ > 0$ such that

$$u_a(x) \le u^+(x) + M^+, \quad \forall x \in \Omega \text{ with } 0 < d < \delta^+.$$

For $x \in \Omega$ with $d(x) \in (0, \delta_1)$, we define $\Psi_x(u) = au - b(x) f(u)$ for each $u > 0$. By Lemma 4.2.4, we deduce

$$\lim_{d(x) \searrow 0} \frac{b(x) f(u^+(x))}{u^+(x)} = \lim_{d \searrow 0} \frac{k^2(d) f(u^+)}{\xi^+ h''(d)} \, \frac{h''(d)}{h(d)} = \infty. \tag{4.49}$$

From this and (4.7), we infer that there exists $\delta_2 \in (0, \delta_1)$ such that, for any $x$ with $0 < d(x) < \delta_2$, we have

$$u \longmapsto \Psi_x(u) \text{ is decreasing on some interval } (u_x, \infty) \text{ with } 0 < u_x < u^+(x).$$

Hence, for each $M > 0$, we have

$$\Psi_x(u^+(x) + M) \le \Psi_x(u^+(x)), \quad \forall x \in \Omega \text{ with } 0 < d(x) < \delta_2. \tag{4.50}$$

Fix $\sigma \in (0, \delta_2/4)$ and set $\mathcal{N}_\sigma := \{x \in \Omega : \ \sigma < d(x) < \delta_2/2\}$.

We define $u^*_\sigma(x) = u^+(d - \sigma, s) + M^+$, where $(d, s)$ are the local coordinates of $x \in \mathcal{N}_\sigma$. We choose $M^+ > 0$ large enough such that

$$u^*_\sigma(\delta_2/2, s) = u^+(\delta_2/2 - \sigma, s) + M^+ \ge u_a(\delta_2/2, s), \quad \forall \sigma \in (0, \delta_2/4), \ \forall s \in \partial\Omega.$$

By (ii), (iii), (4.47) and (4.50), we obtain

$$\begin{aligned} -\Delta u^*_\sigma(x) &\ge a u^+(d - \sigma, s) - (1 - \varepsilon) k^2(d - \sigma) f(u^+(d - \sigma, s)) \\ &\ge a u^+(d - \sigma, s) - b(x) f(u^+(d - \sigma, s)) \\ &\ge a(u^+(d - \sigma, s) + M^+) - b(x) f(u^+(d - \sigma, s) + M^+) \\ &= a u^*_\sigma(x) - b(x) f(u^*_\sigma(x)) \quad \text{in } \mathcal{N}_\sigma. \end{aligned}$$

So, uniformly with respect to $\sigma$, we have

$$\Delta u_\sigma^*(x) + a u_\sigma^*(x) \le b(x) f(u_\sigma^*(x)) \quad \text{in } \mathcal{N}_\sigma. \tag{4.51}$$

Since $u_\sigma^*(x) \to \infty$ as $d \searrow \sigma$, from Lemma 3.1.6, we get $u_a \le u_\sigma^*$ in $\mathcal{N}_\sigma$, for every $\sigma \in (0, \delta_2/4)$. Letting $\sigma \searrow 0$, we achieve the assertion of Step 4.2.2 (with $\delta^+ \in (0, \delta_2/2)$ arbitrarily chosen).

*Step* 4.2.3. There exists $M^-$, $\delta^- > 0$ such that

$$u_a(x) \ge u^-(x) - M^-, \quad \forall x = (d, s) \in \Omega \quad \text{with } 0 < d < \delta^-. \tag{4.52}$$

For every $r \in (0, \delta)$, define $\Omega_r = \{x \in \Omega : \ 0 < d(x) < r\}$.

Fix $\sigma \in (0, \delta_2/4)$. We define $v_\sigma^*(x) = \lambda u^-(d + \sigma, s)$ for $x = (d, s) \in \Omega_{\delta_2/2}$, where $\lambda \in (0, 1)$ is chosen small enough such that

$$v_\sigma^*(\delta_2/4, s) = \lambda u^-(\delta_2/4 + \sigma, s) \le u_a(\delta_2/4, s), \quad \forall \sigma \in (0, \delta_2/4),\ \forall s \in \partial\Omega. \tag{4.53}$$

Notice that $\limsup_{d \searrow 0}(v_\sigma^* - u_a)(x) = -\infty$. By (ii), (iii), (4.47) and (4.7), we have

$$\begin{aligned} \Delta v_\sigma^*(x) + a v_\sigma^*(x) &= \lambda(\Delta u^-(d + \sigma, s) + a u^-(d + \sigma, s)) \\ &\ge \lambda(1 + \varepsilon) k^2(d + \sigma) f(u^-(d + \sigma, s)) \\ &\ge (1 + \varepsilon) k^2(d) f(\lambda u^-(d + \sigma, s)) \\ &\ge b(x) f(v_\sigma^*(x)), \quad \forall x = (d, s) \in \Omega_{\delta_2/4}. \end{aligned}$$

By Lemma 3.3.2, we derive $v_\sigma^* \le u_a$ in $\Omega_{\delta_2/4}$. Letting $\sigma \searrow 0$, we get

$$\lambda u^-(x) \le u_a(x), \quad \forall x \in \Omega_{\delta_2/4}. \tag{4.54}$$

By Lemma 4.2.4, $\lim_{d \searrow 0} k^2(d) f(\lambda^2 u^-)/u^- = \infty$. Thus, there exists $\widetilde{\delta} \in (0, \delta_2/4)$ such that

$$k^2(d) f(\lambda^2 u^-)/u^- \ge \lambda^2 |a|, \quad \forall x \in \Omega \text{ with } 0 < d \le \widetilde{\delta}. \tag{4.55}$$

Choose $\delta_* \in (0, \widetilde{\delta})$, sufficiently close to $\widetilde{\delta}$, such that

$$h(\delta_*)/h(\widetilde{\delta}) < 1 + \lambda. \tag{4.56}$$

For each $\sigma \in (0, \widetilde{\delta} - \delta_*)$, we define $z_\sigma(x) = u^-(d+\sigma, s) - (1-\lambda)u^-(\delta_*, s)$, where $x = (d, s) \in \Omega_{\delta_*}$. We prove that $z_\sigma$ is positive in $\Omega_{\delta_*}$ and

$$\Delta z_\sigma + a z_\sigma \geq b(x) f(z_\sigma) \quad \text{in } \Omega_{\delta_*}. \tag{4.57}$$

By (iv), $u^-(x)$ decreases with $d$ when $d < \widetilde{\delta}$. This and (4.56) imply that

$$1 + \lambda > \frac{u^-(\delta_*, s)}{u^-(\widetilde{\delta}, s)} \geq \frac{u^-(\delta_*, s)}{u^-(d+\sigma, s)}, \qquad \forall x = (d, s) \in \Omega_{\delta_*}. \tag{4.58}$$

Hence, for each $x \in \Omega_{\delta_*}$,

$$z_\sigma(x) = u^-(d+\sigma, s)\left(1 - \frac{(1-\lambda)u^-(\delta_*, s)}{u^-(d+\sigma, s)}\right) \geq \lambda^2 u^-(d+\sigma, s) > 0. \tag{4.59}$$

By (4.47), (ii) and (iii), we see that (4.57) follows if

$$(1+\varepsilon)k^2(d+\sigma)\left[f(u^-(d+\sigma, s)) - f(z_\sigma(d, s))\right] \geq a(1-\lambda)u^-(\delta_*, s). \tag{4.60}$$

for all $(d, s) \in \Omega_{\delta_*}$. The Lagrange mean value theorem and (4.7) show that

$$f(u^-(d+\sigma, s)) - f(z_\sigma(d, s)) \geq (1-\lambda)u^-(\delta_*, s) f(z_\sigma(x))/z_\sigma(x) \tag{4.61}$$

which, combined with (4.55) and (4.59), proves (4.60).

Notice that $\limsup_{d \searrow 0}(z_\sigma - u_a)(x) = -\infty$. By (4.54), we have

$$z_\sigma(x) = u^-(\delta_* + \sigma, s) - (1-\lambda)u^-(\delta_*, s) \leq \lambda u^-(\delta_*, s) \leq u_a(x), \quad \forall x = (\delta_*, s) \in \Omega.$$

By Lemma 3.3.2, $z_\sigma \leq u_a$ in $\Omega_{\delta_*}$, for every $\sigma \in (0, \widetilde{\delta} - \delta_*)$. Letting $\sigma \searrow 0$, we conclude Step 4.2.3. Thus, by Steps 4.2.2 and 4.2.3, we have

$$\xi^- \leq \liminf_{d(x) \searrow 0} \frac{u_a(x)}{h(d(x))} \leq \limsup_{d(x) \searrow 0} \frac{u_a(x)}{h(d(x))} \leq \xi^+.$$

Taking $\varepsilon \longrightarrow 0$, we obtain (4.12).

Let $u_1$ and $u_2$ be two large solutions of (4.6). By (4.12), $\lim_{d(x) \to 0} \frac{u_1(x)}{u_2(x)} = 1$. Let $\epsilon > 0$ be arbitrary and set $(1+\epsilon)u_i = w_i$, $i = 1, 2$. We obtain

$$\begin{cases} -\Delta w_i - a w_i + b(x) f(w_i) \geq 0 & \text{in } \Omega \\ \lim\limits_{d(x) \to 0} (u_1 - w_2)(x) = \lim\limits_{d(x) \to 0} (u_2 - w_1)(x) = -\infty. \end{cases}$$

Therefore, by Lemma 3.3.2, we infer that

$$u_1 \leq (1+\epsilon)u_2 \quad \text{in } \Omega \quad \text{and} \quad u_2 \leq (1+\epsilon)u_1 \quad \text{in } \Omega.$$

Letting $\epsilon \longrightarrow 0$, we get $u_1 = u_2$ in $\Omega$. This ends the proof of Theorem 4.2.1. □

### 4.2.6 Asymptotic Expansion of the Large Solution

#### 4.2.6.1 Proof of Theorem 4.2.2

For $a < \lambda_{\infty,1}$ fixed, let $u_a$ be the unique large solution of (4.6).

Fix $\varepsilon \in (0, 1/2)$ and choose $\delta > 0$ such that (i), (ii), (iv) from §4.2.5.2 hold.

By (4.14) and Remark 4.2.7, we can diminish $\delta > 0$ such that

$$\begin{cases} 1 + (\widetilde{c} - \varepsilon)d^\theta < b(x)/k^2(d) < 1 + (\widetilde{c} + \varepsilon)d^\theta, \ \forall x \in \Omega \text{ with } d \in (0, \delta), \\ k^2(t)\left[1 + (\widetilde{c} - \varepsilon)t^\theta\right] \text{ is increasing on } (0, \delta). \end{cases} \tag{4.62}$$

Set $\chi_\varepsilon^\pm = \chi \pm \varepsilon\,[1 + \text{Heaviside}\,(\zeta - \theta)]/\rho$ and define

$$u^\pm(x) = \xi_0 h(d)(1 + \chi_\varepsilon^\pm d^\varpi), \quad \text{with } d \in (0, \delta).$$

Thus, for small $\delta > 0$, $u^\pm(x) > 0$ for each $x \in \Omega$ with $d \in (0, \delta)$.

By the Lagrange mean value theorem, we obtain

$$f(u^\pm(x)) = f(\xi_0 h(d)) + \xi_0 \chi_\varepsilon^\pm d^\varpi h(d) f'(\Upsilon^\pm(d)),$$

where $\Upsilon^\pm(d) = \xi_0 h(d)(1 + \lambda^\pm(d)\chi_\varepsilon^\pm d^\varpi)$, for some $\lambda^\pm(d) \in [0, 1]$.

As $f \in RV_{\rho+1}$, using Proposition 4.1.1 we deduce

$$\lim_{d \searrow 0} \frac{f(\Upsilon^\pm(d))}{f(\xi_0 h(d))} = \lim_{d \searrow 0} \frac{f(u^\pm(d))}{f(\xi_0 h(d))} = 1. \tag{4.63}$$

*Step* 4.2.4. There exists $\delta_1 \in (0, \delta)$ so that

$$\begin{cases} \Delta u^+ + au^+ - k^2(d)[1 + (\widetilde{c} - \varepsilon)d^\theta] f(u^+) \le 0, & \forall x \in \Omega \text{ with } d < \delta_1, \\ \Delta u^- + au^- - k^2(d)[1 + (\widetilde{c} + \varepsilon)d^\theta] f(u^-) \ge 0, & \forall x \in \Omega \text{ with } d < \delta_1. \end{cases} \tag{4.64}$$

Indeed, for every $x \in \Omega$ with $d \in (0, \delta)$, we have

$$\Delta u^\pm + au^\pm - k^2(d)\left[1 + (\widetilde{c} \mp \varepsilon)d^\theta\right] f(u^\pm) = \xi_0 d^\varpi h''(d)\mathcal{S}^\pm(d), \tag{4.65}$$

where

$$\begin{aligned} \mathcal{S}^\pm(d) :=& \frac{\chi_\varepsilon^\pm h(d)}{d^2 h''(d)}\left(\varpi(\varpi - 1) + ad^2 + \varpi d\Delta d\right) + \frac{ah(d)}{d^\varpi h''(d)} \\ &+ \frac{h'(d)}{d^{1+\zeta} h''(d)}\left(d^{1+\zeta}\chi_\varepsilon^\pm \Delta d + 2\varpi\chi_\varepsilon^\pm d^\zeta + d^{1+\zeta-\varpi}\Delta d\right) + \sum_{j=1}^{3} \mathcal{S}_j^\pm(d) + \mathcal{I}(d). \end{aligned}$$

For brevity, $\mathcal{I}(d)$ is defined by (4.27) and

$$\begin{cases} \mathcal{S}_1^{\pm}(d) := (-\widetilde{c} \pm \varepsilon) d^{\theta-\varpi} \dfrac{k^2(d) f(\xi_0 h(d))}{\xi_0 h''(d)}, \\ \mathcal{S}_2^{\pm}(d) := \chi_\varepsilon^{\pm} \left(1 - \dfrac{k^2(d) h(d) f'(\Upsilon^{\pm}(d))}{h''(d)}\right), \\ \mathcal{S}_3^{\pm}(d) := (-\widetilde{c} \pm \varepsilon) \chi_\varepsilon^{\pm} d^{\theta} \dfrac{k^2(d) h(d) f'(\Upsilon^{\pm}(d))}{h''(d)}. \end{cases}$$

By Lemma 4.2.4, we have

$$\lim_{d \searrow 0} \mathcal{S}_1^{\pm}(d) = (-\widetilde{c} \pm \varepsilon) \text{Heaviside}\,(\zeta - \theta).$$

Using (4.63), we deduce

$$\lim_{d \searrow 0} \frac{k^2(d) h(d) f'(\Upsilon^{\pm}(d))}{h''(d)} = \lim_{d \searrow 0} \frac{\Upsilon^{\pm}(d) f'(\Upsilon^{\pm}(d))}{f(\Upsilon^{\pm}(d))} \frac{k^2(d) f(\xi_0 h(d))}{\xi_0 h''(d)} = \rho + 1.$$

Consequently, we have $\lim_{d \searrow 0} \mathcal{S}_2^{\pm}(d) = -\rho \chi_\varepsilon^{\pm}$ and $\lim_{d \searrow 0} \mathcal{S}_3^{\pm}(d) = 0$.

Using Lemmas 4.2.4 and 4.2.6, we get

$$\lim_{d \searrow 0} \mathcal{S}^{+}(d) = -\varepsilon < 0 \quad \text{and} \quad \lim_{d \searrow 0} \mathcal{S}^{-}(d) = \varepsilon > 0.$$

From this and (4.65), we conclude Step 4.2.4.

*Step* 4.2.5. There exists $M^+$, $\delta^+ > 0$ such that

$$u_a(x) \leq u^+(x) + M^+, \qquad \forall x \in \Omega \text{ with } 0 < d < \delta^+.$$

We only recover (4.51), the rest being similar to the proof of Step 4.2.2 in Theorem 4.2.1. Indeed, by (4.64), (4.62) and (4.50), we obtain

$$\begin{aligned} -\Delta u_\sigma^*(x) &\geq a u^+(d-\sigma, s) - [1 + (\widetilde{c} - \varepsilon)(d-\sigma)^\theta] k^2(d-\sigma) f(u^+(d-\sigma, s)) \\ &\geq a u^+(d-\sigma, s) - [1 + (\widetilde{c} - \varepsilon) d^\theta] k^2(d) f(u^+(d-\sigma, s)) \\ &\geq a u^+(d-\sigma, s) - b(x) f(u^+(d-\sigma, s)) \\ &\geq a(u^+(d-\sigma, s) + M^+) - b(x) f(u^+(d-\sigma, s) + M^+) \\ &= a u_\sigma^*(x) - b(x) f(u_\sigma^*(x)) \quad \text{in } \mathcal{N}_\sigma. \end{aligned}$$

*Step* 4.2.6. There exists $M^-$, $\delta^- > 0$ such that

$$u_a(x) \geq u^-(x) - M^-, \qquad \forall x \in \Omega \text{ with } 0 < d < \delta^-.$$

We proceed in the same way as for proving (4.52). To recover (4.54) (with $\lambda$ given by (4.53)), we show that $\Delta v_\sigma^* + a v_\sigma^* \geq b(x) f(v_\sigma^*)$ in $\Omega_{\delta_2/4}$. Indeed, using (4.62), (4.64) and (4.7), we find

$$\begin{aligned}\Delta v_\sigma^*(x) + a v_\sigma^*(x) &= \lambda(\Delta u^-(d+\sigma, s) + a u^-(d+\sigma, s)) \\ &\geq \lambda k^2(d+\sigma)[1 + (\widetilde{c}+\varepsilon)(d+\sigma)^\theta] f(u^-(d+\sigma, s)) \\ &\geq k^2(d)[1 + (\widetilde{c}+\varepsilon)d^\theta] f(\lambda u^-(d+\sigma, s)) \\ &\geq b(x) f(v_\sigma^*(x)), \quad \forall x = (d, s) \in \Omega_{\delta_2/4}.\end{aligned}$$

Since $\lim_{d \searrow 0} k^2(d) f(\lambda^2 u^-(x))/u^-(x) = \infty$, there is $\widetilde{\delta} \in (0, \delta_2/4)$ such that

$$k^2(d)[1 + (\widetilde{c}+\varepsilon)d^\theta] f(\lambda^2 u^-)/u^- \geq \lambda^2 |a|, \quad \forall x \in \Omega \text{ with } 0 < d \leq \widetilde{\delta}. \tag{4.66}$$

By Lemma 4.2.4, we infer that $u^-(x)$ decreases with $d$ when $d \in (0, \widetilde{\delta})$ (if necessary, $\widetilde{\delta} > 0$ is decreased). Choose $\delta_* \in (0, \widetilde{\delta})$ close enough to $\widetilde{\delta}$ such that

$$\frac{h(\delta_*)(1 + \chi_\varepsilon^- \delta_*^\varpi)}{h(\widetilde{\delta})(1 + \chi_\varepsilon^- \widetilde{\delta}^\varpi)} < 1 + \lambda. \tag{4.67}$$

Hence, we regain (4.58), (4.59) and (4.61).

By (4.62) and (4.64), we see that (4.57) follows if

$$\begin{aligned} &k^2(d+\sigma)[1 + (\widetilde{c}+\varepsilon)(d+\sigma)^\theta] \left[ f(u^-(d+\sigma, s)) - f(z_\sigma(d, s)) \right] \\ &\quad \geq a(1-\lambda) u^-(\delta_*, s), \qquad \forall (d, s) \in \Omega_{\delta_*}. \end{aligned} \tag{4.68}$$

Using (4.61), together with (4.66) and (4.59), we arrive at (4.68). The remaining argument of Step 4.2.3 applies here so that the claim of Step 4.2.6 is proved.

