Quiggin, J. and Chambers, R. G. (2003), Local utility functions and local probability transformations, *Journal of Risk and Uncertainty*, 27(2), 110–20.

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March 19, 2004

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Abstract

This note shows that, under appropriate conditions, preferences may be locally approximated by the linear utility or risk-neutral preference functional associated with a local probability transformation.

1 Introduction

The expected-utility theory of choice under uncertainty provides a simple and tractable model of choice under uncertainty, and has formed the basis of a large and valuable body of economic analysis. However, there is a good deal of evidence to show that the expected-utility model does not provide an adequate description of behavior under uncertainty. A wide range of generalized models have been presented. The most influential class involves the introduction of a transformation of probabilities into decision weights to complement the transformation of outcomes into utilities (Kahneman and Tversky 1979, Quiggin 1982, Yaari 1987).

A crucial issue in generalized expected utility theory is the extent to which results derived in the expected utility framework can be carried ove.r Machina (1982) presented a systematic basis for generalizations of expected utility theory by introducing the idea of local utility functions, which allow general preferences to be approximated in the neighborhood of a given distribution by an expected-utility functional. It is natural to ask whether Machina's local representation of general preferences has an analog in terms of local probability transformations.

In this paper, we address this problem using the characterization of preferences under uncertainty in terms of benefit functions, developed by Chambers and Quiggin (2000) and Quiggin and Chambers (1998). Quiggin and Chambers show that the benefit function provides not only a complete representation of preferences but also a natural characterization of risk premiums and of conditions such as constant absolute risk aversion. Because the benefit function is defined for every risky prospect, considered as a state-contingent income vector, it provides a natural starting point for the consideration of local approximations to preference structures.

This note shows that, under appropriate conditions, preferences may be locally approximated by the linear utility or risk-neutral preference functional associated with a local probability transformation. This idea is developed using a special case of the benefit function, the translation function originally due to Blackorby and Donaldson (1980). Alternative representations using the distance function are also described. Finally, some concluding comments are offered.

2 Notation and background

We are concerned with preferences over state-contingent income or consumption vectors represented as mappings from a state space Ω to an outcome space $Y \subseteq \Re$. We focus on the case where Ω is a finite set $\{1...S\}$, and the space of random variables is thus $Y^S \subseteq \Re^S$. Denote the $S-$ dimensional vector of ones by $\mathbf{1} = (1, 1, ...1)$, and the unit simplex by $\mathcal{P} \subset \Re_+^S$.

Preferences over Y^S are given by a strictly increasing, differentiable function $W : Y^S \rightarrow \Re$. Machina's (1982) analysis is developed for preferences over spaces of cumulative distribution functions. In order to extend a local utility function analysis to preferences expressed in terms of random variables, it is necessary to restrict attention to functions W satisfying symmetry conditions which ensure that any two random variables with the same cumulative distribution function are judged equivalent. Let $\mathbf{y} \preceq \mathbf{y}'$ mean that $\mathbf{y}$ and $\mathbf{y}'$ have the same mean and that $\mathbf{y}$ is less risky than $\mathbf{y}'$ in the sense of Rothschild and Stiglitz (1970), when both the mean and the relative riskiness criterion are calculated with respect to some given probability vector $\boldsymbol{\pi}$. A function $W : \Re^S \rightarrow \Re$ is said to be *generalized Schur concave* with respect to π if $\mathbf{y} \preceq \mathbf{y}'$ implies $W(\mathbf{y}) \geq W(\mathbf{y}')$. Generalized Schur-concavity implies that, if $\mathbf{y}$ and $\mathbf{y}'$ have the same probability distribution, $\mathbf{y} \preceq \mathbf{y}'$ and $\mathbf{y}' \preceq \mathbf{y}$, so that $W(\mathbf{y}) = W(\mathbf{y}')$.