By Steps 4.2.5 and 4.2.6, we get

$$\begin{cases} \chi_\varepsilon^+ \geq \left[ -1 + \dfrac{u_a(x)}{\xi_0 h(d)} \right] d^{-\varpi} - \dfrac{M^+}{\xi_0 d^\varpi h(d)}, & \forall x \in \Omega \text{ with } d \in (0, \delta^+) \\ \chi_\varepsilon^- \leq \left[ -1 + \dfrac{u_a(x)}{\xi_0 h(d)} \right] d^{-\varpi} + \dfrac{M^-}{\xi_0 d^\varpi h(d)}, & \forall x \in \Omega \text{ with } d \in (0, \delta^-). \end{cases}$$

Passing to the limit as $d \to 0$ and using Lemma 4.2.4, we obtain

$$\chi_\varepsilon^- \leq \liminf_{d \to 0} \left[ -1 + \frac{u_a(x)}{\xi_0 h(d)} \right] d^{-\varpi} \leq \limsup_{d \to 0} \left[ -1 + \frac{u_a(x)}{\xi_0 h(d)} \right] d^{-\varpi} \leq \chi_\varepsilon^+.$$

Letting $\varepsilon \to 0$, we conclude the proof of Theorem 4.2.2. □

#### 4.2.6.2 Proof of Theorem 4.2.3

Let $u_a$ denote the unique large solution of (4.6) corresponding to $a < \lambda_{\infty,1}$.

Fix $\varepsilon \in (0, 1/2)$ and choose $\delta > 0$ small as in §4.2.6.1.

Set $\chi_\varepsilon^\pm = \widetilde{\chi} \pm \varepsilon$ and define

$$u^\pm(x) = \xi_0 h(d)\left[1 + \chi_\varepsilon^\pm(-\ln d)^{-\tau}\right] \quad \text{for } x \in \Omega \text{ with } d \in (0, \delta).$$

We can assume $u^\pm(x) > 0$ for every $x \in \Omega$ with $d(x) \in (0, \delta)$.

By the Lagrange mean value theorem, we obtain

$$f(u^\pm(x)) = f(\xi_0 h(d)) + \xi_0 \chi_\varepsilon^\pm \frac{h(d)}{(-\ln d)^\tau} f'(\Psi^\pm(d)),$$

where $\Psi^\pm(d) = \xi_0 h(d)\left[1 + \chi_\varepsilon^\pm \lambda^\pm(d)(-\ln d)^{-\tau}\right]$, for some $\lambda^\pm(d) \in [0, 1]$.

Since $f(u)/u^{\rho+1}$ is slowly varying, by Proposition 4.1.1 we find

$$\lim_{d \searrow 0} \frac{f(\Psi^\pm(d))}{f(\xi_0 h(d))} = \lim_{d \searrow 0} \frac{f(u^\pm(d))}{f(\xi_0 h(d))} = 1. \tag{4.69}$$

*Step* 4.2.7. There exists $\delta_1 \in (0, \delta)$ so that

$$\begin{cases} \Delta u^+ + a u^+ - k^2(d)[1 + (\widetilde{c} - \varepsilon) d^\theta] f(u^+) \le 0, & \forall x \in \Omega \text{ with } d < \delta_1, \\ \Delta u^- + a u^- - k^2(d)[1 + (\widetilde{c} + \varepsilon) d^\theta] f(u^-) \ge 0, & \forall x \in \Omega \text{ with } d < \delta_1. \end{cases} \tag{4.70}$$

For every $x \in \Omega$ with $d \in (0, \delta)$, we have

$$\Delta u^\pm + a u^\pm - k^2(d)\left[1 + (\widetilde{c} \mp \varepsilon) d^\theta\right] f(u^\pm) = \xi_0 \frac{h''(d)}{(-\ln d)^\tau} \mathcal{J}^\pm(d) \tag{4.71}$$

where we denote

$$\begin{aligned} \mathcal{J}^\pm(d) = \Bigg[ & (-\widetilde{c} \pm \varepsilon) d^\theta (-\ln d)^\tau \frac{k^2(d) f(\xi_0 h(d))}{\xi_0 h''(d)} + a \frac{h(d)}{h''(d)} \left(\chi_\varepsilon^\pm + (-\ln d)^\tau\right) \\ & + \frac{\tau \chi_\varepsilon^\pm h(d)}{d^2 h''(d) \ln d} \left(1 + \frac{\tau + 1}{\ln d} - d \Delta d\right) + \frac{h'(d)}{d h''(d)} \left(d(-\ln d)^\tau \Delta d - \frac{2\tau \chi_\varepsilon^\pm}{\ln d}\right) \\ & + (-\widetilde{c} \pm \varepsilon) \chi_\varepsilon^\pm d^\theta \frac{k^2(d) h(d) f'(\Psi^\pm(d))}{h''(d)} + \chi_\varepsilon^\pm \Delta d \frac{h'(d)}{h''(d)} + \mathcal{H}(d) + \mathcal{J}_1^\pm(d) \Bigg]. \end{aligned}$$

Here $\mathcal{H}$ is defined by (4.28), while

$$\mathcal{J}_1^\pm(d) := \chi_\varepsilon^\pm \left(1 - \frac{k^2(d) h(d) f'(\Psi^\pm(d))}{h''(d)}\right).$$

By Lemma 4.2.4 and (4.69), we infer that

$$\lim_{d\searrow 0} \frac{k^2(d)h(d)f'(\Psi^{\pm}(d))}{h''(d)} = \lim_{d\searrow 0} \frac{\Psi^{\pm}(d)f'(\Psi^{\pm}(d))}{f(\Psi^{\pm}(d))} \frac{k^2(d)f(\xi_0 h(d))}{\xi_0 h''(d)} = \rho + 1.$$

This implies that

$$\lim_{d\searrow 0} \mathcal{J}_1^{\pm}(d) = -\rho\chi_\varepsilon^{\pm} := -\rho(\widetilde{\chi} \pm \varepsilon).$$

By Lemmas 4.2.4 and 4.2.7, we get $\lim_{d\searrow 0} \mathcal{J}^{+}(d) = -\rho\varepsilon < 0$ and $\lim_{d\searrow 0} \mathcal{J}^{-}(d) = \rho\varepsilon > 0$. Therefore, by (4.71) we conclude (4.70).

*Step* 4.2.8. There exists $M^+$, $\delta^+ > 0$ such that

$$u_a(x) \le u^+(x) + M^+, \qquad \forall x \in \Omega \text{ with } 0 < d < \delta^+.$$

The claim follows in the same way as for Step 4.2.5 of Theorem 4.2.2.

*Step* 4.2.9. There exists $M^-$, $\delta^- > 0$ such that

$$u_a(x) \ge u^-(x) - M^-, \qquad \forall x \in \Omega \text{ with } 0 < d < \delta^-.$$

The proof goes exactly as in Step 4.2.6, except that $\delta_* \in (0, \widetilde{\delta})$ is chosen sufficiently close to $\widetilde{\delta}$ such that

$$\frac{h(\delta_*)(1 + \chi_\varepsilon^-(-\ln\delta_*)^{-\tau})}{h(\widetilde{\delta})(1 + \chi_\varepsilon^-(-\ln\widetilde{\delta})^{-\tau})} < 1 + \lambda. \tag{4.72}$$

The reasoning for Step 4.2.6 applies now with (4.72) instead of (4.67).

By Steps 4.2.8 and 4.2.9, it follows that

$$\begin{cases} \chi_\varepsilon^{+} \ge \left[-1 + \dfrac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau - \dfrac{M^+(-\ln d)^\tau}{\xi_0 h(d)}, \ \forall x \in \Omega \text{ with } d < \delta^+ \\ \chi_\varepsilon^{-} \le \left[-1 + \dfrac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau + \dfrac{M^-(-\ln d)^\tau}{\xi_0 h(d)}, \ \forall x \in \Omega \text{ with } d < \delta^-. \end{cases} \tag{4.73}$$

Using Lemma 4.2.4, we have

$$\lim_{t\searrow 0} \frac{(-\ln t)^\tau}{h(t)} = \lim_{t\searrow 0} \left(\frac{-\ln t}{\ln h(t)}\right)^\tau \frac{(\ln h(t))^\tau}{h(t)} = \left(\frac{\rho\ell_1}{2}\right)^\tau \lim_{u\to\infty} \frac{(\ln u)^\tau}{u} = 0.$$

Passing to the limit $d \searrow 0$ in (4.73), we obtain

$$\begin{cases} \liminf_{d\searrow 0} \left[-1 + \dfrac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau \ge \chi_\varepsilon^- := \widetilde{\chi} - \varepsilon \\ \limsup_{d\searrow 0} \left[-1 + \dfrac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau \le \chi_\varepsilon^+ := \widetilde{\chi} + \varepsilon. \end{cases}$$

By sending $\varepsilon$ to 0, the proof of Theorem 4.2.3 is finished. □

## 4.3 Degenerate Mixed Boundary Value Problems

### 4.3.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a smooth bounded domain. As in §3.3, $\mathcal{B}$ denotes either the Dirichlet boundary operator $\mathcal{D}u := u$ or the Neumann/Robin boundary operator $\mathcal{R}u = \frac{\partial u}{\partial \nu} + \beta(x)u$, where $\nu$ is the outward unit normal to $\partial\Omega$ and $\beta \geq 0$ is in $C^{1,\mu}(\partial\Omega)$ with $\mu \in (0,1)$.

We are here concerned with the uniqueness of the *large solutions* for the degenerate boundary value problem (3.42) considered in §3.3, namely

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega \setminus D_0, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.74}$$

Recall that non-negative $C^2(\overline{\Omega} \setminus D_0)$-solutions of (4.74) that satisfy

$$u(x) \to \infty \quad \text{as } x \in \Omega \setminus D_0 \quad \text{and } d(x) := \text{dist}\,(x, D_0) \to 0$$

are referred to as *large solutions* of (4.74).

For the reader's convenience, we restate the assumptions on $b$ made in §3.3.1.

Let $b \in C^{0,\mu}(\overline{\Omega})$ satisfy $b \geq 0$, $b \not\equiv 0$ in $\Omega$. Set $\Omega_{0,b} := \{x \in \Omega : \ b(x) = 0\}$.

We assume that $\Omega_{0,b} = D_0 \cup \overline{D}_1$, where $D_0 \neq \emptyset$ is a closed set such that $\Omega \setminus D_0$ is connected with smooth boundary, and $D_1 \subset\subset \Omega \setminus D_0$ is a connected set.

Suppose that $b > 0$ on $\partial\Omega$ if $\mathcal{B} = \mathcal{R}$ and $\partial D_1$ satisfies the exterior cone condition (possibly, $D_1 = \emptyset$). Let $\lambda_{\infty,1}(D_1)$ be the first Dirichlet eigenvalue of $(-\Delta)$ in $H_0^1(D_1)$. Set $\lambda_{\infty,1}(D_1) = \infty$ if $D_1 = \emptyset$.

The purpose of this section is to show that the main results of §4.2 that apply to (4.6) remain, in fact, valid for the problem (4.74).

We only illustrate this point by giving the result that corresponds to Theorem 4.2.1. In §4.3.2 we assert that if $f \in RV_{\rho+1}$ ($\rho > 0$) satisfies (4.7) and (4.9) holds with $d(x) := \text{dist}\,(x, D_0)$, then (4.74) has a unique large solution for any $a < \lambda_{\infty,1}(D_1)$ (see Theorem 4.3.1). The blow-up rate and variation speed of the large solution are also provided. The proof of Theorem 4.3.1 is presented in §4.3.3. Since the asymptotic behavior of the large solution near the boundary is found using a local argument, one can easily formulate Theorems 4.2.2 and 4.2.3 for (4.74) instead of (4.6).

### 4.3.2 Uniqueness of the Large Solution

The main result of section 4.3 is the following (see Theorem 1.3 in Cîrstea and Rădulescu (2004) when $f' \in RV_\rho$).

**Theorem 4.3.1.** *Let* (4.7) *hold and* $f \in RV_{\rho+1}$ $(\rho > 0)$. *Assume that there exists* $k \in \mathcal{K}$ *such that*

$$\lim_{d(x)\to 0} \frac{b(x)}{k^2(d)} = 1, \quad \text{where } d(x) := \operatorname{dist}(x, D_0). \tag{4.75}$$

*Then, for any* $a < \lambda_{\infty,1}(D_1)$, (4.74) *has a unique large solution* $u_a$. *Moreover,*

$$\lim_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} = \xi_0, \tag{4.76}$$

*where* $\xi_0 = \left(\frac{2+\ell_1\rho}{2+\rho}\right)^{1/\rho}$ *and* $h$ *is given by* (4.13).

*If* $\ell_1 \neq 0$, *then* $h(1/u) \in RV_{2/(\rho\ell_1)}$, *i.e., there exists* $L(u) \in RV_0$, *such that*

$$\lim_{d(x)\searrow 0} u_a(x)[d(x)]^{\frac{2}{\rho\ell_1}} L(1/d(x)) = 1, \quad \forall a < \lambda_{\infty,1}(D_1). \tag{4.77}$$

*If* $\ell_1 = 0$, *then* $h(1/u)$ *is* $\Gamma$*-varying at* $u = \infty$ *with auxiliary function*

$$g(u) = \frac{\rho u^2 \int_0^{1/u} k(s)\, ds}{2k(1/u)}.$$

*If, in addition,* $u(\int_0^{1/u} k(s)\, ds)/k(1/u) \in RV_j$ $(j \leq 0)$, *then there exists* $T \in RV_{-2/\rho}$ *and* $R \in RV_{-j}$ *such that*

$$\lim_{d(x)\searrow 0} u_a(x)\, T(e^{R(1/d(x))}) = 1, \quad \forall a < \lambda_{\infty,1}(D_1). \tag{4.78}$$

*Remark* 4.3.1. Note that $k \in \mathcal{K}_0$ satisfies $u(\int_0^{1/u} k(s)\, ds)/k(1/u) \in RV_j$ $(j \leq 0)$, if and only if $k$ is of the form

$$k(t) = \frac{d_0}{t\mathcal{W}(t)} \exp\left\{-\int_t^{d_1} \frac{dx}{x\mathcal{W}(x)}\right\}, \quad (0 < t < d_1),$$

where $d_0, d_1$ are positive constants, while $\mathcal{W}$ is a positive $C^1$-function on $(0, d_1)$ such that (4.38) holds and $\mathcal{W}(1/u) \in RV_j$ (cf. Proposition 4.2.10). Moreover, if $\mathcal{W}(1/u) \in NRV_j$ with $j < 0$, then (4.38) is automatically fulfilled.

### 4.3.3 Proof of Theorem 4.3.1

By Theorem 3.3.1, (4.74) admits large solutions if and only if $a < \lambda_{\infty,1}(D_1)$.

We now prove that (4.76) holds for any large solution of (4.74).

Fix $\varepsilon \in (0, 1/2)$. Let $\delta > 0$ be small such that

(i) $\operatorname{dist}(x, \partial D_0)$ is a $C^2$-function on $\{x \in \Omega \setminus D_0 : \operatorname{dist}(x, \partial D_0) < 2\delta\}$;

(ii) $k$ is non-decreasing on $(0, 2\delta)$;

(iii) $b(x)/k^2(d(x)) \in (1-\varepsilon, 1+\varepsilon)$, for each $x \in \Omega$ with $d(x) \in (0, 2\delta)$;

(iv) $h''(t) > 0$, for each $t \in (0, 2\delta)$ (see Lemma 4.2.4).

Let $\sigma \in (0, \delta)$ be arbitrary. Set $\xi^{\pm} = \left[\frac{2+\ell_1\rho}{(1\mp 2\varepsilon)(2+\rho)}\right]^{1/\rho}$ and define

$$\begin{cases} v_\sigma^+(x) = \xi^+ h(d(x) - \sigma), & \forall x \text{ with } d(x) \in (\sigma, 2\delta) \\ v_\sigma^-(x) = \xi^- h(d(x) + \sigma), & \forall x \text{ with } d(x) + \sigma < 2\delta. \end{cases}$$

As in §4.2.5.1, we can diminish $\delta > 0$ such that

$$\begin{cases} \Delta v_\sigma^+ + a v_\sigma^+ - b(x) f(v_\sigma^+) \le 0, & \forall x \in \Omega \setminus D_0 \text{ with } \sigma < d(x) < 2\delta \\ \Delta v_\sigma^- + a v_\sigma^- - b(x) f(v_\sigma^-) \ge 0, & \forall x \in \Omega \setminus D_0 \text{ with } d(x) + \sigma < 2\delta. \end{cases}$$

Define $\Omega_\delta = \{x \in \Omega : \ d(x) < \delta\}$. Let $\omega \subset\subset D_0$ be such that $a$ is less than the first Dirichlet eigenvalue of $(-\Delta)$ in the smooth domain $\widetilde{D} := \operatorname{int}(D_0 \setminus \omega)$.

Let $p \in C^{0,\mu}(\overline{\Omega}_\delta)$ satisfy $0 < p \le b$ on $\overline{\Omega}_\delta \setminus D_0$, $p \equiv 0$ on $D_0 \setminus \omega$ and $p > 0$ in $\omega$. By Theorem 3.1.1, there exists a large solution of $\Delta w + aw = p(x) f(w)$ in $\Omega_\delta$.

Let $u_a$ be an arbitrary large solution of (4.74). Then $v := u_a + w$ satisfies $\Delta v + av - b(x) f(v) \le 0$ in $\Omega_\delta \setminus D_0$. Lemma 3.3.3 yields $u_a + w \ge v_\sigma^-$ on $\Omega_\delta \setminus D_0$. Similarly, $v_\sigma^+ + w \ge u_a$ on $\Omega_\delta \setminus \overline{\Omega}_\sigma$. Letting $\sigma \to 0$, we find

$$h(d)\xi^+ + 2w \ge u_a + w \ge h(d)\xi^- \quad \text{on } \Omega_\delta \setminus D_0.$$

It follows that

$$\xi^- \le \liminf_{d(x) \searrow 0} \frac{u_a(x)}{h(d(x))} \le \limsup_{d(x) \searrow 0} \frac{u_a(x)}{h(d(x))} \le \xi^+.$$

Letting $\varepsilon \to 0$, we reach (4.76).

Let $u_1$ and $u_2$ be two arbitrary large solutions of (4.74). For any $\varepsilon > 0$, define $\widetilde{u}_i = (1+\varepsilon)u_i$, $i = 1, 2$. Using (4.76), we get

$$\lim_{d(x)\searrow 0} \frac{u_1(x) - \widetilde{u}_2(x)}{h(d(x))} = \lim_{d(x)\searrow 0} \frac{u_2(x) - \widetilde{u}_1(x))}{h(d(x))} = -\varepsilon\xi_0,$$

which implies that $\lim_{d(x)\searrow 0}[u_1(x) - \widetilde{u}_2(x)] = \lim_{d(x)\searrow 0}[u_2(x) - \widetilde{u}_1(x)] = -\infty$. Using (4.7), we find $\Delta\widetilde{u}_i \leq b(x)f(\widetilde{u}_i) - a\widetilde{u}_i$ on $\Omega \setminus D_0$. Since $\mathcal{B}\widetilde{u}_i = \mathcal{B}u_i = 0$ on $\partial\Omega$, by Lemma 3.3.3 we deduce $u_1 \leq \widetilde{u}_2$ on $\overline{\Omega} \setminus D_0$ and $u_2 \leq \widetilde{u}_1$ on $\overline{\Omega} \setminus D_0$. Letting $\varepsilon \to 0$, we conclude the uniqueness of the large solution of (4.74).

By Remark 4.2.8, it only remains to prove (4.78) provided that $k \in \mathcal{K}_0$ satisfies $u(\int_0^{1/u} k(s)\,ds)/k(1/u) \in RV_j$ $(j \leq 0)$.

Define $U_1(u) = 1/\int_u^\infty [2F(s)]^{-1/2}\,ds$ for $u > 0$ and $U_2(u) = 1/\int_0^{1/u} k(s)\,ds$, for $u > 0$ sufficiently large. We see that $U_1 : (0,\infty) \to (0,\infty)$ is a $C^1$-increasing and bijective function. Thus, for each $y > 0$, $U_1^{\leftarrow}(y) = \inf\{s : U_1(s) \geq y\}$ coincides with the inverse function of $U_1$ at $y$. Hence, $h(1/u) = U_1^{\leftarrow}(U_2(u))$ for $u > 0$ large enough. Clearly, $\lim_{u\to\infty} U_1(u) = \lim_{u\to\infty} U_2(u) = \infty$ and $U_1(u) \in RV_{\rho/2}$. Thus, $U_1^{\leftarrow} \in RV_{2/\rho}$, cf. Proposition 4.1.6.

Since $\mathcal{W}(1/u) = u(\int_0^{1/u} k(s)\,ds)/k(1/u) \in RV_j$, we obtain $R(u) := \ln U_2(u) \in RV_{-j}$. We let $T(u) = 1/[\xi_0 U_1^{\leftarrow}(u)]$ for $u > 0$, which concludes (4.78). □

# Chapter 5

# Further Uniqueness Results on Logistic-type Equations

**“Real success is finding your lifework in the work that you love.” (David McCullough)**

In Chapter 4 we established the uniqueness and asymptotics of the large solutions for elliptic problems such as (3.1) with either complete boundary blow-up or mixed boundary conditions; the analysis applies when the variation of $f$ at infinity is *regular* of index greater than 1 and the decay rate of $b$ is expressed in terms of a ratio whose limit near the boundary is finite.

The objective of Chapter 5 is to relax the vanishing condition imposed on $b$ near the boundary by allowing the above ratio to be bounded and bounded away from zero. This question will be treated for nonlinearities of $f(u)$ whose variation at infinity is *regular* (as in Chapter 4, see §5.2) as well as *rapid* (see §5.3).

The feature of §5.2 is to prove the uniqueness of the large solution without determining precisely its blow-up rate. The argument, which refines that of Chapter 4, relies as a novelty on local blow-up estimates jointly with a modified version of Safonov's iteration technique. The uniqueness of the large solution when the variation of $f(u)$ becomes rapid, thus loosing its regular character, is examined in §5.3. In contrast to §5.2, the asymptotic behavior of any large solution can be described using a different approach based on de Haan theory that extends regular variation theory. The variation speed of the large solution is demonstrated to slow down significantly when the variation of $f(u)$ changes from regular to rapid.

## 5.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a smooth bounded domain and $\Gamma_\infty$ be a non-empty open and closed subset of $\partial\Omega$ (possibly, $\Gamma_\infty = \partial\Omega$). Set $\Gamma_{\mathcal{B}} := \partial\Omega \setminus \Gamma_\infty$ when $\Gamma_\infty \neq \partial\Omega$. We denote by $\mathcal{B}$ either the Dirichlet boundary operator $\mathcal{D}u = u$ or the Neumann/Robin boundary operator $\mathcal{R}u = \frac{\partial u}{\partial \nu} + \beta(x)u$. Here $\nu$ is the outward unit normal to $\partial\Omega$ and $\beta \geq 0$ is in $C^{1,\mu}(\partial\Omega)$, $0 < \mu < 1$.

We are interested in the uniqueness and asymptotic behavior of the *large solutions* to the equation

$$-\Delta u = au - b(x)f(u) \quad \text{in } \Omega, \tag{5.1}$$

if $\Gamma_\infty = \partial\Omega$, and to the boundary value problem

$$\begin{cases} -\Delta u = au - b(x)f(u) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \Gamma_{\mathcal{B}}, \end{cases} \tag{5.2}$$

if $\Gamma_\infty \neq \partial\Omega$, where $f \in C[0,\infty)$ is locally Lipschitz, $a \in \mathbb{R}$ is a parameter and $b \in C^{0,\mu}(\overline{\Omega})$ is non-negative.

A $C^2(\Omega)$-solution of (5.1) and $C^2(\Omega \cup \Gamma_{\mathcal{B}})$-solution of (5.2), respectively satisfying $u(x) \geq 0$ in $\Omega$ and $u(x) \to \infty$ as $\operatorname{dist}(x, \Gamma_\infty) \to 0$ is called a *large solution* of (5.1) and (5.2), respectively.

Set $\Omega_{0,b} := \{x \in \Omega : \ b(x) = 0\}$ and denote by $\Omega_0$ the interior of $\Omega_{0,b}$.

We assume, throughout this chapter, that $\partial\Omega_0$ satisfies the exterior cone condition (possibly, $\Omega_0 = \emptyset$), $\Omega_0$ is connected, $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ on $\Omega \setminus \overline{\Omega}_0$. If $\Gamma_\infty \neq \partial\Omega$, then we require $b > 0$ on $\Gamma_{\mathcal{B}}$ if $\mathcal{B} = \mathcal{R}$. Note that we allow $b \geq 0$ on $\Gamma_\infty$ as well as on $\Gamma_{\mathcal{B}}$ when $\mathcal{B} = \mathcal{D}$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of $(-\Delta)$ in $\Omega_0$ ($\lambda_{\infty,1} = \infty$ if $\Omega_0 = \emptyset$).

Assume that $f$ satisfies

$$f \geq 0 \text{ is locally Lipschitz continuous and } f(u)/u \text{ is increasing on } (0,\infty). \tag{5.3}$$

Then, necessarily $f(0) = 0$, and by the strong maximum principle, any non-negative classical solution of (5.1) or (5.2) is positive in $\Omega$ unless it is identically zero. Consequently, any large solution of (5.1) or (5.2) is positive.

By using an iteration technique due to Safonov, Du (2004) shows that for the special case $f(u) = u^p$ $(p > 1)$, (5.1) or (5.2) (when $\mathcal{B} = \mathcal{D}$) has a unique large solution provided that, for some constant $\alpha \geq 0$,

$$0 < \liminf_{d(x,\Gamma_\infty)\to 0} \frac{b(x)}{d(x,\Gamma_\infty)^\alpha} \quad \text{and} \quad \limsup_{d(x,\Gamma_\infty)\to 0} \frac{b(x)}{d(x,\Gamma_\infty)^\alpha} < \infty. \tag{5.4}$$

The main purpose of this chapter is two-fold: to establish the uniqueness and blow-up rate of the large solution of (5.1) and (5.2) for a more general version of (5.4) assuming that

(a) $f$ varies *regularly* at infinity of index greater than 1 (in §5.2) (thus covered by Karamata's theory), in the spirit of Chapter 4;

(b) $f$ varies *rapidly* (at infinity) with index $\infty$ (in §5.3), that is

$$\lim_{u\to\infty} \frac{f(\lambda u)}{f(u)} = \begin{cases} \infty, & \text{if } \lambda > 1 \\ 1, & \text{if } \lambda = 1 \\ 0, & \text{if } 0 < \lambda < 1. \end{cases}$$

To achieve these aims we need techniques beyond those of Chapter 4. The approach we put forward will deepen the interplay between the blow-up topic in PDEs and the extensions of regular variation theory in applied probability (Resnick (1987)).

## 5.2 Case I: Regularly Varying Nonlinearities

### 5.2.1 Main Results

As in §4.2, $\mathcal{K}$ comprises all positive, non-decreasing $C^1$-functions $k$ defined on $(0, \nu)$, for some $\nu > 0$, which satisfy

$$\lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{k(t)} = \ell_0 \quad \text{and} \quad \lim_{t\to 0^+} \left(\frac{\int_0^t k(s)\,ds}{k(t)}\right)' = \ell_1.$$

Our first result shows that the uniqueness assertion of Theorems 4.2.1 and 4.3.1 is valid when the assumptions on $b$ are weakened. More precisely, we have (see Theorem 1.1 in Cîrstea and Du (2005)).

**Theorem 5.2.1.** *Let (5.3) hold and $f \in RV_{\rho+1}$ with $\rho > 0$. Assume that for each connected open and closed subset, say $\Gamma_\infty^c$, of $\Gamma_\infty$ there exists $k \in \mathcal{K}$ such that*

$$0 < \liminf_{d(x)\to 0} \frac{b(x)}{k^2(d(x))} \quad \text{and} \quad \limsup_{d(x)\to 0} \frac{b(x)}{k^2(d(x))} < \infty, \quad \text{where } d(x) := d(x, \Gamma_\infty^c), \tag{5.5}$$

*then (5.1) (resp., (5.2)) has a unique large solution for any $a < \lambda_{\infty,1}$.*

*Remark* 5.2.1. We note that this result improves Theorem 3.2 in Du (2004), where $f(u) = u^{\rho+1}$, $b > 0$ on $\overline{\Omega} \setminus \Gamma_\infty$, $\mathcal{B}u = u$ on $\Gamma_\mathcal{B}$ and $k(t) = t^\alpha$, $\alpha \geq 0$.

Next we provide the blow-up rate of $u_a$ when (5.5) is replaced by (5.6) below (cf. Theorem 1.2 in Cîrstea and Du (2005)).

**Theorem 5.2.2.** *Let (5.3) hold and $f \in RV_{\rho+1}$, for some $\rho > 0$. Suppose that for each connected open and closed subset $\Gamma_\infty^c$ of $\Gamma_\infty$ there exists $k \in \mathcal{K}$ and a positive continuous function $c(x)$ on $\Gamma_\infty$ such that*

$$\lim_{x\to y\in\Gamma_\infty^c} \frac{b(x)}{k^2(d(x))} = c(y), \quad \text{uniformly for } y \in \Gamma_\infty^c \quad (\text{where } d(x) = d(x, \Gamma_\infty^c)). \tag{5.6}$$

*Suppose that $a \in (-\infty, \lambda_{\infty,1})$. Then the blow-up rate of the unique blow-up solution $u_a$ of problem (5.1) (resp., (5.2)) is given by*

$$\lim_{x\to y\in\Gamma_\infty^c} \frac{u_a(x)}{\Psi(d(x))} = \left(\frac{2+\rho\ell_1}{2c(y)}\right)^{\frac{1}{\rho}}, \quad \text{uniformly for } y \in \Gamma_\infty^c \tag{5.7}$$

*where $\Psi$ is uniquely determined by*

$$\int_{\Psi(t)}^{\infty} \frac{dy}{\sqrt{yf(y)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0,\tau), \quad \text{for } \tau > 0 \text{ small enough.} \tag{5.8}$$

The behavior of $\Psi(t)$ for small $t > 0$ will be described in §5.2.2. In particular, if $k$ has corresponding $\ell_1 \neq 0$, then $\Psi(1/u)$ is a function in $RV_{2/(\rho\ell_1)}$.

*Remark* 5.2.2. If $f(u) = u^p$ ($p > 1$), then we get $\Psi(t) = \left(\frac{p-1}{2}\int_0^t k(s)\,ds\right)^{2/(1-p)}$.

*Remark* 5.2.3. If we see $c \in C(\Gamma_\infty)$ as extended by continuity to a positive function on $\overline{\Omega}$, then (5.6) and (5.7) are equivalent to, respectively,

$$\lim_{d(x,\Gamma_\infty^c)\to 0} \frac{b(x)}{c(x)k^2(d(x))} = 1, \tag{5.9}$$

and

$$\lim_{d(x,\Gamma_\infty^c)\to 0} \frac{u_a(x)}{\Psi(d(x))[c(x)]^{-1/\rho}} = \left(\frac{2+\rho\ell_1}{2}\right)^{1/\rho}. \tag{5.10}$$

*Remark* 5.2.4. Theorem 5.2.2 improves Theorems 4.2.1 and 4.3.1, where the positive function $c(y)$ in (5.6) and (5.7) is required to be a positive constant.

The blow-up rate is *local* in nature. We demonstrate this point by considering the positive solutions to the problem

$$\begin{cases} -\Delta u = au - b(x)f(u) & \text{in } \Omega \cap \mathbb{B}, \\ u = \infty & \text{on } \Gamma_\infty \cap \mathbb{B}, \end{cases} \tag{5.11}$$

where $\mathbb{B}$ denotes an open ball in $\mathbb{R}^N$ such that $\Gamma_\infty \cap \mathbb{B} \neq \emptyset$.

The following results (see Theorems 1.3 and 1.4 in Cîrstea and Du (2005)) will also be used in the proof of Theorem 5.2.1.

**Theorem 5.2.3.** *Let (5.3) hold and $f \in RV_{\rho+1}$ $(\rho > 0)$. Suppose that $x_* \in \Gamma_\infty \cap \mathbb{B}$ and there exists $k \in \mathcal{K}$ such that*

$$\limsup_{x \to x_*,\, x \in \Omega} \frac{b(x)}{k^2(d(x,\Gamma_\infty))} \leq c_*, \quad \text{for some constant } c_* > 0. \tag{5.12}$$

*Then, any positive solution $U$ of (5.11) satisfies*

$$\liminf_{x \to x_*,\, x \in \Omega} \frac{U(x)}{\Psi(d(x,\Gamma_\infty))} \geq \left( \frac{2 + \rho \ell_1}{2c_*} \right)^{\frac{1}{\rho}}, \tag{5.13}$$

*where $\Psi$ is given by (5.8).*

**Theorem 5.2.4.** *Let (5.3) hold and $f \in RV_{\rho+1}$ $(\rho > 0)$. Suppose that $x_* \in \Gamma_\infty \cap \mathbb{B}$ and there exists $k \in \mathcal{K}$ such that*

$$\liminf_{x \to x_*,\, x \in \Omega} \frac{b(x)}{k^2(d(x,\Gamma_\infty))} \geq d_*, \quad \text{for some constant } d_* > 0, \tag{5.14}$$

*then, any positive solution $U$ of (5.11) satisfies*

$$\limsup_{x \to x_*,\, x \in \Omega} \frac{U(x)}{\Psi(d(x,\Gamma_\infty))} \leq \left( \frac{2 + \rho \ell_1}{2d_*} \right)^{\frac{1}{\rho}}, \tag{5.15}$$

*with $\Psi$ given by (5.8).*

**Corollary 5.2.5.** *Let* (5.3) *hold and* $f \in RV_{\rho+1}$ $(\rho > 0)$*. Suppose that* $x_* \in \Gamma_\infty \cap \mathbb{B}$ *and there exists* $k \in \mathcal{K}$ *such that*

$$\lim_{x \to x_*, x\in\Omega} \frac{b(x)}{k^2(d(x,\Gamma_\infty))} = c_*, \quad \textit{for some constant } c_* > 0, \tag{5.16}$$

*then any positive solution of* (5.11) *satisfies*

$$\lim_{x \to x_*, x\in\Omega} \frac{u(x)}{\Psi(d(x,\Gamma_\infty))} = \left(\frac{2+\rho\ell_1}{2c_*}\right)^{\frac{1}{\rho}}, \tag{5.17}$$

*where* $\Psi$ *is given by* (5.8).

*Remark* 5.2.5. The above local estimates improve the corresponding ones in Du (2004) even for the special function classes considered there. In Du (2004) (motivated by López-Gómez (2003)) for the particular case $f(u) = u^{\rho+1}$ and $k(t) = t^\alpha$, it is proved that the limits in (5.13) and (5.15) hold under the extra restriction

$$x \in C_{x_*,\omega} = \{x \in \Omega : \ \text{angle}(x - x_*, -n_{x_*}) \leq \pi/2 - \omega\}, \quad \forall \omega \in (0, \pi/2), \tag{5.18}$$

where $n_{x_*}$ is the outward unit normal of $\partial\Omega$ at $x_*$.

The rest of §5.2 is organized as follows. In §5.2.2, we describe the behavior of the function $\Psi(t)$ used in our main results. The proof of Theorem 5.2.2 is given in §5.2.3. By invoking Theorem 5.2.2, we prove Theorems 5.2.3 and 5.2.4 in §5.2.4 and §5.2.5, respectively. In §5.2.6 we provide the proof of Theorem 5.2.1, where Theorems 5.2.2–5.2.4 and a variant of Safonov's iteration technique are employed.

### 5.2.2 A Preliminary Result

The following result (see Proposition 2.8 in Cîrstea and Du (2005)) comprises the properties of the function $\Psi$ that plays an important role in our main results. We deduce these properties by invoking regular variation theory (see §4.1).

**Proposition 5.2.6 (Properties of $\Psi$).** *Suppose that* $f \in RV_{\rho+1}$ $(\rho > 0)$ *is continuous and* $k \in \mathcal{K}$*, then the function* $\Psi = \Psi_f$*, given by* (5.8)*, is well defined. Moreover,* $\Psi \in C^1(0,\tau)$ *satisfies* $\lim_{t\to 0^+} \Psi(t) = \infty$ *and*

(i) *$\Psi(1/u)$ belongs to $RV_{2/(\rho\ell_1)}$ if $\ell_1 \neq 0$; if $\ell_1 = 0$, then $\Psi(1/u)$ is $\Gamma$-varying at $u = \infty$ with auxiliary function $\chi(u) = \rho u \mathcal{W}(1/u)/2$, where*

$$\mathcal{W}(1/u) = \frac{u \int_0^{1/u} k(s)\, ds}{k(1/u)};$$

(ii) $\lim_{t \to 0^+} \frac{\Psi_f(t)}{\Psi_g(t)} = 1$ *if* $\lim_{u \to \infty} \frac{g(u)}{f(u)} = 1$;

(iii) $\lim_{t \to 0^+} \frac{\Psi(t)}{\Psi'(t)} = 0$;

(iv) $\lim_{t \to 0^+} \frac{f(\Psi(t))}{\Psi(t)} \left( \int_0^t k(s)\, ds \right)^2 = \frac{4}{\rho^2}$;

(v) $\lim_{t \to 0^+} \frac{t^2 k^2(t) f(\Psi(t))}{\Psi(t)} = \frac{4}{\ell_1^2 \rho^2}$ *if* $\ell_1 \neq 0$.

*If we assume further that $f \in NRV_{\rho+1}$, then $\Psi$ is $C^2$ and*

(vi) $\lim_{t \to 0^+} \frac{\Psi'(t)}{\Psi''(t)} = \lim_{t \to 0^+} \frac{\Psi(t)}{\Psi''(t)} = 0$;

(vii) $\lim_{t \to 0^+} \frac{\Psi''(t)}{k^2(t) f(\Psi(t))} = \lim_{t \to 0^+} \frac{\Psi''(t)\Psi(t)}{[\Psi'(t)]^2} = 1 + \frac{\rho \ell_1}{2}$.

*Proof.* Since $f \in RV_{\rho+1}$, we see that $\lim_{z \to \infty} z^{1+r}/\sqrt{z f(z)} = 0$, for any $r \in (0, \rho/2)$. This shows that, for some $D_1 > 0$,

$$\varsigma(x) := \int_x^\infty \frac{dy}{\sqrt{y f(y)}} < \infty, \quad \forall x > D_1. \tag{5.19}$$

Obviously, $\varsigma : (D_1, \infty) \to (0, \varsigma(D_1))$ is bijective and $\lim_{t \to 0^+} \int_0^t k(s)\, ds = 0$. Hence, we can define $\Psi(t) = \varsigma^{-1}(\int_0^t k(s)\, ds)$, for $t \in (0, \tau)$ if $\tau > 0$ is chosen small enough ($\varsigma^{-1}$ denotes the inverse of $\varsigma$). Notice that $\lim_{t \to 0^+} \Psi(t) = \infty$. The fact that $\Psi$ is $C^1$ follows by direct differentiation (see below).