The translation function, $B : Y^S \times \Re \rightarrow \Re$, is defined by:

$$B(\mathbf{y}, w) = \max\{\beta \in \Re : W(\mathbf{y} - \beta\mathbf{1}) \geq w\}$$

if $W(\mathbf{y} - \beta\mathbf{1}) \geq w$ for some β, and $-\infty$ otherwise (Blackorby and Donaldson 1980). The translation function is a special case of the benefit function introduced by Luenberger (1992), which is defined with an arbitrary vector $\mathbf{g} \in \Re^S$ in place of $\mathbf{1}$.

The properties of the translation function are well known (Blackorby and Donaldson 1980; Luenberger 1992; Chambers, Chung, and Färe 1996; and Chambers and Quiggin 2000), and are presented in lemma form for later use:

Lemma 1: $B(\mathbf{y}, w)$ satisfies:

a) $B(\mathbf{y}, w)$ is nonincreasing in w and nondecreasing in $\mathbf{y}$;

b) $B(\mathbf{y} + \alpha\mathbf{1}, w) = B(\mathbf{y}, w) + \alpha$, $\alpha \in \Re$ (the translation property);

c) $B(\mathbf{y}, w) \geq 0 \Leftrightarrow W(\mathbf{y}) \geq w$, and $B(w, \mathbf{y}) = 0 \Leftrightarrow W(\mathbf{y}) = w$; and

d) If W is generalized Schur concave, then $\mathbf{y} \preceq_{\boldsymbol{\pi}} \mathbf{y}' \Rightarrow B(\mathbf{y}', w) \leq B(\mathbf{y}, w)$.

Two of these properties prove particularly important in what follows: Property 1.c implies that B is a complete function representation of preferences. Property 1.b is central to the dual relationship between the local utility functions and the local probability transformations. In particular, for differentiable preferences, property 1.b implies

$$\sum_{s \in \Omega} B_s(\mathbf{y}, w) = 1,$$

where subscripts on functions denote partial derivatives. Chambers and Quiggin (2002) generalize this property to superdifferentiable preferences. The certainty equivalent is an important canonical representation of preferences which is conveniently defined in terms of the translation function as

$$\begin{aligned} e(\mathbf{y}) &= \min\{W : W(e\mathbf{1}) \geq W(\mathbf{y})\} \\ &= -B(\mathbf{0}, W(\mathbf{y})). \end{aligned}$$

No true loss of generality is involved in replacing $W(\mathbf{y})$ by its transform $e(\mathbf{y})$, and in what follows we always do so, writing $B(\mathbf{y}, e)$ in place of $B(\mathbf{y}, w)$.

3 Local utility functions

We now consider the local representation of e in terms of a probability distribution π^0, assumed objectively given, and a family of local utility functions $u(\bullet; \mathbf{y})$.[1] A function

[1]This representation is more general than the local utility function $u(\bullet; F)$ defined by Machina (1982). Machina's representation incorporates the assumption of probabilistic sophistication, namely that if two state-contingent income vectors $\mathbf{y}, \mathbf{y}'$ induce the same cumulative probability distribution F for the objectively-given probability vector $\boldsymbol{\pi}^0$, then $W(\mathbf{y}) = W(\mathbf{y}')$. Generalized Schur-concavity with respect to π^0 is sufficient, but not necessary, for probabilistic sophistication.

$u(\bullet;\mathbf{y}):\Re\rightarrow\Re$, is a local utility function for e at $\mathbf{y}$ if and only if, for all $\Delta\mathbf{y}$

$$e(\mathbf{y}+\Delta\mathbf{y})-e(\mathbf{y})=\sum_{s=1}^{S}\pi_s^0 u'(y_s;\mathbf{y})\Delta y_s+o\|\Delta\mathbf{y}\|.$$

The local utility function $u(\bullet;\mathbf{y})$ is determined in a neighborhood of the support of $\mathbf{y}$ by the condition:

$$\frac{\pi_s^0 u'(y_s;\mathbf{y})}{\pi_t^0 u'(y_t;\mathbf{y})}=\frac{e_s(\mathbf{y})}{e_t(\mathbf{y})}\ \forall s,t\in S.$$

Machina (1982) shows that, for any e, there exists a local utility function representation such that

$$u'(y_s;\mathbf{y})=\frac{e_s(\mathbf{y})}{\pi_s^0}\ \forall s$$

and

$$\sum_{s=1}^{S}\pi_s^0 u(y_s;\mathbf{y})=e(\mathbf{y})$$

with the latter equality holding approximately in a neighborhood of $\mathbf{y}$.