We define $U_1(u) = 1/\varsigma(u)$ for $u > D_1$, with $\varsigma$ given by (5.19). Set $U_1(u) = U_1(D_1)$, $\forall u \leq D_1$. Obviously, $U_1$ is increasing on $(D_1, \infty)$ and, for each $y > U_1(D_1)$, $U_1^{\leftarrow}(y) := \inf\{s : \ U_1(s) \geq y\}$ coincides with the inverse of $U_1$ at $y$.

Set $U_2(u) = 1/\int_0^{1/u} k(s)\, ds$ for $u \geq 1/\tau$, where $\tau > 0$ is chosen small enough such that $U_2(1/\tau) > U_1(D_1)$. Hence, $\Psi(1/u) = U_1^{\leftarrow}(U_2(u))$ for $u > 1/\tau$. Clearly,

$\lim_{u\to\infty} U_1(u) = \lim_{u\to\infty} U_2(u) = \infty$. Moreover, since $f \in RV_{\rho+1}$, $[uf(u)]^{-1/2}$ belongs to $RV_{-(1+\rho/2)}$. By Proposition 4.1.4, $\varsigma(u)$ and $u[uf(u)]^{-1/2}$ are regularly varying functions with the same index $-\rho/2$. Therefore $U_1(u) \in RV_{\rho/2}$.

Suppose that $\ell_1 \neq 0$. By Proposition 4.2.8, $k(1/u)$ belongs to $RV_{(\ell_1-1)/\ell_1}$. Thus, by Proposition 4.1.4, we deduce $U_2 \in RV_{1/\ell_1}$. Using Proposition 4.1.6, we find $U_1^{\leftarrow} \in RV_{2/\rho}$ and $U_1^{\leftarrow} \circ U_2 \in RV_{2/(\rho\ell_1)}$. Hence, $\Psi(1/u)$ is in $RV_{2/(\rho\ell_1)}$.

Assume that $\ell_1 = 0$. Then, from Remark 4.2.8, $U_2(u)$ is $\Gamma$-varying at $u = \infty$ with auxiliary function $u\mathcal{W}(1/u) := \dfrac{u^2\int_0^{1/u} k(s)\,ds}{k(1/u)}$. Since $U_1^{\leftarrow}(u)$ is monotone at infinity and $U_1^{\leftarrow} \in RV_{2/\rho}$, it follows that $\Psi(1/u)$ is $\Gamma$-varying at $u = \infty$ with auxiliary function $\rho u\mathcal{W}(1/u)/2$ (see (Resnick, 1987, p. 36)). This proves (i).

Suppose that $g(u)/f(u) \to 1$ as $u \to \infty$. Then $g \in RV_{\rho+1}$ and, for any given small $\epsilon > 0$, we can find small $t_0 > 0$ such that

$$f[(1+\epsilon)y]/f(y) > (1+\epsilon)^{1+\rho/2},\ g(y) < (1+\epsilon)^{\rho/2} f(y),\ \forall y > \Psi_g(t),\ \forall t \in (0, t_0).$$

It follows that, for $t \in (0, t_0)$,

$$\int_{\Psi_f(t)}^{\infty} \frac{dy}{\sqrt{yf(y)}} = \int_{\Psi_g(t)}^{\infty} \frac{dy}{\sqrt{yg(y)}} > \int_{\Psi_g(t)}^{\infty} \frac{dy}{\sqrt{y(1+\epsilon)^{\rho/2} f(y)}}$$
$$> \int_{\Psi_g(t)}^{\infty} \frac{dy}{\sqrt{y(1+\epsilon)^{-1} f[(1+\epsilon)y]}} = \int_{(1+\epsilon)\Psi_g(t)}^{\infty} \frac{dz}{\sqrt{zf(z)}}.$$

This implies that

$$\Psi_f(t) < (1+\epsilon)\Psi_g(t),\ \forall t \in (0, t_0).$$

Similarly, we can show that there exists $t_1 > 0$ such that

$$\Psi_g(t) < (1+\epsilon)\Psi_f(t),\ \forall t \in (0, t_1).$$

Therefore, $\lim_{t\to 0^+} \Psi_g(t)/\Psi_f(t) = 1$, which proves (ii).

By Proposition 4.1.4, we obtain

$$\lim_{u\to\infty} \frac{\sqrt{u}}{\sqrt{f(u)}} \frac{1}{\int_u^{\infty} \frac{dy}{\sqrt{yf(y)}}} = \frac{\rho}{2}$$

which, together with (5.8), yields

$$\lim_{t\to 0^+} \frac{\sqrt{\Psi(t)}}{\sqrt{f(\Psi(t))}} \frac{1}{\int_0^t k(s)\,ds} = \frac{\rho}{2}. \tag{5.20}$$

Thus (iv) follows.

By differentiating (5.8), we find

$$\Psi'(t) = -k(t)\sqrt{\Psi(t)}\sqrt{f(\Psi(t))}, \quad \forall t \in (0,\tau). \tag{5.21}$$

Hence, by (5.20) and (5.21)

$$\lim_{t\to 0^+} \frac{\Psi(t)}{\Psi'(t)} = -\lim_{t\to 0^+} \frac{\sqrt{\Psi(t)}}{k(t)\sqrt{f(\Psi(t))}} = -\frac{\rho}{2}\lim_{t\to 0^+} \frac{\int_0^t k(s)\,ds}{k(t)} = 0. \tag{5.22}$$

This proves (iii).

If $\ell_1 \neq 0$, then by (iv), we have

$$\lim_{t\to 0^+} \frac{t^2k^2(t)f(\Psi(t))}{\Psi(t)} = \lim_{t\to 0^+} \frac{k^2(t)t^2}{\left(\int_0^t k(s)\,ds\right)^2} \frac{\left(\int_0^t k(s)\,ds\right)^2 f(\Psi(t))}{\Psi(t)} = \frac{4}{\rho^2\ell_1^2},$$

which proves (v).

We now assume that $f \in NRV_{\rho+1}$. This implies that

$$\lim_{t\to 0^+} \frac{\Psi(t)f'(\Psi(t))}{f(\Psi(t))} = \rho + 1.$$

By differentiating (5.21), we deduce

$$\begin{aligned}\Psi''(t) &= -k'(t)\sqrt{\Psi(t)f(\Psi(t))} + \frac{k^2(t)}{2}f(\Psi(t)) + \frac{k^2(t)}{2}\Psi(t)f'(\Psi(t))\\ &= k^2(t)f(\Psi(t))\left[-\frac{k'(t)}{k^2(t)}\frac{\sqrt{\Psi(t)}}{\sqrt{f(\Psi(t))}} + \frac{1}{2} + \frac{\Psi(t)f'(\Psi(t))}{2f(\Psi(t))}\right].\end{aligned}$$

By (5.20), we obtain

$$\lim_{t\to 0^+} \frac{k'(t)}{k^2(t)}\frac{\sqrt{\Psi(t)}}{\sqrt{f(\Psi(t))}} = \frac{\rho}{2}\lim_{t\to 0^+} \frac{k'(t)}{k^2(t)}\left(\int_0^t k(s)\,ds\right) = \frac{\rho(1-\ell_1)}{2}.$$

Hence, in view of (5.21), we have

$$\lim_{t\to 0^+} \frac{\Psi''(t)}{k^2(t)f(\Psi(t))} = \lim_{t\to 0^+} \frac{\Psi''(t)\Psi(t)}{[\Psi'(t)]^2} = 1 + \frac{\rho\ell_1}{2},$$

and we conclude (vii).

Using (vii) and (5.22), we infer that

$$\lim_{t\to 0^+} \frac{\Psi'(t)}{\Psi''(t)} = \frac{2}{2+\rho\ell_1} \lim_{t\to 0^+} \frac{\Psi(t)}{\Psi'(t)} = 0. \tag{5.23}$$

By (5.22) and (5.23), it follows that $\lim_{t\to 0^+} \Psi(t)/\Psi''(t) = 0$. This concludes (vi). The proof of Proposition 5.2.6 is now complete. □

*Remark* 5.2.6. Suppose that $g$ and $\Psi_g$ are as in (ii) of Proposition 5.2.6. Then, $\Psi_g(1/u)$ and $\Psi_f(1/u)$ belong to $RV_{2/(\rho\ell_1)}$ when $\ell_1 \neq 0$. If $\ell_1 = 0$, then from conclusion (ii) and the definition of Γ-varying functions, both $\Psi_g(1/u)$ and $\Psi_f(1/u)$ are Γ-varying at $u = \infty$ with the same auxiliary function $\chi(u) = \dfrac{\rho u^2 \int_0^{1/u} k(s)\, ds}{2k(1/u)}$.

## 5.2.3 Proof of Theorem 5.2.2

### 5.2.3.1 A Comparison Principle

We start by recalling a comparison principle, which plays an important role in the proof of Theorem 5.2.2, and will also be used in later sections.

**Proposition 5.2.7.** *Let $f$ be continuous on $(0,\infty)$ such that $f(u)/u$ is increasing for $u > 0$. Let $0 \not\equiv p \in C(\overline{\Omega} \setminus \Gamma_\infty)$ be a non-negative function. Assume that $u_1, u_2 \in C^2(\overline{\Omega} \setminus \Gamma_\infty)$ are positive such that*

$$\begin{cases} \limsup\limits_{d(x,\Gamma_\infty)\to 0} (u_2 - u_1)(x) \le 0 \\ -\Delta u_1 - au_1 + p(x)f(u_1) \ge 0 \ge -\Delta u_2 - au_2 + p(x)f(u_2) \quad \text{in } \Omega. \end{cases}$$

*When $\Gamma_\infty \neq \partial\Omega$, then we suppose, in addition, that*

$$\text{either } \mathcal{B}u_1 \ge \mathcal{B}u_2 \text{ on } \Gamma_{\mathcal{B}} \text{ if } \mathcal{B} = \mathcal{D} \quad \text{or } \mathcal{B}u_1 \ge 0 \ge \mathcal{B}u_2 \text{ on } \Gamma_{\mathcal{B}} \text{ if } \mathcal{B} = \mathcal{R},$$

*then we have $u_1 \ge u_2$ on $\overline{\Omega} \setminus \Gamma_\infty$.*

*Proof.* When $\Gamma_\infty = \partial\Omega$, then Proposition 5.2.7 reduces to Lemma 3.3.2 (see also Du and Guo (2003) for the version corresponding to the $p$-Laplacian). The proof of Proposition 5.2.7 when $\Gamma_\infty \neq \partial\Omega$ follows exactly as in Lemma 3.3.3.

Note that this proposition also follows from Lemma 2.3 in Du and Li (2002), since by the maximum principle, $u_1$ and $u_2$ being positive and satisfying the differential inequalities imply that $a < \lambda_{\infty,1}$. □

#### 5.2.3.2 Preparation

From $f \in RV_{\rho+1}$ ($\rho > 0$) we find $\lim_{u\to\infty} f(u)/u^q = \infty$, for every $q \in (1, \rho+1)$. Hence $f$ satisfies the Keller–Osserman condition (3.6).

By Theorem 3.1.1 if $\Gamma_\infty = \partial\Omega$ or Theorem 3.3.1 if $\Gamma_\infty \neq \partial\Omega$, we conclude that (5.1) (resp., (5.2)) possesses large solutions if and only if $a < \lambda_{\infty,1}$.

We fix $a < \lambda_{\infty,1}$. Let $\Gamma_\infty^c$ be an arbitrary connected open and closed subset of $\Gamma_\infty$. In what follows we denote $d(x) = d(x, \Gamma_\infty^c)$.

By the Dugunji extension theorem, we may assume that $c(x)$ is a positive continuous function on $\overline{\Omega}$. Given any small $\varepsilon \in (0, \min_{\overline{\Omega}} c/4)$ we can find a smooth function on $\overline{\Omega}$, say $\widetilde{c}(x)$, such that

$$|\widetilde{c}(x) - c(x)| < \varepsilon/2, \quad \forall x \in \overline{\Omega}. \tag{5.24}$$

In view of (5.6), there exists $\delta_0 = \delta_0(\varepsilon)$, which is independent of $y$, so that

$$c(y) - \frac{\varepsilon}{2} < \frac{b(x)}{k^2(d(x))} < c(y) + \frac{\varepsilon}{2}, \quad \forall x \in \Omega \text{ with } |x-y| < \delta_0, \forall y \in \Gamma_\infty^c. \tag{5.25}$$

Since $c \in C(\overline{\Omega})$ is uniformly continuous, there exists $\delta \in (0, \delta_0/2)$ such that

$$|c(y) - c(x)| < \varepsilon/2, \quad \forall x,\, y \in \overline{\Omega} \text{ with } |x-y| < 2\delta. \tag{5.26}$$

Therefore,

$$c(x) - \varepsilon < \frac{b(x)}{k^2(d(x))} < c(x) + \varepsilon, \quad \forall x \in \Omega \text{ with } d(x) < 2\delta. \tag{5.27}$$

In what follows we will need properties (vi) and (vii) for $\Psi$ in Proposition 5.2.6, yet we only have $f \in RV_{\rho+1}$. To overcome this difficulty, we choose $g \in NRV_{\rho+1}$ such that $g(u)/f(u) \to 1$ as $u \to \infty$; such $g$ always exists (see Remark 4.1.2). We then replace $\Psi_f$ by $\Psi_g$ and still denote it by $\Psi$. By Remark 5.2.6 and conclusion (ii) in Proposition 5.2.6, we know that this does not change the validity of the conclusions in Theorem 5.2.2.

We now diminish $\delta > 0$ to ensure that

(i) $d(x)$ is a $C^2$-function on $\{x \in \Omega : \ d(x) < 2\delta\}$;

(ii) $k$ is non-decreasing on $(0, 2\delta)$;

(iii) $\Psi''$ is positive on $(0, 2\delta)$.

Set $m := (1 + \rho\ell_1/2)^{1/\rho}$. Let $\sigma \in (0, \delta)$ be arbitrary. We define

$$\begin{cases} v_\sigma^+(x) = m\Psi(d(x) - \sigma)[\widetilde{c}(x) - 2\varepsilon]^{-1/\rho}, & \forall x \in \Omega \text{ with } \sigma < d(x) < 2\delta; \\ v_\sigma^-(x) = m\Psi(d(x) + \sigma)[\widetilde{c}(x) + 2\varepsilon]^{-1/\rho}, & \forall x \in \Omega \text{ with } d(x) + \sigma < 2\delta. \end{cases} \tag{5.28}$$

#### 5.2.3.3 Upper and Lower Solutions Near the Boundary

We prove that by diminishing $\delta > 0$ if necessary, we have

$$\begin{cases} -\Delta v_\sigma^+ - av_\sigma^+ + b(x)f(v_\sigma^+) \geq 0, & \forall x \in \Omega \text{ with } \sigma < d(x) < 2\delta; \\ -\Delta v_\sigma^- - av_\sigma^- + b(x)f(v_\sigma^-) \leq 0, & \forall x \in \Omega \text{ with } d(x) + \sigma < 2\delta. \end{cases} \tag{5.29}$$

A simple calculation yields

$$\begin{aligned} \nabla v_\sigma^\pm(x) = {} & m\Psi'(d(x) \mp \sigma)[\widetilde{c}(x) \mp 2\varepsilon]^{-\frac{1}{\rho}}\nabla d(x) \\ & - \frac{m}{\rho}\Psi(d(x) \mp \sigma)[\widetilde{c}(x) \mp 2\varepsilon]^{-\frac{1}{\rho}-1}\nabla\widetilde{c}(x). \end{aligned}$$

It follows that (using $|\nabla d(x)| \equiv 1$)

$$\Delta v_\sigma^\pm(x) = m\Psi''(d(x) \mp \sigma)[\widetilde{c}(x) \mp 2\varepsilon]^{-\frac{1}{\rho}} + \eta_\sigma^\pm(x)\Psi'(d(x) \mp \sigma), \tag{5.30}$$

where we denote

$$\begin{aligned} \eta_\sigma^\pm(x) = {} & -\frac{2m}{\rho}[\widetilde{c}(x) \mp 2\varepsilon]^{-\frac{1}{\rho}-1}\nabla d(x) \cdot \nabla\widetilde{c}(x) + m[\widetilde{c}(x) \mp 2\epsilon]^{-\frac{1}{\rho}}\Delta d(x) \\ & + \frac{m(\rho+1)}{\rho^2}\frac{\Psi(d(x) \mp \sigma)}{\Psi'(d(x) \mp \sigma)}[\widetilde{c}(x) \mp 2\varepsilon]^{-\frac{1}{\rho}-2}|\nabla\widetilde{c}(x)|^2 \\ & - \frac{m}{\rho}\frac{\Psi(d(x) \mp \sigma)}{\Psi'(d(x) \mp \sigma)}[\widetilde{c}(x) \mp 2\varepsilon]^{-\frac{1}{\rho}-1}\Delta\widetilde{c}(x). \end{aligned}$$

Using Proposition 5.2.6 (iii), we can find a constant $C_1 = C_1(\varepsilon) > 0$ such that

$$\begin{cases} |\eta_\sigma^+(x)| \leq C_1, & \forall x \in \Omega \text{ with } d(x) \in (\sigma, 2\delta); \\ |\eta_\sigma^-(x)| \leq C_1, & \forall x \in \Omega \text{ with } d(x) + \sigma < 2\delta. \end{cases} \tag{5.31}$$

By (5.27), we obtain

$$\begin{aligned} b(x) & > (c(x) - \varepsilon)k^2(d(x)) \geq (c(x) - \varepsilon)k^2(d(x) - \sigma) \\ & \geq M_\varepsilon(\widetilde{c}(x) - 2\varepsilon)k^2(d(x) - \sigma), \quad \forall x \in \Omega \text{ with } \sigma < d(x) < 2\delta, \end{aligned} \tag{5.32}$$

and

$$\begin{aligned} b(x) &< (c(x)+\varepsilon)k^2(d(x)) \leq (c(x)+\varepsilon)k^2(d(x)+\sigma) \\ &\leq N_\varepsilon(\widetilde{c}(x)+2\varepsilon)k^2(d(x)+\sigma), \quad \forall x \in \Omega \text{ with } d(x)+\sigma < 2\delta, \end{aligned} \tag{5.33}$$

where $M_\varepsilon := \min_{x\in\overline{\Omega}} \left(\frac{c(x)-\varepsilon}{\widetilde{c}(x)-2\varepsilon}\right)$ and $N_\varepsilon := \max_{x\in\overline{\Omega}} \left(\frac{c(x)+\varepsilon}{\widetilde{c}(x)+2\varepsilon}\right)$. By (5.24), we have $M_\varepsilon > 1$ and $N_\varepsilon < 1$.