The local utility function can be derived from the translation function. By Lemma 1.c, $e(\mathbf{y}+\Delta\mathbf{y})\geq e(\mathbf{y})$ if and only if

$$\begin{aligned}0 &\leq B(\mathbf{y}+\Delta\mathbf{y},e(\mathbf{y})) \qquad (1)\\ &= B(\mathbf{y}+\Delta\mathbf{y},e(\mathbf{y}))-B(\mathbf{y},e(\mathbf{y}))\\ &= \sum_{s=1}^{S}B_s(\mathbf{y},e(\mathbf{y}))\Delta y_s+o||\Delta\mathbf{y}||\\ &= \sum_{s=1}^{S}\pi_s^0\frac{B_s(\mathbf{y},e(\mathbf{y}))}{\pi_s^0}\Delta y_s+o||\Delta\mathbf{y}||.\end{aligned}$$

This fact and the observation that

$$B_s(\mathbf{y},e(\mathbf{y}))=\frac{e_s(\mathbf{y})}{\sum_t e_t(\mathbf{y})}$$

implies that the local utility function can be written as

$$u'(y_s;\mathbf{y})=\frac{B_s(\mathbf{y},e(\mathbf{y}))}{\pi_s^0}\sum_t e_t(\mathbf{y}).$$

This normalization by the derivative of $e(\mathbf{y})$ in the direction of $\mathbf{1}$, $\sum_t e_t(\mathbf{y})$, characterizes marginal utility $u'(y;\mathbf{y})$. As Machina (1988) notes, the local utility function at $\mathbf{y}$ is defined up to an arbitrary additive constant, which may vary with $\mathbf{y}$. In addition, all the local utility functions may be scaled by a multiplicative constant, common to all distributions, $\mathbf{y},\mathbf{y}'$ such that $e(\mathbf{y}') \geq e(\mathbf{y})$.

By Lemma 1.b:

$$\begin{aligned}\sum_s \pi_s^0 u'(y_s;\mathbf{y}) &= \sum_{s=1}^{S} B_s(\mathbf{y}, e(\mathbf{y})) \sum_t e_t(\mathbf{y}) \\ &= \sum_t e_t(\mathbf{y}).\end{aligned}$$

Example 1 *If preferences are consistent with expected utility maximization for some strictly increasing* u,

$$B_s(\mathbf{y}, e(\mathbf{y})) = \frac{\pi_s u'(y_s)}{\sum_t \pi_t u'(y_t)}$$

for all $\mathbf{y}$.

Example 2 *Preferences are characterized by constant absolute risk aversion (Chambers and Quiggin 2000) if for all* $\mathbf{y}$

$$e(\mathbf{y}+\delta\mathbf{1}) = e(\mathbf{y}) + \delta,$$

whence $\sum_s e_s(\mathbf{y}) = 1$ *implying that* $B_s(\mathbf{y}, e(\mathbf{y})) = e_s(\mathbf{y})$ *and hence* $u'(y_s;\mathbf{y}) = \frac{e_s(\mathbf{y})}{\pi_s}$.