Using (5.30), (5.32) and (5.33), we arrive at

$$\begin{aligned} -\Delta v_\sigma^+ - av_\sigma^+ + b(x)f(v_\sigma^+) \geq \Big\{ & M_\varepsilon \frac{k^2(d-\sigma)}{\Psi''(d-\sigma)} [\widetilde{c}(x)-2\varepsilon]^{\frac{1}{\rho}+1} f(v_\sigma^+) - m \\ &+ Q^+(d-\sigma)\Big\} \Psi''(d-\sigma)[\widetilde{c}(x)-2\varepsilon]^{-\frac{1}{\rho}} \end{aligned} \tag{5.34}$$

for all $x \in \Omega$ with $\sigma < d(x) < 2\delta$. Similarly, we have

$$\begin{aligned} -\Delta v_\sigma^- - av_\sigma^- + b(x)f(v_\sigma^-) \leq \Big\{ & N_\varepsilon \frac{k^2(d+\sigma)}{\Psi''(d+\sigma)} [\widetilde{c}(x)+2\varepsilon]^{\frac{1}{\rho}+1} f(v_\sigma^-) - m \\ &+ Q^-(d+\sigma)\Big\} \Psi''(d+\sigma)[\widetilde{c}(x)+2\varepsilon]^{-\frac{1}{\rho}} \end{aligned} \tag{5.35}$$

for all $x \in \Omega$ with $d(x)+\sigma < 2\delta$. In (5.34) and (5.35) we used the notation

$$Q^\pm(d(x)\mp\sigma) = -\eta_\sigma^\pm(x)\frac{\Psi'(d(x)\mp\sigma)}{\Psi''(d(x)\mp\sigma)}[\widetilde{c}(x)\mp 2\varepsilon]^{\frac{1}{\rho}} - am\frac{\Psi(d(x)\mp\sigma)}{\Psi''(d(x)\mp\sigma)}.$$

By (5.31) and Proposition 5.2.6 (vi), we have

$$\lim_{d(x)\searrow\sigma} Q^+(d(x)-\sigma) = 0 \quad \text{and} \quad \lim_{d(x)+\sigma\to 0} Q^-(d(x)+\sigma) = 0. \tag{5.36}$$

Since $f \in RV_{\rho+1}$, we find $L_f(z) := \frac{f(z)}{z^{\rho+1}}$ is a slowly varying function.

We claim that

$$\lim_{d(x)\searrow\sigma} \frac{L_f(v_\sigma^+(x))}{L_f(\Psi(d(x)-\sigma))} = 1 \quad \text{and} \quad \lim_{d(x)+\sigma\to 0} \frac{L_f(v_\sigma^-(x))}{L_f(\Psi(d(x)+\sigma))} = 1. \tag{5.37}$$

Using (5.24), together with $\varepsilon \in (0, \min_{\overline{\Omega}} c/4)$, we find

$$\xi_0(\varepsilon) := (\max_{\overline{\Omega}} c - \varepsilon)^{-\frac{1}{\rho}} < (\widetilde{c}(x)-2\varepsilon)^{-\frac{1}{\rho}} < \varepsilon^{-\frac{1}{\rho}} := \xi_1(\varepsilon), \quad \forall x \in \overline{\Omega}.$$

Since $\lim_{t\to 0^+} \Psi(t) = \infty$ and, by Proposition 4.1.1, $\lim_{u\to\infty} \frac{L_f(m\xi u)}{L_f(u)} = 1$, uniformly with respect to $\xi \in [\xi_0(\varepsilon), \xi_1(\varepsilon)] \subset (0,\infty)$, it follows that

$$\lim_{d(x)\searrow\sigma} \frac{L_f(v_\sigma^+(x))}{L_f(\Psi(d(x)-\sigma))} = \lim_{d(x)\searrow\sigma} \frac{L_f(m\Psi(d(x)-\sigma)(\widetilde{c}(x)-2\varepsilon)^{-\frac{1}{\rho}})}{L_f(\Psi(d(x)-\sigma))} = 1.$$

A similar argument can be used for the remaining limit in (5.37).

Since

$$\begin{aligned} f(v_\sigma^\pm(x)) &= [v_\sigma^\pm(x)]^{\rho+1} L_f(v_\sigma^\pm(x)) \\ &= m^{\rho+1}(\widetilde{c}(x) \mp 2\varepsilon)^{-\frac{\rho+1}{\rho}} [\Psi(d(x) \mp \sigma)]^{\rho+1} L_f(v_\sigma^\pm(x)), \end{aligned}$$

using (5.37) and Proposition 5.2.6 (vii), we arrive at

$$\begin{cases} \lim\limits_{d(x)\searrow\sigma} \dfrac{k^2(d(x)-\sigma) f(v_\sigma^+(x))[\widetilde{c}(x)-2\varepsilon]^{\frac{1}{\rho}+1}}{\Psi''(d(x)-\sigma)} = m; \\ \lim\limits_{d(x)+\sigma\to 0} \dfrac{k^2(d(x)+\sigma) f(v_\sigma^-(x))[\widetilde{c}(x)+2\varepsilon]^{\frac{1}{\rho}+1}}{\Psi''(d(x)+\sigma)} = m. \end{cases} \tag{5.38}$$

The inequalities in (5.29) now follow from (5.34)–(5.36) and (5.38).

#### 5.2.3.4 Asymptotic Behavior of the Large Solution

Let $\zeta > 0$ be small such that $a$ is less than the first eigenvalue of $(-\Delta)$ in the domain $E_\zeta := \{x \in \mathbb{R}^N \setminus \overline{\Omega} :\ d(x) < \zeta\}$. Set $I_\delta = \{x \in \Omega :\ d(x) < \delta\}$.

Define $\Omega_1 = E_{2\zeta} \cup \{x \in \overline{\Omega} :\ d(x) < \delta\}$, where $\delta > 0$ is as in (5.29).

Let $p \in C^{0,\mu}(\overline{\Omega}_1)$ be such that $0 < p(x) \le b(x)$ for $x \in \Omega$ with $d(x) \le \delta$, $p = 0$ in $\overline{E}_\zeta$ and $p > 0$ in $\overline{E}_{2\zeta} \setminus \overline{E}_\zeta$.

Denote by $w$ a large solution of $-\Delta u = au - p(x) f(u)$ in $\Omega_1$. The existence of $w$ is given by Theorem 3.1.1. Note that $w$ is uniformly bounded on $\Gamma_\infty^c$ and $w = \infty$ on $\partial I_\delta \cap \Omega$.

Let $u_a$ be an arbitrary large solution of (5.1) (resp., (5.2)). By (5.29) and (5.3), we find

$$\begin{cases} -\Delta(u_a + w) - a(u_a + w) + b(x) f(u_a + w) \ge 0 \quad \text{in } I_\delta \\ -\Delta v_\sigma^- - a v_\sigma^- + b(x) f(v_\sigma^-) \le 0 \quad \text{in } I_\delta \\ (u_a + w)|_{\partial I_\delta} = \infty > v_\sigma^-|_{\partial I_\delta} \end{cases}$$

and

$$\begin{cases} -\Delta(v_\sigma^+ + w) - a(v_\sigma^+ + w) + b(x) f(v_\sigma^+ + w) \ge 0 \quad \text{in } I_\delta \setminus \overline{I}_\sigma \\ -\Delta u_a - a u_a + b(x) f(u_a) = 0 \quad \text{in } I_\delta \setminus \overline{I}_\sigma \\ (v_\sigma^+ + w)|_{\partial(I_\delta \setminus \overline{I}_\sigma)} = \infty > u_a|_{\partial(I_\delta \setminus \overline{I}_\sigma)}. \end{cases}$$

By Proposition 5.2.7, we get

$$\begin{cases} u_a + w \geq v_\sigma^- & \text{in } I_\delta, \\ v_\sigma^+ + w \geq u_a & \text{in } I_\delta \setminus \overline{I}_\sigma. \end{cases}$$

Letting $\sigma \to 0$, we arrive at

$$\begin{cases} u_a + w \geq m\Psi(d(x))[\widetilde{c}(x) + 2\varepsilon]^{-\frac{1}{\rho}}, & \forall x \in \Omega \text{ with } 0 < d(x) < \delta \\ m\Psi(d(x))[\widetilde{c}(x) - 2\varepsilon]^{-\frac{1}{\rho}} + w \geq u_a, & \forall x \in \Omega \text{ with } 0 < d(x) < \delta. \end{cases}$$

Since $|\widetilde{c}(x) - c(x)| \leq \varepsilon$, we deduce

$$m\Psi(d(x))[c(x) + 3\varepsilon]^{-\frac{1}{\rho}} - w(x) \leq u_a(x) \leq m\Psi(d(x))[c(x) - 3\varepsilon]^{-\frac{1}{\rho}} + w(x)$$

for all $x \in \Omega$ with $0 < d(x) < \delta$. It follows that

$$m[c(y) + 3\varepsilon]^{-\frac{1}{\rho}} \leq \liminf_{x \to y \in \Gamma_\infty^c} \frac{u_a(x)}{\Psi(d(x))} \leq \limsup_{x \to y \in \Gamma_\infty^c} \frac{u_a(x)}{\Psi(d(x))} \leq m[c(y) - 3\varepsilon]^{-\frac{1}{\rho}}$$

uniformly for $y \in \Gamma_\infty^c$. Recall that $\varepsilon \in (0, \min_{\overline{\Omega}} c/4)$ is arbitrary. Hence, passing to the limit $\varepsilon \to 0$, we find

$$\lim_{x \to y \in \Gamma_\infty^c} \frac{u_a(x)}{\Psi(d(x))} = m[c(y)]^{-\frac{1}{\rho}} \quad \text{uniformly for } y \in \Gamma_\infty^c,$$

which proves (5.7).

#### 5.2.3.5 Uniqueness of the Large Solution

Let $u_1$ and $u_2$ be two large solutions of (5.1) (resp., (5.2)). By the asymptotic behavior (5.7) we deduce $\lim_{d(x,\Gamma_\infty) \to 0} \frac{u_1(x)}{u_2(x)} = 1$. The uniqueness conclusion follows from this and Proposition 5.2.7 using a standard argument.

Indeed, for $\theta > 0$ arbitrary, set $(1+\theta)u_i = w_i$, $i = 1, 2$. We obtain

$$\begin{cases} -\Delta w_i - a w_i + b(x) f(w_i) \geq 0 & \text{in } \Omega \\ \lim_{d(x,\Gamma_\infty) \to 0} (u_1 - w_2)(x) = \lim_{d(x,\Gamma_\infty) \to 0} (u_2 - w_1)(x) = -\infty. \end{cases}$$

If $\Gamma_\infty \neq \partial\Omega$, then $\mathcal{B} w_i = 0$ on $\Gamma_{\mathcal{B}}$. Therefore, by Proposition 5.2.7, we infer that

$$u_1 \leq (1+\theta)u_2 \;\text{ in } \Omega \quad \text{and} \quad u_2 \leq (1+\theta)u_1 \;\text{ in } \Omega.$$

Passing to the limit $\theta \to 0$, we get $u_1 = u_2$ on $\Omega$. This completes the proof of Theorem 5.2.2. □

### 5.2.4 Proof of Theorem 5.2.3

Let $\varepsilon > 0$ be fixed. We denote by $\Gamma_\infty^*$ the unique connected open and closed subset of $\Gamma_\infty$ that contains $x_*$. By (5.12) we can assume that

$$0 < b(x) \leq (c_* + \varepsilon) k^2(d(x, \Gamma_\infty^*)), \quad \forall x \in B_{r_0}(x_*) \cap \Omega$$

for some $r_0 > 0$ small enough such that

$$B_{r_0}(x_*) := \{x \in \mathbb{R}^N : \ |x - x_*| < r_0\} \subset \mathbb{B} \quad \text{and} \quad B_{r_0}(x_*) \cap \partial\Omega \subseteq \Gamma_\infty^*. \tag{5.39}$$

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two smooth domains such that $\mathcal{O}_1 \subset\subset \mathcal{O}_2 \subset\subset B_{r_0}(x_*)$ and $\overline{\mathcal{O}}_1 \cap \overline{\Omega} = I_* \subset \Gamma_\infty^*$ with $x_*$ belonging to the interior of $I_*$, namely, $\mathcal{O}_1$ is outside $\Omega$ but its boundary and $\partial\Omega$ has a common part $I_*$ which contains $x_*$ in its interior. Set $D_* = \mathcal{O}_2 \setminus \overline{\mathcal{O}}_1$.

By Lemma 3.3.4 and Theorem 5.2.2, the boundary value problem

$$\begin{cases} -\Delta u = au - (c_* + \varepsilon) k^2(d(x, \partial\mathcal{O}_1)) f(u) & \text{in } D_*, \\ u = 0 & \text{on } \partial\mathcal{O}_2, \end{cases} \tag{5.40}$$

subject to $u = n \geq 1$ (resp., $u = \infty$) on $\partial\mathcal{O}_1$ has a *unique* positive solution $w_n$ (resp., $W$). Moreover, we have

$$\lim_{d(x,\partial\mathcal{O}_1) \to 0} \frac{W(x)}{\Psi(d(x, \partial\mathcal{O}_1))} = \left( \frac{2 + \rho \ell_1}{2(c_* + \varepsilon)} \right)^{\frac{1}{\rho}}. \tag{5.41}$$

Notice that $d(x, \partial\mathcal{O}_1) \geq d(x, \Gamma_\infty^*)$, for each $x \in D_* \cap \Omega$. Hence, for all $n \geq 1$,

$$\begin{cases} -\Delta w_n \leq a\, w_n - b(x) f(w_n) & \text{in } D_* \cap \Omega, \\ w_n = 0 \ \ \text{on } \partial\mathcal{O}_2, \\ U|_{\overline{D}_* \cap \Gamma_\infty^*} = \infty > w_n|_{\overline{D}_* \cap \Gamma_\infty^*}. \end{cases}$$

Using Proposition 5.2.7, we infer that

$$\begin{cases} w_n \leq w_{n+1} \leq W & \text{in } D_*, \quad \forall n \geq 1, \\ w_n \leq U & \text{in } D_* \cap \Omega, \quad \forall n \geq 1. \end{cases} \tag{5.42}$$

Here $U$ denotes an arbitrary positive solution of (5.11). Standard regularity arguments show that $w_\infty$ defined by $w_\infty(x) := \lim_{n \to \infty} w_n(x)$, $\forall x \in D_*$, is a positive solution of (5.40) satisfying $w_\infty = \infty$ on $\partial\mathcal{O}_1$. It follows that $w_\infty \equiv W$ in $D_*$. By (5.42), we obtain $U \geq W$ in $D_* \cap \Omega$. Hence,

$$\frac{U(x)}{\Psi(d(x,\Gamma_\infty^*))} \geq \frac{W(x)}{\Psi(d(x,\Gamma_\infty^*))}, \quad \forall x \in D_* \cap \Omega.$$

By our choice of $\mathcal{O}_1$, $d(x,\partial\mathcal{O}_1) = d(x,\Gamma_\infty^*)$ if $x \in \Omega$ is sufficiently close to $x_*$. Hence, letting $x \to x_*$ and using (5.41), we find

$$\liminf_{x\to x_*,\, x\in\Omega} \frac{U(x)}{\Psi(d(x,\Gamma_\infty))} \geq \left(\frac{2+\rho\ell_1}{2(c_*+\varepsilon)}\right)^{\frac{1}{\rho}}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude (5.13) by letting $\epsilon \to 0$. □

### 5.2.5 Proof of Theorem 5.2.4

Let $\varepsilon \in (0, d_*)$ be arbitrary and $\Gamma_\infty^*$ be the same as in the proof of Theorem 5.2.3. By (5.14), we can choose $r_0 > 0$ small such that (5.39) holds and

$$b(x) \geq (d_* - \varepsilon)k^2(d(x,\Gamma_\infty^*)), \quad \forall x \in B_{r_0}(x_*) \cap \Omega.$$

Let $V$ be a smooth domain such that $\overline{V} \subset \overline{\Omega} \cap B_{r_0}(x_*)$ and $I_* := \partial V \cap \Gamma_\infty^*$ contains $x_*$ in its interior.

For $n \geq 1$, set $V_n = \{x \in V:\ 1/n < d(x,\partial V)\}$. Obviously, $V = \cup_{n=1}^\infty V_n$.

Let $Z$ be the *unique* positive solution (see Theorem 5.2.2) of

$$-\Delta u = au - (d_* - \varepsilon)k^2(d(x,\partial V))f(u) \quad \text{in } V \tag{5.43}$$

subject to $u = \infty$ on $\partial V$. Let $Z_n$ be the unique positive solution of the above problem with $V$ replaced by $V_n$. By Theorem 5.2.2, we have

$$\lim_{d(x,\partial V)\to 0} \frac{Z(x)}{\Psi(d(x,\partial V))} = \left(\frac{2+\rho\ell_1}{2(d_*-\varepsilon)}\right)^{\frac{1}{\rho}}. \tag{5.44}$$

Clearly $d(x,\partial V_n) \leq d(x,\partial V_{n+1}) \leq d(x,\Gamma_\infty^*)$ for each $x \in V_n$, and

$$\begin{cases} -\Delta Z_n \geq aZ_n - b(x)f(Z_n) & \text{in } V_n, \quad \forall n \geq 1, \\ Z_n|_{\partial V_n} = \infty > U|_{\partial V_n}, & \qquad\quad\ \ \forall n \geq 1. \end{cases}$$

By Proposition 5.2.7, we deduce

$$\begin{cases} Z_n \geq Z_{n+1} \quad \text{in } V_n, \quad \forall n \geq 1, \\ Z_n \geq U \quad \text{in } V_n, \qquad \forall n \geq 1, \end{cases} \tag{5.45}$$

where $U$ is an arbitrary positive solution of (5.11). For each $x \in V$, there exists an integer $m(x) \geq 1$ such that $x \in V_n$, for each $n \geq m(x)$. By virtue of (5.45), $Z_\infty(x) = \lim_{n\to\infty} Z_n(x)$ is well defined. Standard regularity arguments imply that $Z_\infty$ is a positive solution of (5.43) in $V$ satisfying $Z_\infty = \infty$ on $\partial V$. Since there is only one such solution we conclude that $Z_\infty \equiv Z$ in $V$. It follows that

$$\frac{U(x)}{\Psi(d(x,\Gamma_\infty^*))} \leq \frac{Z(x)}{\Psi(d(x,\Gamma_\infty^*))}, \quad \forall x \in V.$$

Since $d(x,\partial V) = d(x,\Gamma_\infty^*)$ if $x \in V$ is close to $x_*$, letting $x \to x_*$ and using (5.44), we obtain

$$\limsup_{x\to x_*,\, x\in\Omega} \frac{U(x)}{\Psi(d(x,\Gamma_\infty))} \leq \left(\frac{2+\rho\ell_1}{2(d_*-\varepsilon)}\right)^{\frac{1}{\rho}}.$$

Passing to the limit $\varepsilon \to 0$, we arrive at (5.15). This finishes the proof. □

### 5.2.6 Proof of Theorem 5.2.1

Suppose by contradiction that (5.1) (resp., (5.2)) has two distinct large solutions $U_1$ and $U_2$. We observe that there exist some constants $0 < \gamma_1 < \gamma_2$ and $\delta_+ > 0$ such that, for each connected open and closed subset $\Gamma_\infty^c$ of $\Gamma_\infty$,

$$\gamma_1 \Psi(d(x)) \leq U_1(x), U_2(x) \leq \gamma_2 \Psi(d(x)), \quad \forall x \in \Omega \text{ with } d(x) < \delta_+, \tag{5.46}$$

where $d(x) := d(x,\Gamma_\infty^c)$. Indeed, by (5.5), there exist some constants $0 < \beta_1 < \beta_2$ such that, for any $x_* \in \Gamma_\infty^c$,

$$\limsup_{x\to x_*,\, x\in\Omega} \frac{b(x)}{k^2(d(x))} \leq \beta_2, \quad \liminf_{x\to x_*,\, x\in\Omega} \frac{b(x)}{k^2(d(x))} \geq \beta_1.$$

Applying Theorems 5.2.3 and 5.2.4, we see that $U_i$ $(i = 1, 2)$ satisfies

$$\limsup_{x\to x_*, x\in\Omega} \frac{U_i(x)}{\Psi(d(x))} \leq \left(\frac{2+\rho\ell_1}{2\beta_1}\right)^{\frac{1}{\rho}}, \quad \liminf_{x\to x_*, x\in\Omega} \frac{U_i(x)}{\Psi(d(x))} \geq \left(\frac{2+\rho\ell_1}{2\beta_2}\right)^{\frac{1}{\rho}}.$$

Since $\Gamma_\infty$ is compact, we deduce (5.46). By §5.2.3.5 in the proof of Theorem 5.2.2, we must have $\limsup_{d(x)\to 0}(U_2/U_1)(x) > 1$ or $\liminf_{d(x)\to 0}(U_2/U_1)(x) < 1$.

We can assume, without loss of generality, that $\limsup_{d(x)\to 0}(U_2/U_1)(x) > 1$. Hence, there exist a constant $\varepsilon_0 > 0$ and a sequence $(x_n)_{n\geq 1} \subset \Omega$ such that

$$d(x_n) \to 0 \quad \text{as } n \to \infty \quad \text{and} \quad U_2(x_n) > (1+\varepsilon_0)U_1(x_n), \qquad \forall n \geq 1. \tag{5.47}$$

A key step in our proof is the following result, which is the key to Safonov's iteration technique.

**Proposition 5.2.8.** *There exist some constants $\theta_\# > 1$ and $\delta_\# \in (0, \delta_+/2)$ such that for any $\widetilde{x} \in \Omega$ satisfying*

$$d(\widetilde{x}) \leq \delta_\# \quad \textit{and} \quad U_2(\widetilde{x}) > (1+\widetilde{\varepsilon})U_1(\widetilde{x}) \quad \textit{with } \widetilde{\varepsilon} \geq \varepsilon_0 \tag{5.48}$$

*we can find $\widetilde{y} \in \Omega$ with the property*

$$|\widetilde{x} - \widetilde{y}| < d(\widetilde{x}) \quad \textit{and} \quad U_2(\widetilde{y}) > \theta_\#(1+\widetilde{\varepsilon})U_1(\widetilde{y}).$$

Suppose, for the moment, that the above result is proved. By (5.46), we have

$$\frac{U_2(x)}{U_1(x)} \leq \frac{\gamma_2}{\gamma_1}, \quad \text{for any } x \in \Omega \ \text{ with } d(x, \Gamma_\infty^c) \leq \delta_\#. \tag{5.49}$$

Since $\theta_\# > 1$, we can choose a large integer $m \geq 2$ such that $\theta_\#^m(1+\varepsilon_0) > \gamma_2/\gamma_1$. By (5.47), we can select $n \geq 1$ such that $2^m d(x_n) < \delta_\#$. By applying Proposition 5.2.8 with $\widetilde{x} = x_n$ and $\widetilde{\varepsilon} = \varepsilon_0$, we obtain $\widetilde{y} = z_1$ satisfying

$$|z_1 - x_n| < d(x_n) \quad \text{and} \quad U_2(z_1) > \theta_\#(1+\varepsilon_0)U_1(z_1). \tag{5.50}$$

We see that $d(z_1) \leq 2d(x_n) \leq 2^m d(x_n) < \delta_\#$. So, by (5.50) we can invoke again Proposition 5.2.8 with $\widetilde{x} = z_1$ and $1+\widetilde{\varepsilon} = \theta_\#(1+\varepsilon_0)$. Thus, we get $z_2 \in \Omega$ such that

$$|z_2 - z_1| < d(z_1) \quad \text{and} \quad U_2(z_2) > \theta_\#^2(1+\varepsilon_0)U_1(z_2).$$

Clearly, $d(z_2) \leq 2d(z_1) \leq 2^2 d(x_n) < \delta_\#$. We can reiterate this process until we obtain $z_m \in \Omega$ that fulfills

$$|z_m - z_{m-1}| < d(z_{m-1}) \quad \text{and} \quad U_2(z_m) > \theta_\#^m(1+\varepsilon_0)U_1(z_m).$$

Hence, we found $z_m \in \Omega$ such that

$$d(z_m) \leq 2^m d(x_n) < \delta_\# \quad \text{and} \quad (U_2/U_1)(z_m) > \gamma_2/\gamma_1.$$

This contradiction with (5.49) proves the assertion of Theorem 5.2.1.

*Proof of Proposition* 5.2.8. We define $\theta_\#$ as follows

$$\theta_\# = \begin{cases} 1 + \dfrac{\varepsilon_0 \beta_1}{16 N \rho \ell_1^2 \gamma_2} \left(\dfrac{\gamma_1}{3}\right)^{\rho+1} (3^{\rho \ell_1/2} - 1)^2 & \text{if } \ell_1 \neq 0, \\ 1 + \dfrac{\rho \varepsilon_0 \beta_1}{32 N \gamma_2} \left(\dfrac{\gamma_1}{3}\right)^{\rho+1} & \text{if } \ell_1 = 0. \end{cases} \tag{5.51}$$

Recall that $\Psi$ is decreasing on $(0,\tau)$ for $\tau > 0$ small and $\lim_{t\to 0^+} \Psi(t) = \infty$. Hence, $\Psi^{-1} : (\Psi(\tau), \infty) \to (0, \tau)$ is decreasing, too. For $t > 0$ near the origin, set

$$\mathcal{M}(t) = \Psi^{-1}\Big(\frac{1}{3t}\Big) - \Psi^{-1}\Big(\frac{1}{t}\Big). \tag{5.52}$$

*Claim* 5.2.1. There exists $\tau_\# > 0$ small such that, for any $t \in (0, \tau_\#)$, we have

$$t\, k^2(\Psi^{-1}(1/t)) f\left(\frac{\gamma_1}{3t}\right) [\mathcal{M}(t)]^2 \geq \frac{32 N \gamma_2}{\rho \varepsilon_0 \beta_1} (\theta_\# - 1). \tag{5.53}$$

By (5.51), we have

$$\frac{32 N \gamma_2}{\rho \varepsilon_0 \beta_1} (\theta_\# - 1) = \begin{cases} \left(\dfrac{\gamma_1}{3}\right)^{\rho+1} \dfrac{2}{\ell_1^2 \rho^2} (3^{\rho \ell_1/2} - 1)^2 & \text{if } \ell_1 \neq 0, \\ \left(\dfrac{\gamma_1}{3}\right)^{\rho+1}, & \text{if } \ell_1 = 0. \end{cases}$$

We now divide the proof of (5.53) into two cases:

*Case* 5.2.1. $k \in \mathcal{K}$ with $\ell_1 \neq 0$.

By Proposition 5.2.6 (v), we have

$$\lim_{t \to 0^+} t\, k^2(\Psi^{-1}(1/t))\, f(1/t) [\Psi^{-1}(1/t)]^2 = \frac{4}{\rho^2 \ell_1^2}. \tag{5.54}$$

By Proposition 5.2.6 (i), $\Psi_*(u) := \Psi(1/u)$ belongs to $RV_{2/(\rho \ell_1)}$. By Proposition 4.1.6, we infer that $\Psi_*^{-1} \in RV_{\rho \ell_1/2}$. Since $\Psi_*^{-1}(u) = 1/\Psi^{-1}(u)$, we deduce

$$\lim_{t \to 0^+} \frac{\Psi^{-1}(1/(3t))}{\Psi^{-1}(1/t)} = 3^{\rho \ell_1/2}. \tag{5.55}$$

By (5.52), (5.54) and (5.55), we arrive at

$$\lim_{t\to 0^+} t\,k^2(\Psi^{-1}(1/t))\, f\left(\frac{\gamma_1}{3t}\right)[\mathcal{M}(t)]^2 = \left(\frac{\gamma_1}{3}\right)^{\rho+1} \frac{4}{\ell_1^2\rho^2}(3^{\rho\ell_1/2}-1)^2,$$

which proves (5.53).

*Case* 5.2.2. $k \in \mathcal{K}$ with $\ell_1 = 0$.

By Proposition 5.2.6 (i), $\Psi_*(u) := \Psi(1/u)$ is $\Gamma$-varying at $u = \infty$ with auxiliary function $\chi(u) = \rho u \mathcal{W}(1/u)/2$. Applying Proposition 4.1.8, we find $\Psi_*^{-1}$ is $\Pi$-varying with auxiliary function $\chi(\Psi_*^{-1}(u))$. Since $\Psi_*^{-1}(u) = 1/\Psi^{-1}(u)$, by Definition 4.1.3 we have

$$\lim_{u\to\infty} \frac{\Psi^{-1}(u) - \Psi^{-1}(\lambda u)}{\frac{\rho}{2}\Psi^{-1}(\lambda u)\mathcal{W}(\Psi^{-1}(u))} = \log\lambda, \quad \forall \lambda > 0. \tag{5.56}$$

Hence by the definition of $\mathcal{W}$ and Proposition 5.2.6 (iv), we obtain

$$\begin{aligned} &t\,k^2(\Psi^{-1}(1/t))\, f(1/t)\left[\mathcal{W}(\Psi^{-1}(1/t))\Psi^{-1}(1/t)\right]^2 \\ &\quad = t\, f(1/t)\left[\int_0^{\Psi^{-1}(1/t)} k(s)\,ds\right]^2 \to \frac{4}{\rho^2} \quad \text{as } t\to 0^+. \end{aligned} \tag{5.57}$$

Combining (5.56) and (5.57), we deduce

$$\lim_{t\to 0^+} t\,k^2(\Psi^{-1}(1/t))\, f\left(\frac{\gamma_1}{3t}\right)[\mathcal{M}(t)]^2\left[\frac{\Psi^{-1}(1/t)}{\Psi^{-1}(1/(3t))}\right]^2 = \left(\frac{\gamma_1}{3}\right)^{\rho+1}(\log 3)^2.$$

Since $\Psi^{-1}(1/t) \le \Psi^{-1}(1/(3t))$, we conclude the proof of (5.53).

From (5.46) and Proposition 4.1.1, it follows that

$$\lim_{d(x)\to 0} \frac{L_f(U_2(x))}{L_f(U_1(x))} = 1, \quad \text{where } L_f(u) := \frac{f(u)}{u^{\rho+1}} \text{ for } u > 0. \tag{5.58}$$

Using Proposition 5.2.6 (v), we have

$$\lim_{t\to 0^+} \frac{\Psi(t)}{k^2(t) f(\gamma_1 \Psi(t))} = \frac{1}{\gamma_1^{\rho+1}} \lim_{t\to 0^+} \frac{\Psi(t)}{k^2(t) f(\Psi(t))} = 0. \tag{5.59}$$

In view of (5.58) and (5.59), we can choose $\delta_\# \in (0, \mathcal{M}(\tau_\#)/2)$ small such that

$$\begin{cases} \dfrac{L_f(U_2(x))}{L_f(U_1(x))} \ge (1+\varepsilon_0)^{-\rho/2}, & \forall x \in \Omega \text{ with } d(x) \le 2\delta_\# \\ \dfrac{|a|\gamma_2}{(1+\varepsilon_0)}\, \dfrac{\Psi(t)}{k^2(t) f(\gamma_1\Psi(t))} \le \dfrac{\rho\varepsilon_0\beta_1}{4}, & \forall t \in (0, 2\delta_\#). \end{cases} \tag{5.60}$$

Let $\widetilde{x} \in \Omega$ satisfy (5.48). Since $d(\widetilde{x}) \le \delta_\# < \mathcal{M}(\tau_\#)/2$, the equation

$$\Psi^{-1}\left(\frac{1}{t}\right) + \Psi^{-1}\left(\frac{1}{3t}\right) = 2d(\widetilde{x})$$

has a unique solution $r \in (0, \tau_\#)$. We define

$$S_0 := \{x \in \Omega : \; U_2(x) > (1+\widetilde{\varepsilon})U_1(x)\} \cap B_{\widetilde{r}}(\widetilde{x}),$$

where

$$\widetilde{r} = \frac{\mathcal{M}(r)}{2} = \frac{1}{2}\left[\Psi^{-1}\left(\frac{1}{3r}\right) - \Psi^{-1}\left(\frac{1}{r}\right)\right].$$

For each $x \in B_{\widetilde{r}}(\widetilde{x})$, we have

$$\Psi^{-1}\left(\frac{1}{r}\right) \le d(x) \le \Psi^{-1}\left(\frac{1}{3r}\right) < \Psi^{-1}\left(\frac{1}{r}\right) + \Psi^{-1}\left(\frac{1}{3r}\right) \le 2\delta_\# < \delta_+. \quad (5.61)$$

Using (5.60), we obtain

$$\begin{aligned} \frac{f(U_2(x))}{(1+\widetilde{\varepsilon})f(U_1(x))} &= \frac{U_2^{\rho+1}L_f(U_2(x))}{(1+\widetilde{\varepsilon})U_1^{\rho+1}L_f(U_1(x))} \ge (1+\widetilde{\varepsilon})^\rho \frac{L_f(U_2(x))}{L_f(U_1(x))} \\ &\ge (1+\varepsilon_0)^{\frac{\rho}{2}} \ge 1 + \frac{\rho\varepsilon_0}{2}, \quad \forall x \in S_0. \end{aligned} \quad (5.62)$$

Using (5.46) and (5.61), we infer that, for any $x \in B_{\widetilde{r}}(\widetilde{x})$,

$$\frac{\gamma_1}{3r} \le \gamma_1\Psi(d(x)) \le U_i(x) \le \gamma_2\Psi(d(x)) \le \frac{\gamma_2}{r}, \quad i = 1, 2. \quad (5.63)$$

By (5.5), (5.60), (5.62) and (5.63) we find

$$\begin{aligned} &\Delta(U_2 - (1+\widetilde{\varepsilon})U_1) \\ &\quad = -a(U_2 - (1+\widetilde{\varepsilon})U_1) + b(x)[f(U_2) - (1+\widetilde{\varepsilon})f(U_1)] \\ &\quad \ge -|a|U_2 + (1+\widetilde{\varepsilon})\beta_1 k^2(d(x))f(U_1)\left[\frac{f(U_2)}{(1+\widetilde{\varepsilon})f(U_1)} - 1\right] \\ &\quad \ge -|a|\gamma_2\Psi(d(x)) + (1+\widetilde{\varepsilon})\beta_1 k^2(d(x))f(\gamma_1\Psi(d(x)))\left[\frac{f(U_2)}{(1+\widetilde{\varepsilon})f(U_1)} - 1\right] \\ &\quad = (1+\widetilde{\varepsilon})k^2(d(x))f(\gamma_1\Psi(d(x)))\left\{\beta_1\left[\frac{f(U_2)}{(1+\widetilde{\varepsilon})f(U_1)} - 1\right]\right. \\ &\qquad\qquad \left. - \frac{|a|\gamma_2}{(1+\widetilde{\varepsilon})}\frac{\Psi(d(x))}{k^2(d(x))f(\gamma_1\Psi(d(x)))}\right\} \\ &\quad \ge (1+\widetilde{\varepsilon})k^2(\Psi^{-1}(1/r))f\left(\frac{\gamma_1}{3r}\right)\left[\frac{\rho\varepsilon_0\beta_1}{2} - \frac{|a|\gamma_2}{(1+\varepsilon_0)}\frac{\Psi(d(x))}{k^2(d(x))f(\gamma_1\Psi(d(x)))}\right] \\ &\quad \ge (1+\widetilde{\varepsilon})\frac{\rho\varepsilon_0\beta_1}{4}k^2(\Psi^{-1}(1/r))f\left(\frac{\gamma_1}{3r}\right), \qquad \forall x \in S_0. \end{aligned}$$

For any $x \in S_0$, we define

$$w(x) = (2N)^{-1}(1+\tilde{\varepsilon})\frac{\rho\varepsilon_0\beta_1}{4}\, k^2(\Psi^{-1}(1/r))f\left(\frac{\gamma_1}{3r}\right)(\tilde{r}^2 - |x-\tilde{x}|^2).$$

Obviously, we have

$$\Delta w = -(1+\tilde{\varepsilon})\frac{\rho\varepsilon_0\beta_1}{4}\, k^2(\Psi^{-1}(1/r))f(\gamma_1/(3r)) \quad \text{in } S_0.$$

Thus, we get

$$\Delta(U_2 - (1+\tilde{\varepsilon})U_1 + w) \geq 0 \quad \text{in } S_0.$$

Applying the maximum principle for sub-harmonic functions, we have

$$U_2(\tilde{x}) - (1+\tilde{\varepsilon})U_1(\tilde{x}) + w(\tilde{x}) \leq \max_{\partial S_0}(U_2 - (1+\tilde{\varepsilon})U_1 + w).$$

Note that $\max_{\partial S_0}(U_2-(1+\tilde{\varepsilon})U_1+w)$ cannot be achieved on $\partial S_0 \cap B_{\tilde{r}}(\tilde{x})$. Indeed, we see that

$$U_2(y) = (1+\tilde{\varepsilon})U_1(y), \quad \text{for each } y \in \partial S_0 \cap B_{\tilde{r}}(\tilde{x}).$$

This yields that

$$\begin{aligned} U_2(y) - (1+\tilde{\varepsilon})U_1(y) + w(y) = w(y) &\leq w(\tilde{x}) \\ &< U_2(\tilde{x}) - (1+\tilde{\varepsilon})U_1(\tilde{x}) + w(\tilde{x}) \end{aligned}$$

for each $y \in \partial S_0 \cap B_{\tilde{r}}(\tilde{x})$. Therefore, we have

$$\max_{\partial S_0}(U_2 - (1+\tilde{\varepsilon})U_1 + w)$$

is reached at some point $\tilde{y} \in \partial S_0 \cap \partial B_{\tilde{r}}(\tilde{x})$. It follows that

$$\begin{aligned} U_2(\tilde{y}) - (1+\tilde{\varepsilon})U_1(\tilde{y}) &= U_2(\tilde{y}) - (1+\tilde{\varepsilon})U_1(\tilde{y}) + w(\tilde{y}) \\ &\geq U_2(\tilde{x}) - (1+\tilde{\varepsilon})U_1(\tilde{x}) + w(\tilde{x}) > w(\tilde{x}). \end{aligned} \tag{5.64}$$

Since $r < \tau_\#$, we use (5.61), (5.63) and (5.53) to obtain

$$\begin{aligned} w(\tilde{x}) &= \frac{1}{2N}(1+\tilde{\varepsilon})\frac{\rho\varepsilon_0\beta_1}{4}\,(k^2\circ\Psi^{-1})\left(\frac{1}{r}\right) f\left(\frac{\gamma_1}{3r}\right)\frac{[\mathcal{M}(r)]^2}{4} \\ &\geq \frac{(\theta_\# - 1)(1+\tilde{\varepsilon})\gamma_2}{r} = (\theta_\# - 1)(1+\tilde{\varepsilon})\gamma_2\Psi(\Psi^{-1}(1/r)) \\ &\geq (\theta_\# - 1)(1+\tilde{\varepsilon})\gamma_2\Psi(d(\tilde{y})) \\ &\geq (\theta_\# - 1)(1+\tilde{\varepsilon})U_1(\tilde{y}). \end{aligned} \tag{5.65}$$

By (5.64) and (5.65), we obtain $U_2(\tilde{y}) > \theta_\#(1+\tilde{\varepsilon})U_1(\tilde{y})$. Hence, $\tilde{y} \in \partial S_0 \cap \partial B_{\tilde{r}}(\tilde{x})$ has all the properties stated in Proposition 5.2.8. This concludes the proof of Proposition 5.2.8 and Theorem 5.2.1. □

## 5.3 Case II: Rapidly Varying Nonlinearities

### 5.3.1 Introduction and Main Results

Bieberbach (1916) initiated the topic of large solutions for $\Delta u = e^u$ in a smooth bounded domain $\Omega \subset \mathbb{R}^2$. He showed that there is a unique positive solution $u \in C^2(\Omega)$ such that $u(x) - \ln(d(x)^{-2})$ is bounded as $d(x) = \operatorname{dist}(x, \partial\Omega) \to 0$. Problems of this type arise in Riemannian geometry; if a Riemannian metric of the form $|ds|^2 = e^{2u(x)}|dx|^2$ has constant Gaussian curvature $-c^2$, then $\Delta u = c^2 e^{2u}$. Rademacher (1943) extended the result of Bieberbach on smooth bounded domains in $\mathbb{R}^3$.