Machina's definition is applicable to the case when e is Frechet differentiable. More generally, the concept of a local utility function is applicable in the case when e has one-sided directional Gateaux derivatives:

$$e'(\mathbf{y};\Delta\mathbf{y}) = \lim_{t\to 0^+}\left\{\frac{e(\mathbf{y}+t\Delta\mathbf{y}) - e(\mathbf{y})}{t}\right\}.$$

The methods of this paper can be adapted to that case by relying on the corresponding Gateaux derivatives of the translation function

$$B'(\mathbf{y},e;\Delta\mathbf{y}) = \lim_{t\to 0^+}\left\{\frac{B(\mathbf{y}+t\Delta\mathbf{y},e) - B(\mathbf{y},e)}{t}\right\},$$

evaluated at $e = e(\mathbf{y})$.

4 The Local Probability Transformation

Wakker (1989) and others have noted the existence of a duality between utility functions, which transform monetary outcomes into utilities, and functions which transform probabilities into 'decision weights'. Yaari's (1987) version of rank-dependent utility with linear utility makes this duality explicit. We now ask whether general preferences may be approximated by a local version of a probability weighting function.

The most general local probability transformation is a mapping $\boldsymbol{\pi}$ associating to each $\mathbf{y} \in \Re^S$, a vector of probabilities $\boldsymbol{\pi}(\mathbf{y}) \in \mathcal{P}$, such that, in a neighborhood of $\mathbf{y}$, $e(\mathbf{y}+\Delta\mathbf{y}) - e(\mathbf{y}) \geq 0$ if and only if

$$\sum_{s=1}^{S} \pi_s(\mathbf{y}) \Delta y_s \geq 0.$$

Noting the requirement that $\sum_{\mathbf{s}} \pi_s = 1$, the third line of (1) implies that

$$\pi_s(\mathbf{y}) = B_s(\mathbf{y}, e(\mathbf{y}))$$

provides an appropriate local probability transformation. Preferences in a neighborhood of any $\mathbf{y}$ may be locally approximated by a 'corrected arithmetic mean' of the form

$$e(\mathbf{y}+\Delta\mathbf{y}) - e(\mathbf{y}) = \sum_{s=1}^{S} e_s(\mathbf{y}) \sum_{s=1}^{S} \pi_s(\mathbf{y}) \Delta y_s.$$

In the case of constant absolute risk-aversion, we have

$$\sum_{s=1}^{S} e_s(\mathbf{y}) = 1$$

and the corrected arithmetic mean is the ordinary mean calculated with respect to $\boldsymbol{\pi}(\mathbf{y})$.

The local utility function is a mapping of $\Re$ into itself, and it is desirable to have a similarly compact local representation of probability transformations. The rank-dependent approach provides such a representation, based on $\mathbf{y}_{[]}$, the increasing rearrangement of $\mathbf{y}$, denoted by a subscript $[]$, and such that $y_{[1]} \leq y_{[2]} \leq ... \leq y_{[S]}$, and a known vector of 'objective probabilities' $\boldsymbol{\pi}^0$. A local probability weighting function representing the probability transformation $\boldsymbol{\pi}$ is a mapping $q(p; \mathbf{y})$, such that $q(\bullet; \mathbf{y}) : [0,1] \to [0,1]$ with $q(0; \mathbf{y}) = 0$ and $q(1; \mathbf{y}) = 1$, such that

$$\pi_{[s]}(\mathbf{y}) = q\left(\sum_{t=1}^{s} \pi^0_{[t]}; \mathbf{y}\right) - q\left(\sum_{t=1}^{s-1} \pi^0_{[t]}; \mathbf{y}\right).$$

Since the value of the transformation function q is determined only at S points in the unit interval, some canonical choice is required. For our purposes, the simplest assumption is that q is piecewise linear.

5 Risk aversion

Having derived the local utility function and the local probability transformation, we turn attention to the extent to which global preferences inherit the properties of the local transformations. Machina's (1982) central insight was that when all the local utility functions shared certain properties, such as preservation of first-order or second-order stochastic dominance, these properties were inherited by the global preference functional e. The equivalent result on first-order stochastic dominance for fixed probability transformations is proved by Quiggin (1982) and Chew, Karni and Safra (1987).