The goal of section 5.3 is to give the uniqueness and asymptotic behavior of large solutions to (5.1) (resp., (5.2)) in the setting of §5.1 for a class of functions $f$ rapidly varying (at infinity) with index $\infty$, that is

$$\lim_{u\to\infty} \frac{f(\lambda u)}{f(u)} = \begin{cases} \infty, & \text{if } \lambda > 1 \\ 1, & \text{if } \lambda = 1 \\ 0, & \text{if } 0 < \lambda < 1. \end{cases}$$

We establish a subtle connection between the blow-up rate of the solution and the rapid variation of $f$ by using de Haan theory (see §5.3.2).

Among functions rapidly varying with index $\infty$ one can distinguish the proper subclass of $\Gamma$-varying functions (cf. Proposition 3.10.3 in Bingham et al. (1987)). For ease of reference, we restate here Definition 4.1.2 of Chapter 4.

**Definition 5.3.1.** A non-decreasing function $f$ is $\Gamma$-varying at $\infty$ (written $f \in \Gamma$) if $f$ is defined on some interval $(D, \infty)$ with $D > 0$, $\lim_{u\to\infty} f(u) = \infty$ and there exists $g : (D, \infty) \to (0, \infty)$ (called an *auxiliary function*) such that

$$\lim_{y\to\infty} \frac{f(y + \lambda g(y))}{f(y)} = e^{\lambda}, \ \forall \lambda \in \mathbb{R}.$$

The auxiliary function $g$ is unique up to asymptotic equivalence.

For a non-decreasing function $H$ defined on $\mathbb{R}$, we denote by $H^{\leftarrow}$ the (left continuous) inverse of $H$, that is

$$H^{\leftarrow}(y) = \inf\{s : \ H(s) \geq y\}.$$

Our first main result is the following (see Theorem 1.2 in Cîrstea (2004*b*)).

**Theorem 5.3.1.** *Let (5.3) hold and $f \in \Gamma$ with auxiliary function $g$. Assume that for each connected open and closed subset $\Gamma_\infty^c$ of $\Gamma_\infty$ there exists $k \in \mathcal{K}$ with $\ell_1 \neq 0$ such that (5.5) is fulfilled.*

*Then, for any $a < \lambda_{\infty,1}$, (5.1) (resp., (5.2)) has a unique large solution $u_a$, which satisfies*

$$\frac{u_a(x)}{\phi(d(x))} \to 1 \quad \textit{as } d(x) := \operatorname{dist}(x, \Gamma_\infty^c) \to 0, \tag{5.66}$$

*where $\phi$ is given by*

$$\phi(t) = \psi^{\leftarrow}(1/[tk(t)]^2) \qquad \textit{for } t > 0 \textit{ small}, \tag{5.67}$$

*and $\psi$ is defined on some interval $[\alpha, \infty) \subset (0, \infty)$ by*

$$\psi(u) = \sup\{f(y)/g(y) : \ \alpha \le y \le u\}, \quad \forall u \ge \alpha. \tag{5.68}$$

**Corollary 5.3.2.** *If $f(u) = e^{cu} - 1$ $(c > 0)$ in Theorem 5.3.1, then the unique large solution $u_a$ satisfies*

$$\lim_{d(x)\to 0} \frac{u_a(x)}{\ln d(x)} = \frac{-2}{c\,\ell_1}.$$

Indeed, $\lim_{t\to 0^+} \ln k(t)/\ln t = 1/\ell_1 - 1$ (cf. Proposition 4.2.8) and

$$\lim_{t\to 0^+} \frac{\phi(t)}{\ln t + \ln k(t)} = \frac{-2}{c}.$$

We point out that Theorem 5.3.1 does not concern the quotient of $u_a(x)$ and $\Upsilon(d(x))$, as established in Bandle and Marcus (1992*a*) (for $a = 0$ and $b = 1$), where $\Upsilon$ is a chosen solution of the singular problem

$$\begin{cases} u''(r) = f(u(r)) \ \text{ on } (0, \tau) \text{ for some } \tau > 0, \\ u(r) \to \infty \text{ as } r \to 0^+. \end{cases}$$

In contrast, the function $\phi$ in (5.66) does not have enough regularity to use it directly in constructing upper and lower solutions near $\Gamma_\infty^c$. The idea is to build smoother versions of $\phi$ which are asymptotically equivalent to $\phi$ at the origin. This will be achieved in Lemmas 5.3.11 and 5.3.12 via de Haan theory.

We note an extreme variation phenomenon given that the solution $u_a$ blows-up at $\Gamma_\infty$ in a slow fashion (cf. Remark 5.3.2), while $f$ varies rapidly at infinity.

The following results (see Cîrstea (2004*a*)) reiterate the fact that the blow-up rate is *local* in nature. We consider the positive solutions to the problem

$$\begin{cases} -\Delta u = au - b(x)f(u) & \text{in } \Omega \cap \mathbb{B}, \\ u = \infty & \text{on } \Gamma_\infty \cap \mathbb{B}, \end{cases} \tag{5.69}$$

where $\mathbb{B}$ denotes an open ball in $\mathbb{R}^N$ such that $\Gamma_\infty \cap \mathbb{B} \neq \emptyset$.

**Theorem 5.3.3.** *Let* (5.3) *hold and* $f \in \Gamma$ *with auxiliary function* $g$. *Assume that* $x_* \in \Gamma_\infty \cap \mathbb{B}$ *and there exists* $k \in \mathcal{K}$ *with* $\ell_1 \neq 0$ *such that*

$$\limsup_{x \to x_*,\, x \in \Omega} \frac{b(x)}{k^2(d(x, \Gamma_\infty))} < \infty, \tag{5.70}$$

*then, any positive solution* $U$ *of* (5.69) *satisfies*

$$\liminf_{x \to x_*,\, x \in \Omega} \frac{U(x)}{\phi(d(x, \Gamma_\infty))} \geq 1, \tag{5.71}$$

*where* $\phi$ *is given by* (5.67).

**Theorem 5.3.4.** *Let* (5.3) *hold and* $f \in \Gamma$ *with auxiliary function* $g$. *Suppose that* $x_* \in \Gamma_\infty \cap \mathbb{B}$ *and there exists* $k \in \mathcal{K}$ *with* $\ell_1 \neq 0$ *such that*

$$\liminf_{x \to x_*,\, x \in \Omega} \frac{b(x)}{k^2(d(x, \Gamma_\infty))} > 0, \tag{5.72}$$

*then, any positive solution* $U$ *of* (5.69) *satisfies*

$$\limsup_{x \to x_*,\, x \in \Omega} \frac{U(x)}{\phi(d(x, \Gamma_\infty))} \leq 1, \tag{5.73}$$

*where* $\phi$ *is given by* (5.67).

Combining Theorems 5.3.3 and 5.3.4, we obtain:

**Corollary 5.3.5.** *Let* (5.3) *hold and* $f \in \Gamma$ *with auxiliary function* $g$. *Suppose that* $x_* \in \Gamma_\infty \cap \mathbb{B}$ *and there exists* $k \in \mathcal{K}$ *with* $\ell_1 \neq 0$ *such that*

$$0 < \liminf_{x \to x_*,\, x \in \Omega} \frac{b(x)}{k^2(d(x, \Gamma_\infty))} \quad \text{and} \quad \limsup_{x \to x_*,\, x \in \Omega} \frac{b(x)}{k^2(d(x, \Gamma_\infty))} < \infty, \tag{5.74}$$

*then, any positive solution* $U$ *of* (5.69) *satisfies*

$$\lim_{x \to x_*,\, x \in \Omega} \frac{U(x)}{\phi(d(x, \Gamma_\infty))} = 1, \tag{5.75}$$

*with* $\phi$ *given by* (5.67).

*Remark* 5.3.1. The above local estimates differentiate from those of Theorems 5.2.3 and 5.2.4, where it is assumed that $f \in RV_{\rho+1}$ ($\rho > 0$) instead of $f \in \Gamma$.

The rest of section 5.3 is organized as follows. In §5.3.2 we gather some results which are extensions of regular variation. Based on these, we construct in §5.3.3 smoother versions of $\phi$ which are asymptotically equivalent to $\phi$ at the origin. The proofs of Theorems 5.3.1, 5.3.3 and 5.3.4, respectively are presented in §5.3.4–§5.3.6, respectively.

### 5.3.2 Preliminaries: Extensions of Regular Variation

We recall some concepts and results due to de Haan (1970, 1974, 1976) which appear in the extreme value theory (see Resnick (1987) or Bingham et al. (1987)). For convenience, we include here Definition 4.1.3 and Proposition 4.1.8 of Chapter 4.

**Definition 5.3.2 (p. 27 in Resnick (1987)).** A non-negative, non-decreasing function $V$ defined on $(z, \infty)$ is $\Pi$-varying (written $V \in \Pi$) if there exists a function $\theta(u) > 0$ (called an *auxiliary function*) such that

$$\lim_{u \to \infty} \frac{V(\lambda u) - V(u)}{\theta(u)} = \log \lambda, \quad \text{for } \lambda > 0.$$

The auxiliary function $\theta$ is unique up to asymptotic equivalence.

**Proposition 5.3.6 (Proposition 0.9 in Resnick (1987)).** *The following statements hold:*

1. *If $U \in \Gamma$ with auxiliary function $\chi$, then $U^{\leftarrow} \in \Pi$ with auxiliary function $\theta(u) = \chi \circ U^{\leftarrow}(u)$.*

2. *If $V \in \Pi$ with auxiliary function $\theta(u)$, then $V^{\leftarrow} \in \Gamma$ with auxiliary function $\chi(u) = \theta \circ V^{\leftarrow}(u)$.*

**Proposition 5.3.7 (Proposition 0.12 in Resnick (1987)).** *If $V \in \Pi$ with auxiliary function $\theta(u)$, then $\theta \in RV_0$.*

**Proposition 5.3.8 (Proposition 0.15 in Resnick (1987)).** *We have $V \in \Pi$ if and only if*

$$R(x) := \int_x^\infty u^{-1} V(du) = \int_x^\infty u^{-2} V(u)\, du - x^{-1} V(x)$$

*is finite and $-1$ varying. In this case the auxiliary function $\theta$ satisfies*

$$\lim_{u\to\infty} \frac{\theta(u)}{uR(u)} = 1$$

*and we have the representation*

$$V(x) - V(1) = \int_1^x R(u)\, du - xR(x) + K(1).$$

**Proposition 5.3.9 (de Haan (1970) or p. 35 in Resnick (1987)).** *If $V \in \Pi$ is a monotone function, then $V \in RV_0$ and $V(u)/\theta(u) \to \infty$, where $\theta(u)$ is the auxiliary function of $V$.*

**Definition 5.3.3 (p. 33 in Resnick (1987)).** If $V_1 \in \Pi$ with auxiliary function $\theta(u)$, we say $V_1$ and $V_2$ are $\Pi$-equivalent (written $V_1 \overset{\Pi}{\sim} V_2$) if

$$\lim_{u\to\infty} \frac{V_1(u) - V_2(u)}{\theta(u)} = c \in \mathbb{R}.$$

In this case $V_2 \in \Pi$ with auxiliary function $\theta(u)$.

The next result shows that if $V \in \Pi$, then we may construct smoother versions which are $\Pi$-equivalent to $V$.

**Proposition 5.3.10 (Proposition 0.16 in Resnick (1987)).** *If $V \in \Pi$ there exists a continuous, strictly increasing $V_1 \overset{\Pi}{\sim} V$ such that $V_1(u) > V(u)$ and*

$$\lim_{u\to\infty} \frac{V_1(u) - V(u)}{\theta(u)} = 1.$$

*In fact, there exists a twice differentiable $V_2 \overset{\Pi}{\sim} V$ with $V_2(u) > V(u)$ and*

$$-\frac{1}{uV_2''(u)} \in RV_1, \qquad \lim_{u\to\infty} \frac{-uV_2''(u)}{V_2'(u)} = 1.$$

### 5.3.3 Auxiliary Results

We first build a $C^1$-function which is asymptotically equivalent to $\phi$ at zero (see Lemma 2.2 in Cîrstea (2004b)). By $f_1(u) \sim f_2(u)$ as $u \to \infty$ we mean $\lim_{u\to\infty} f_1(u)/f_2(u) = 1$.

**Lemma 5.3.11.** *If $f \in \Gamma$ with auxiliary function $g$, then there exists a twice differentiable $V_2 \overset{\Pi}{\sim} f^{\leftarrow}$ with $V_2(u) > f^{\leftarrow}(u)$, $V_2' \in RV_{-1}$, $\lim_{u\to\infty} -uV_2''(u)/V_2'(u) = 1$, and $\lim_{u\to\infty} V_2(u)/f^{\leftarrow}(u) = 1$. Furthermore, if $f$ is continuous and increasing on $(D,\infty)$, then $\lim_{u\to\infty} f(V_2(u))/u = C(\text{Const.}) > 0$ and*

$$(V_2 \circ (1/V_2')^{\leftarrow})(u) \sim \psi^{\leftarrow}(u) \quad \text{as } u \to \infty, \tag{5.76}$$

*where $\psi$ is defined by (5.68).*

If $f \in \Gamma$ and $k \in \mathcal{K}$, set

$$\chi(t) = (1/V_2')^{\leftarrow}(1/[tk(t)]^2), \quad \text{for } t > 0 \text{ small},$$

where $V_2$ is from Lemma 5.3.11. Thus, under the assumptions of Theorem 5.3.1, $\chi(t)$ is a $C^1$-function such that $(V_2 \circ \chi)(t) \sim \phi(t)$ as $t \to 0^+$.

*Proof of Lemma 5.3.11.* By Propositions 5.3.6 and 5.3.7, $f^{\leftarrow} \in \Pi$ with auxiliary function $g \circ f^{\leftarrow} \in RV_0$. Thus, by Proposition 5.3.10, there exists a twice differentiable $V_2 \overset{\Pi}{\sim} f^{\leftarrow}$ with $V_2(u) > f^{\leftarrow}(u)$, $V_2' \in RV_{-1}$, $\lim_{u\to\infty} -uV_2''(u)/V_2'(u) = 1$.

Since $V_2 \in \Pi$ is increasing, by Proposition 5.3.9 we have

$$\lim_{u\to\infty} \frac{V_2(u)}{(g \circ f^{\leftarrow})(u)} = \infty \quad \text{and} \quad V_2 \in RV_0.$$

Using $V_2 \overset{\Pi}{\sim} f^{\leftarrow}$, we deduce $\lim_{u\to\infty} V_2(u)/f^{\leftarrow}(u) = 1$.

Assuming that $f$ is continuous and increasing on $(D,\infty)$, then $f^{\leftarrow}(u)$ coincides with $f^{-1}(u)$ (the inverse of $f$ at $u$) for $u > 0$ large.

By $V_2 \overset{\Pi}{\sim} f^{\leftarrow}$, we have

$$\lim_{u\to\infty} \frac{V_2(u) - f^{\leftarrow}(u)}{(g \circ f^{\leftarrow})(u)} = c \in \mathbb{R}.$$

By Definition 5.3.1, we get $\lim_{u\to\infty} f(V_2(u))/u = e^c > 0$. By (5.68), we infer that

$$(\psi \circ f^{\leftarrow})(u) = \sup\{z/(g \circ f^{\leftarrow})(z) : \quad f(\alpha) \le z \le u\} \quad (\alpha > 0 \text{ is large})$$

so that, by Theorem 4.1.7, we deduce

$$\psi \circ f^{\leftarrow} \in RV_1 \quad \text{and} \quad (\psi \circ f^{\leftarrow})(u) \sim u/(g \circ f^{\leftarrow})(u) \quad \text{as } u \to \infty.$$

By the construction of $V_2$ in (Resnick, 1987, p. 34) and Proposition 5.3.8, we get $\lim_{u\to\infty} uV_2'(u)/(g \circ f^{\leftarrow})(u) = 1$. Consequently,

$$(\psi \circ f^{\leftarrow})(u) \sim 1/V_2'(u) \quad \text{as } u \to \infty.$$

It follows that

$$(\psi \circ f^{\leftarrow})^{\leftarrow}(u) = (f \circ \psi^{\leftarrow})(u) \sim (1/V_2')^{\leftarrow}(u) \quad \text{as } u \to \infty.$$

By Proposition 4.1.1 and $V_2(u) \sim f^{-1}(u)$ as $u \to \infty$, we achieve (5.76). □

Recall that $\widehat{Z}(u)$, defined for $u > D$, is a *normalized* regularly varying function of index $q$ (in short, $\widehat{Z} \in NRV_q$) if $\widehat{Z}$ is a positive $C^1$-function such that $\lim_{u\to\infty} u\widehat{Z}'(u)/\widehat{Z}(u) = q$. By Remark 4.1.2, for each $Z \in RV_q$ there exists $\widehat{Z} \in NRV_q$ such that $\widehat{Z}(u) \sim Z(u)$ as $u \to \infty$.

We now refine Lemma 5.3.11 by constructing a $C^2$-function which is equivalent to $\phi$ at the origin (see Lemma 2.3 in Cîrstea (2004*b*)).

**Lemma 5.3.12.** *Suppose $f \in \Gamma$ is continuous and increasing on some interval $(D, \infty)$. If $k \in \mathcal{K}$ with $\ell_1 \neq 0$, then there exists $\widehat{\chi} \in C^2(0, \tau)$ satisfying $\lim_{t\to 0^+} \widehat{\chi}(t)/\chi(t) = 1$ and the following:*

(i) $\displaystyle \lim_{t\to 0^+} \frac{\widehat{\chi}(t)}{\widehat{\chi}'(t)} = \lim_{t\to 0^+} \frac{\widehat{\chi}'(t)}{\widehat{\chi}''(t)} = 0$ *and* $\displaystyle \lim_{t\to 0^+} \frac{\widehat{\chi}(t)\widehat{\chi}''(t)}{[\widehat{\chi}'(t)]^2} = \frac{2+\ell_1}{2}$;

(ii) $\displaystyle \lim_{t\to 0^+} P_1(t) := \lim_{t\to 0^+} \frac{V_2(\widehat{\chi}(t))}{V_2'(\widehat{\chi}(t))} \frac{\widehat{\chi}(t)}{[\widehat{\chi}'(t)]^2} = 0$;

(iii) $\displaystyle \lim_{t\to 0^+} P_2(t) := \lim_{t\to 0^+} \frac{k^2(t)(f \circ V_2)(\widehat{\chi}(t))}{\widehat{\chi}''(t)V_2'(\widehat{\chi}(t))} = \frac{C\ell_1^2}{2(2+\ell_1)}$.

*Proof.* By Lemma 5.3.11, $1/V_2'(u) \in NRV_1$ so that $(1/V_2')^{\leftarrow}(u) \in NRV_1$. Since $k \in \mathcal{K}$ with $\ell_1 \neq 0$, we have $k(1/u) \in NRV_{1-1/\ell_1}$ (cf. Proposition 4.2.8). Therefore, $\chi(1/u) \in NRV_{2/\ell_1}$. By Karamata's Theorem (Proposition 4.1.5), we get

$\frac{d}{du}[\chi(1/u)] \in RV_{-1+2/\ell_1}$. Hence, $-\chi'(1/u) \in RV_{1+2/\ell_1}$. By Remark 4.1.2, there exists $\widehat{\chi} \in C^2(0,\tau)$ such that

$$-\widehat{\chi}'(1/u) \in NRV_{1+2/\ell_1} \quad \text{and} \quad \widehat{\chi}'(1/u) \sim \chi'(1/u) \text{ as } u \to \infty.$$

It follows that

$$\lim_{t\to 0^+} \frac{\widehat{\chi}'(t)}{\chi'(t)} = 1 = \lim_{t\to 0^+} \frac{\widehat{\chi}(t)}{\chi(t)} \quad \text{and} \quad \lim_{t\to 0^+} \frac{t\widehat{\chi}''(t)}{\widehat{\chi}'(t)} = -\left(1+\frac{2}{\ell_1}\right).$$

Consequently, $\widehat{\chi}(1/u) \in NRV_{2/\ell_1}$ (that is, $\lim_{t\to 0^+} t\widehat{\chi}'(t)/\chi(t) = -2/\ell_1$). Thus, (i) follows. Moreover, we have

$$\lim_{t\to 0^+} \frac{\log \widehat{\chi}(t)}{\log t} = -2/\ell_1 \quad \text{and} \quad \lim_{t\to 0^+} \frac{\log(-\widehat{\chi}'(t))}{\log t} = -\left(1+\frac{2}{\ell_1}\right).$$

Since $\lim_{u\to\infty} \log V_2'(u)/\log u = -1$ and $\lim_{u\to\infty} \log V_2(u)/\log u = 0$, we find $\lim_{t\to 0^+} \log P_1(t) = -\infty$. Thus, (ii) is derived.

Using $V_2' \in NRV_{-1}$ and $\lim_{t\to 0^+} \widehat{\chi}(t)/\chi(t) = 1$, by Proposition 4.1.1, we get

$$t^2k^2(t)/V_2'(\widehat{\chi}(t)) \sim t^2k^2(t)/V_2'(\chi(t)) = 1 \quad \text{as } t \to 0^+.$$

From this and Lemma 5.3.11, we infer that

$$\lim_{t\to 0^+} P_2(t) = \lim_{t\to 0^+} \frac{\widehat{\chi}(t)}{t^2\widehat{\chi}''(t)} \, \frac{(f\circ V_2)(\widehat{\chi}(t))}{\widehat{\chi}(t)} = \frac{C\ell_1^2}{2(2+\ell_1)}.$$

This concludes the proof. □

*Remark* 5.3.2. If $f \in \Gamma$ is continuous and increasing on $(D,\infty)$ and $k \in \mathcal{K}$ with $\ell_1 \neq 0$, then by Lemmas 5.3.11 and 5.3.12, we have $\lim_{t\to 0^+}(V_2\circ\widehat{\chi})(t)/\phi(t) = 1$, where $\phi$ is given by (5.67) and $(V_2\circ\widehat{\chi})(1/u)$ belongs to $RV_0$.

### 5.3.4 Proof of Theorem 5.3.1

By Lemma 5.3.11, $f(V_2(u)) \sim Cu$ as $u \to \infty$ and $(V_2(u))^q \in RV_0$, for any $q \in \mathbb{R}$. Thus, $\lim_{u\to\infty} f(u)/u^2 = \infty$ so that the Keller–Osserman condition (3.6) holds. Hence, (5.1) (resp., (5.2)) possesses large solutions if and only if $a < \lambda_{\infty,1}$ (see Theorem 3.1.1 if $\Gamma_\infty = \partial\Omega$ resp., Theorem 3.3.1 if $\Gamma_\infty \neq \partial\Omega$).

Fix $a < \lambda_{\infty,1}$. Let $\Gamma_\infty^c$ be an arbitrary connected open and closed subset of $\Gamma_\infty$. Set $d(x) = \mathrm{dist}(x, \Gamma_\infty^c)$.

By (5.5), there exist some positive constants $\gamma_-, \gamma_+$ and $\delta$ such that

$$\gamma_- \le b(x)/k^2(d(x)) \le \gamma_+, \quad \text{for all } x \in \Omega \text{ with } d(x) \le 2\delta.$$

Choose $\beta_- \in (0, \gamma_-/2)$ and $\beta_+ \in (2\gamma_+, \infty)$. We diminish $\delta > 0$ such that:

(i) $d(x)$ is a $C^2$-function on $\{x \in \Omega : \ d(x) < 2\delta\}$;

(ii) $k$ is non-decreasing on $(0, 2\delta)$;

(iii) $\widehat{\chi}''(t) > 0$ on $(0, 2\delta)$, where $\widehat{\chi}$ is provided by Lemma 5.3.12.

Let $\sigma \in (0, \delta)$ be arbitrary. With $V_2$ given by Lemma 5.3.11, we define

$$u_\sigma^\pm(x) := V_2(m(\beta_\mp)^{-1}\widehat{\chi}(d(x) \mp \sigma)) > 0, \tag{5.77}$$

for each $x \in \Omega$ with $\sigma/2 < d(x) \mp \sigma/2 < 2\delta - \sigma/2$, where $m := (C\ell_1/2)^{-1}$ and $C > 0$ is from Lemma 5.3.11. For brevity, we put

$$J^\pm(x) := m(\beta_\mp)^{-1}\widehat{\chi}(d(x) \mp \sigma).$$

We prove that, by diminishing $\delta > 0$, $u_\sigma^+$ and $u_\sigma^-$ become upper and lower solutions near the boundary:

$$\pm[-\Delta u_\sigma^\pm - a u_\sigma^\pm + b(x) f(u_\sigma^\pm)] \ge 0, \tag{5.78}$$

for each $x \in \Omega$ with $\sigma/2 < d(x) \mp \sigma/2 < 2\delta - \sigma/2$. One can see that

$$\begin{aligned}\Delta u_\sigma^\pm =& m(\beta_\mp)^{-1}\widehat{\chi}''(d(x) \mp \sigma) V_2'(J^\pm) \\ &\times \left[1 + \frac{J^\pm V_2''(J^\pm)}{V_2'(J^\pm)} \frac{[\widehat{\chi}']^2}{\widehat{\chi}\widehat{\chi}''}(d(x) \mp \sigma) + \Delta d(x) \frac{\widehat{\chi}'}{\widehat{\chi}''}(d(x) \mp \sigma)\right].\end{aligned} \tag{5.79}$$

Denote $S^\pm(d \mp \sigma)$ the last factor in the right-hand side of (5.79). It follows that

$$\pm[-\Delta u_\sigma^\pm - a u_\sigma^\pm + b(x) f(u_\sigma^\pm)] \ge \pm m(\beta_\mp)^{-1}\widehat{\chi}''(d \mp \sigma) V_2'(J^\pm) K^\pm(d \mp \sigma),$$

where we put

$$\begin{aligned}K^\pm(d \mp \sigma) =& \frac{\gamma_\mp \beta_\mp}{m} \frac{k^2(d \mp \sigma)}{\widehat{\chi}''(d \mp \sigma)} \frac{f(u_\sigma^\pm)}{V_2'(J^\pm(x))} - \frac{a}{m} \frac{\beta_\mp}{\widehat{\chi}''(d \mp \sigma)} \frac{V_2(J^\pm(x))}{V_2'(J^\pm(x))} \\ &- S^\pm(d \mp \sigma) =: T_1(d \mp \sigma) + T_2(d \mp \sigma) - S^\pm(d \mp \sigma).\end{aligned}$$

By Lemmas 5.3.11 and 5.3.12, we deduce $\lim_{d\mp\sigma\to 0} T_2(d\mp\sigma)=0$ and

$$\lim_{d\mp\sigma\to 0} T_1(d\mp\sigma)=\frac{\gamma_\mp}{\beta_\mp}\,\frac{\ell_1}{(2+\ell_1)},\quad \lim_{d\mp\sigma\to 0} S^\pm(d\mp\sigma)=\frac{\ell_1}{2+\ell_1}.$$

Hence,

$$\lim_{d\mp\sigma\to 0} K^\pm(d\mp\sigma)=\left(\frac{\gamma_\mp}{\beta_\mp}-1\right)\frac{\ell_1}{2+\ell_1}.$$

This proves (5.78).

*Proof of* (5.66). Let $\zeta>0$ be small such that $a$ is less than the first Dirichlet eigenvalue of $(-\Delta)$ in the domain $E_\zeta:=\{x\in\mathbb{R}^N\setminus\overline{\Omega}:\ d(x)<\zeta\}$.

For $\delta>0$ as in (5.78), set

$$I_\delta=\{x\in\Omega:\ d(x)<\delta\}\ \text{ and }\ \Omega_1:=E_{2\zeta}\cup\{x\in\overline{\Omega}:\ d(x)<\delta\}.$$

Let $p\in C^{0,\mu}(\overline{\Omega}_1)$ be such that $0<p(x)\le b(x)$ for $x\in\Omega$ with $d(x)\le\delta$, $p=0$ in $\overline{E}_\zeta$ and $p>0$ in $\overline{E}_{2\zeta}\setminus\overline{E}_\zeta$. Denote by $w$ a large solution of

$$-\Delta u=au-p(x)f(u)\quad\text{in }\ \Omega_1.$$

Note that $w$ is uniformly bounded on $\Gamma^c_\infty$ and $w=\infty$ on $\partial I_\delta\cap\Omega$.

Let $u_a$ be an arbitrary large solution of (5.1) (resp., (5.2)). By (5.78) and (5.3), we find

$$\begin{cases} -\Delta(u_a+w)-a(u_a+w)+b(x)f(u_a+w)\ge 0 & \text{in } I_\delta,\\ -\Delta(u^+_\sigma+w)-a(u^+_\sigma+w)+b(x)f(u^+_\sigma+w)\ge 0 & \text{in } I_\delta\setminus\overline{I}_\sigma,\\ (u_a+w)|_{\partial I_\delta}=\infty>u^-_\sigma|_{\partial I_\delta}\ \text{ and }\ (u^+_\sigma+w)|_{\partial(I_\delta\setminus\overline{I}_\sigma)}=\infty>u_a|_{\partial(I_\delta\setminus\overline{I}_\sigma)}. \end{cases}$$

By Lemma 5.2.7, we get

$$u_a+w\ge u^-_\sigma\ \text{ in } I_\delta\quad\text{and}\quad u^+_\sigma+w\ge u_a\ \text{ in } I_\delta\setminus\overline{I}_\sigma.$$

Letting $\sigma\to 0$, we arrive at

$$V_2(m(\beta_+)^{-1}\widehat{\chi}(d(x)))-w(x)\le u_a\le V_2(m(\beta_-)^{-1}\widehat{\chi}(d(x)))+w(x),$$

for each $x\in\Omega$ with $0<d(x)<\delta$. Since $V_2\in RV_0$, by Proposition 4.1.1 we derive

$$1\le\liminf_{d(x)\to 0}\frac{u_a(x)}{V_2(\widehat{\chi}(d(x)))}\le\limsup_{d(x)\to 0}\frac{u_a(x)}{V_2(\widehat{\chi}(d(x)))}\le 1.$$

By Remark 5.3.2, we conclude (5.66). The uniqueness of the large solution follows now in a standard way (see §5.2.3.5). $\square$

### 5.3.5 Proof of Theorem 5.3.3

Denote by $\Gamma_\infty^*$ the unique connected open and closed subset of $\Gamma_\infty$ that contains $x_*$.

Since (5.70) holds, we can take $c_* \in \mathbb{R}$ such that

$$c_* > \limsup_{x \to x_*, x \in \Omega} \frac{b(x)}{k^2(d(x, \Gamma_\infty))}.$$

Hence, we can assume that

$$0 < b(x) \leq c_* \, k^2(d(x, \Gamma_\infty^*)), \quad \forall x \in B_{r_0}(x_*) \cap \Omega,$$

for some $r_0 > 0$ small enough such that

$$B_{r_0}(x_*) := \{x \in \mathbb{R}^N : \; |x - x_*| < r_0\} \subset \mathbb{B} \quad \text{and} \quad B_{r_0}(x_*) \cap \partial\Omega \subseteq \Gamma_\infty^*. \tag{5.80}$$

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two smooth domains such that $\mathcal{O}_1 \subset\subset \mathcal{O}_2 \subset\subset B_{r_0}(x_*)$ and $\overline{\mathcal{O}}_1 \cap \overline{\Omega} = I_* \subset \Gamma_\infty^*$ with $x_*$ belonging to the interior of $I_*$. In other words, $\mathcal{O}_1$ is outside $\Omega$, but there is a part common to $\partial\mathcal{O}_1$ and $\partial\Omega$, denoted by $I_*$, which contains $x_*$ in its interior.

Set $D_* = \mathcal{O}_2 \setminus \overline{\mathcal{O}}_1$. By Lemma 3.3.4, the boundary value problem

$$\begin{cases} -\Delta u = au - c_* \, k^2(d(x, \partial\mathcal{O}_1)) f(u) & \text{in } D_* \\ u = 0 & \text{on } \partial\mathcal{O}_2 \end{cases} \tag{5.81}$$

subject to $u = n \geq 1$ on $\partial\mathcal{O}_1$ has a *unique* positive solution $w_n$.

By Theorem 5.3.1, (5.81) subject to $u = \infty$ on $\partial\mathcal{O}_1$ has a unique positive, say $W$, which satisfies

$$\lim_{d(x,\partial\mathcal{O}_1) \to 0} \frac{W(x)}{\phi(d(x, \partial\mathcal{O}_1))} = 1. \tag{5.82}$$

Notice that $d(x, \partial\mathcal{O}_1) \geq d(x, \Gamma_\infty^*)$, for each $x \in D_* \cap \Omega$. Hence, for all $n \geq 1$,

$$\begin{cases} -\Delta w_n \leq a \, w_n - b(x) f(w_n) \quad \text{in } D_* \cap \Omega, \\ w_n = 0 \quad \text{on } \partial\mathcal{O}_2, \\ U|_{\overline{D}_* \cap \Gamma_\infty^*} = \infty > w_n|_{\overline{D}_* \cap \Gamma_\infty^*}, \end{cases}$$

where $U$ denotes an arbitrary positive solution of (5.69).

By Proposition 5.2.7, we infer that

$$\begin{cases} w_n \le w_{n+1} \le W & \text{in } D_*, \quad \forall n \ge 1, \\ w_n \le U \quad \text{in } D_* \cap \Omega, & \forall n \ge 1. \end{cases} \tag{5.83}$$

Standard regularity arguments show that $w_\infty$ defined by

$$w_\infty(x) := \lim_{n\to\infty} w_n(x), \quad \forall x \in D_*,$$

is a positive solution of (5.81) satisfying $w_\infty = \infty$ on $\partial \mathcal{O}_1$. Hence,

$$w_\infty \equiv W \quad \text{in } D_*.$$

By (5.83), we obtain $U \ge W$ in $D_* \cap \Omega$. It follows that

$$\frac{U(x)}{\phi(d(x,\Gamma_\infty^*))} \ge \frac{W(x)}{\phi(d(x,\Gamma_\infty^*))}, \quad \forall x \in D_* \cap \Omega. \tag{5.84}$$

By our choice of $\mathcal{O}_1$, $d(x, \partial\mathcal{O}_1) = d(x, \Gamma_\infty^*)$ if $x \in \Omega$ is sufficiently close to $x_*$. Hence, letting $x \to x_*$ in (5.84) and using (5.82), we find

$$\liminf_{x\to x_*,\, x\in\Omega} \frac{U(x)}{\phi(d(x,\Gamma_\infty))} \ge 1,$$

which concludes the proof. □

### 5.3.6 Proof of Theorem 5.3.4

Let $\Gamma_\infty^*$ be the same as in the proof of Theorem 5.3.3. Since we assume (5.72), we can take

$$0 < d_* < \liminf_{x\to x_*, x\in\Omega} \frac{b(x)}{k^2(d(x,\Gamma_\infty))}$$

and find $r_0 > 0$ small such that (5.80) holds and

$$b(x) \ge d_*\, k^2(d(x,\Gamma_\infty^*)), \quad \forall x \in B_{r_0}(x_*) \cap \Omega.$$

Let $\Omega_*$ be a smooth domain such that

$$\overline{\Omega}_* \subset \overline{\Omega} \cap B_{r_0}(x_*) \ \text{ and } I_* := \partial\Omega_* \cap \Gamma_\infty^* \text{ contains } x_* \text{ in its interior.} \tag{5.85}$$

For $n \ge 1$, set $\Omega_{*,n} = \{x \in \Omega_* : \ 1/n < d(x, \partial\Omega_*)\}$. Obviously, $\Omega_* = \cup_{n=1}^\infty \Omega_{*,n}$.

By Theorem 5.3.1, there is a *unique* large solution $Z_*$ of the equation

$$-\Delta u = au - d_* k^2(d(x, \partial\Omega_*))f(u) \quad \text{in } \Omega_*. \tag{5.86}$$

Let $Z_n$ be the unique large solution of (5.86) with $\Omega_*$ replaced by $\Omega_{*,n}$.

Applying Theorem 5.3.1, we get

$$\lim_{d(x,\partial\Omega_*)\to 0} \frac{Z_*(x)}{\phi(d(x, \partial\Omega_*))} = 1. \tag{5.87}$$

Clearly $d(x, \partial\Omega_{*,n}) \leq d(x, \partial\Omega_{*,n+1}) \leq d(x, \Gamma_\infty^*)$ for each $x \in \Omega_{*,n}$, and

$$\begin{cases} -\Delta Z_n \geq aZ_n - b(x)f(Z_n) & \text{in } \Omega_{*,n}, \quad \forall n \geq 1, \\ Z_n|_{\partial\Omega_{*,n}} = \infty > U|_{\partial\Omega_{*,n}}, & \qquad\qquad \forall n \geq 1. \end{cases}$$

By Proposition 5.2.7, we deduce

$$\begin{cases} Z_n \geq Z_{n+1} & \text{in } \Omega_{*,n}, \quad \forall n \geq 1, \\ Z_n \geq U & \text{in } \Omega_{*,n}, \quad \forall n \geq 1, \end{cases} \tag{5.88}$$

where $U$ is an arbitrary positive solution of (5.69). For each $x \in \Omega_*$, there exists an integer $m(x) \geq 1$ such that $x \in \Omega_{*,n}$, for each $n \geq m(x)$. By (5.88), $Z_\infty(x) = \lim_{n\to\infty} Z_n(x)$ is well defined. Standard regularity arguments imply that $Z_\infty$ is a positive solution of (5.86) in $\Omega_*$ satisfying $Z_\infty = \infty$ on $\partial\Omega_*$. Since there is only one such solution, we conclude that

$$Z_\infty \equiv Z_* \quad \text{in } \Omega_*. \tag{5.89}$$

By (5.88) and (5.89), it follows that

$$\frac{U(x)}{\phi(d(x, \Gamma_\infty^*))} \leq \frac{Z_*(x)}{\phi(d(x, \Gamma_\infty^*))}, \quad \forall x \in \Omega_*. \tag{5.90}$$

By (5.85), we have $d(x, \partial\Omega_*) = d(x, \Gamma_\infty^*)$ if $x \in \Omega_*$ is close to $x_*$. Thus, letting $x \to x_*$ in (5.90) and using (5.87), we obtain

$$\limsup_{x\to x_*,\, x\in\Omega} \frac{U(x)}{\phi(d(x, \Gamma_\infty))} \leq 1.$$

This finishes the proof. □

# References


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