These issues may be addressed more directly by consideration of the local probability transformation. For differentiable preferences generalized Schur concave with respect to π^0 (Marshall and Olkin 1979), it is true that:

$$\left[\frac{B_s(\mathbf{y},e)}{\pi_s^0} - \frac{B_k(\mathbf{y},e)}{\pi_k^0}\right](y_s - y_k) \leq 0, \quad s,k \in \Omega$$

for all $\mathbf{y}$, e. From the definitions of the local utility function and local probability transformation, this condition holds if and only if

$$(u'(y_s;\mathbf{y}) - u'(y_k;\mathbf{y}))(y_s - y_k) \leq 0$$
$$\left(\frac{\pi_s(\mathbf{y})}{\pi_s^0} - \frac{\pi_k(\mathbf{y})}{\pi_s^0}\right)(y_s - y_k) \leq 0.$$

The first of these conditions has the straightforward and well-known interpretation that the local utility function must be concave (Machina 1982). The second condition becomes more transparent with the observation that the local probability vector, $\boldsymbol{\pi}(\mathbf{y})$, supports the decisionmaker's indifference curve at $\mathbf{y}$. Hence, it has a natural intepretation as a vector of price-dependent demands consistent with the state-claim $\mathbf{y}$. The second condition then requires that if $y_s > y_k$ then $\frac{\pi_s(\mathbf{y})}{\pi_k(\mathbf{y})} \leq \frac{\pi_s^0}{\pi_s^0}$. Cast in this context, the second condition is the generalization to generalized Schur concave preferences of the Peleg–Yaari (1975) result characterizing risk-aversely efficient points over general convex choice sets. Because choice over a differentiable convex set will lead such a decisionmaker to equate the slope of his indifference curve to the choice set's boundary, in equilibrium relative Peleg-Yaari efficiency prices must equal $\frac{\pi_s(\mathbf{y})}{\pi_k(\mathbf{y})}$ for all s and k.

By the intermediate value theorem,

$$\begin{aligned} \pi_{[s]}(\mathbf{y}) &= q'(c;\mathbf{y})\left(\sum_{t=1}^{s}\pi_{[t]}^0 - \sum_{t=1}^{s-1}\pi_{[t]}^0\right) \\ &= q'(c;\mathbf{y})\,\pi_{[s]}^0 \end{aligned}$$

for some $c \in \left[\sum_{t=1}^{s-1}\pi_{[t]}^0, \sum_{t=1}^{s}\pi_{[t]}^0\right]$. Under the assumption of piecewise linearity, q' is constant on intervals of the form $\left[\sum_{t=1}^{s-1}\pi_{[t]}^0, \sum_{t=1}^{s}\pi_{[t]}^0\right]$, and this allows us to obtain:

Proposition 3 *The following are equivalent:*

(i) the local utility function $u(;\mathbf{y})$ is concave for all $\mathbf{y}$;

(ii) the local probability transformation function $q(;\mathbf{y})$ is concave for all $\mathbf{y}$;

(iii) $(u'(y_s;\mathbf{y}) - u'(y_k;\mathbf{y}))\,(y_s - y_k) \leq 0$ for all s, k;

(iv) $\left(\frac{\pi_s(\mathbf{y})}{\pi_s^0} - \frac{\pi_k(\mathbf{y})}{\pi_s^0}\right)(y_s - y_k) \leq 0$ for all s, k; and

(v) preferences are generalized Schur-concave for $\boldsymbol{\pi}^0$.

Condition (i) is the characterization derived by Machina (1982). Condition (ii) is the obvious analogue for the local probability transformation, since, for the Yaari model, risk-aversion in the sense of generalized Schur-concavity is equivalent to concavity of the probability transformation (Chew, Karni and Safra 1987). Condition (iii) is trivially equivalent to (i) and (iv) is similarly equivalent to (ii). Finally, (v) reflects the fact that conditions (iii) and (iv) characterize the local property of generalized Schur-concavity for the two representations.

6 Alternative Formulations

The translation function is only one of many possible alternative function representations of preferences. It has the 'nice' property that its gradient in $\mathbf{y}$ is non-negative and always sums to one. Thus, it is naturally and conveniently interpretable as a vector of probabilities.

However, the general procedure outlined here is applicable both for the more general benefit function and for radial representations of preferences such as the distance function.[2] We briefly illustrate by considering the distance function defined by

$$D(\mathbf{y},e) = \sup\left\{\lambda : e\left(\frac{\mathbf{y}}{\lambda}\right) \geq e\right\},$$

where we specifically restrict attention to $\mathbf{y} \in \Re^S_+$. Among other properties the distance function is a complete function representation of preferences in the sense that $e(\mathbf{y}) \geq e \Leftrightarrow D(\mathbf{y},e) \geq 1$, and it is positively linearly homogeneous in $\mathbf{y}$ (Shephard 1970).

Because D is a complete function representation of preferences, $e(\mathbf{y}+\Delta\mathbf{y}) \geq e(\mathbf{y})$ if and only if

$$0 \leq \ln D(\mathbf{y}+\Delta\mathbf{y},e(\mathbf{y})) - \ln D(\mathbf{y},e(\mathbf{y})).$$

Making a slight change of variables and the associated renormalizations of the distance function and the certainty equivalent, we define

$$\begin{aligned} D^*(\ln\mathbf{y},e) &= D(\mathbf{y},e) \\ e^*(\ln\mathbf{y}) &= e(\mathbf{y}). \end{aligned}$$

This renormalization allows us to apply an exactly identical argument to that used in (1) to identify the local utility function:

$$\begin{aligned} 0 &\leq \ln D^*(\ln\mathbf{y}+\Delta\ln\mathbf{y},e(\mathbf{y})) - \ln D^*(\ln\mathbf{y},e(\mathbf{y})) \\ &= \sum_{s=1}^{S} \frac{\partial \ln D^*(\ln\mathbf{y},e(\mathbf{y}))}{\partial \ln y_s} \triangle \ln y_s + o||\triangle\ln\mathbf{y}||. \end{aligned}$$

By parallel arguments used in establishing (1), we obtain the alternative local representation $v(\ln\mathbf{y};\mathbf{y})$ using the normalization $\sum e^*_s(\ln\mathbf{y})$:

$$v'(\ln y_s;\mathbf{y}) = \frac{1}{\pi_s}\frac{\partial \ln D^*(\ln\mathbf{y},e(\mathbf{y}))}{\partial \ln y_s}\sum e^*_s(\ln\mathbf{y}).$$

By the positive linear homogeneity of the distance function, $\sum_{s=1}^{S} \frac{\partial \ln D^*(\ln\mathbf{y},e(\mathbf{y}))}{\partial \ln y_s} = 1$, an exactly parallel argument to that used in defining the local probability structure implies

[2] In fact, it is applicable for all function representations of preferences.

that preferences in a neighborhood of $\ln \mathbf{y}$ may be locally approximated by a a 'corrected geometric mean':

$$e^{*}\left(\ln \mathbf{y}+\Delta \ln \mathbf{y}\right)-e^{*}\left(\ln \mathbf{y}\right)=\sum e_{s}^{*}\left(\ln \mathbf{y}\right)\sum_{s=1}^{S}\pi_{s}\left(\mathbf{y}\right)\Delta \ln y_{s}.$$

If preferences are generalized Schur concave, D is also generalized Schur concave for each e (Chambers and Quiggin 2000) allowing the trivial extension of our main proposition to this case.

7 Concluding comments

The central object of this paper has been the exploration of the links between local characterizations of preferences based on concepts familiar from simple parametric representations such as the expected-utility and Yaari dual models and those derived from translation and distance functions. Because the translation function is a special case of the more general benefit function, our results can be generalized by considering directions in Y^{S} other than $\mathbf{1}$. This observations implies the existence of a wide range of local representations of risk attitudes consistent, for example, with state-dependent preferences or uninsurable background risk.

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