The firm under uncertainty with general risk-averse preferences: a state-contingent approach

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1. Introduction

The limitations of expected-utility theory as a descriptive model of choice under uncertainty are well known. To address these limitations, a number of generalized expected-utility models have been developed (Chew 1983, Machina 1982, Quiggin 1982). In particular, considerable attention has been focused on problems involving two or more pairwise choices, in which some choice patterns are inconsistent with the assumptions of expected utility.

Although choice problems of this kind yield insights into the strengths and weaknesses of expected-utility theory, they are of little direct relevance to economic analysis. Most economic analysis of uncertainty relates to economic choice problems involving the determination of the value of one or more control variables, which decision-makers are free to vary over a range of positive, and sometimes negative values. The return is determined by interaction between the chosen control variables and some random variable. General formulations of the problem are discussed by Feder (1977) and Machina (1989). However, to understand the underlying issues, it is perhaps more useful to focus on a concrete problem such as that of the firm under uncertainty. Since the work of Sandmo (1971), a large literature analyzing the problem of the output decisions of the firm under uncertainty has developed. Sandmo assumed expected-utility maximization. Most applied work based on his approach has maintained this assumption. Quiggin (1991) showed that Sandmo's main results may be extended to the case of rank-dependent preferences. However, there has been little analysis of the comparative static properties of more general preference structures.

While the limitations of the expected-utility hypothesis are widely recognized, the problems relating to the typical specification of stochastic technologies have been less remarked, but turn out to be equally fundamental. Microeconomic models of production under uncertainty are normally presented in terms of a stochastic production function formulation that degenerates to a family of random variables indexed by effort input. Thus, given inputs x and random shocks θ, output is determined by a function $f(x, \theta)$. The decision problem, therefore, reduces to choosing the most attractive random variable $f(x, \theta)$. By contrast, in the general-equilibrium theory of uncertainty, production is typically represented in terms of the state-space representation of Arrow (1953) and Debreu (1952). This approach is also standard in finance theory (Milne 1995). However, neither general-equilibrium theorists nor finance theorists have been much concerned with the details of choices involving state-contingent production or with comparative-static responses to changes in variables such as input and output prices.

Chambers and Quiggin (2000) argue that, despite its seeming simplicity, the stochastic production function approach is a highly unsatisfactory way of modelling production decisions under uncertainty. Its crucial failing is that the stochastic production function approach does not allow decision-makers to allocate resources in a way that improves outcomes in one state of nature, while leaving them unchanged or reduced in another state. As a result, risk-management options such as irrigation cannot be modelled properly. More generally, risk-management options that require individuals to prepare differentially for different states of nature are assumed to be technically infeasible. Individuals are, thus, assumed to respond passively to risk, simply scaling down their input levels more the more risk-averse are their preferences. By contrast, Chambers and Quiggin (2000) argue, a general state-contingent representation of production under uncertainty allows for a realistic representation of individual and market responses to risk.

. The state-contingent approach has already proved useful in analyzing hedging behavior (Chambers and Quiggin 1997) and principal-agent problems (Chambers and Quiggin 1996; Quiggin and Chambers 1998a). Moreover, the state-contingent approach permits the application of standard duality mappings to state-contingent technologies and their associated cost functions. The existence of the duality mappings has important comparative-static implications. Perhaps most crucially, however, the state-contingent approach makes the expected-utility hypothesis largely superfluous in considering productive decision-making under uncertainty. Just as the Arrow-Debreu proof of competitive equilibrium only requires the assumption of convex preferences, the state-contingent approach admits a very general specification of preferences that has the expected-utility model and many others as special cases. Quiggin and Chambers (1998b) show that concepts of constant absolute and relative risk aversion can be characterized independently of the expected-utility hypothesis.

In this paper, the problem of the firm under uncertainty is analyzed using a general state-contingent production technology. The paper is organized as follows. Section 1, which follows the analysis of Quiggin and Chambers (1998b), presents the model of preferences. Section 2 presents the state-contingent model of production. The general duality between production under uncertainty and choice under uncertainty, obscured by expected utility analysis of stochastic production function models, is made explicit. In particular, concepts of absolute and relative production risk premiums, analogous to the absolute and relative risk premiums familiar from the theory of choice under uncertainty, are defined and characterized..

The main results of the paper are presented in Section 3, which is an analysis of the choice problem for producers under uncertainty with general preferences. The crucial development is the concept of the efficient frontier, the set of state-contingent output and revenue vectors that are potentially optimal for some weakly risk-averse decision-maker. It is shown that, under constant absolute riskiness, all elements of the efficient frontier have the same production cost. It follows for such a technology that a risk-neutral producer will choose an output vector that yields a higher mean revenue than that chosen by a risk-averter, but that the risk-averter will view the risk-neutral optimal output as excessively risky. Our focus on the efficient frontier contrasts with the Sandmo model in which the only choice variable is a scalar output or effort variable. Nevertheless, in our final result, the crucial property of the Sandmo model, that risk-aversion results in a reduction in the scale of any given risk-taking activity, is extended to the more general framework. Thus, the state-contingent model represents a proper generalization of the Sandmo model with respect to technology as well as preferences.

2. Preferences

As in Quiggin and Chambers (1998b), we analyze general preferences of the form $W : Y^S \to \Re$, where $Y \subseteq \Re_+$ and $\Omega = \{1, 2, ..., S\}$ is a state space. Thus the analysis is concerned with preferences over state-contingent income vectors $\mathbf{y} \in \Re_+^S$. It is assumed that preferences are subjectively risk-averse in the sense that there exists a (not necessarily unique) vector $\pi \in \Re^S$, with $\sum_{s=1}^S \pi_s = 1$ and

$$W(\bar{y}\mathbf{1}_S) \geq W(\mathbf{y}), \forall \mathbf{y}$$

where $\bar{y} = \sum_{s=1}^S \pi_s y_s$, and $\bar{y}\mathbf{1}_S$ is the state-contingent outcome vector with $\bar{y}$ occurring in every state of nature. Thus, the elements of π may be considered as subjective probabilities. The special case of expected utility is

$$W(\mathbf{y}) = \sum_{s=1}^{S} \pi_s u(y_s)$$

where u is a von Neumann-Morgenstern utility function. The representation of preferences considered here is general enough to encompass not only expected utility, but also mean-variance preferences (Markowitz 1959), weighted utility (Chew 1983) rank-dependent expected utility (Quiggin 1982) and general smooth preferences (Machina 1982).

A canonical representation of preferences is given by the certainty equivalent

$$e(\mathbf{y}) = \min\{t : W(t\mathbf{1}_S) \geq W(\mathbf{y})\}$$

Quiggin and Chambers (1998b) define the absolute and relative risk premiums for any given $\mathbf{y}$

$$\begin{aligned} r(\mathbf{y}) &= \bar{y} - e(\mathbf{y}). \\ v(\mathbf{y}) &= \bar{y}/e(\mathbf{y}) \end{aligned}$$

and say that W displays *constant absolute risk aversion* (CARA) if, for all $\mathbf{y}$,

t, $r(\mathbf{y} + t\mathbf{1}) = r(\mathbf{y})$ or *constant relative risk aversion* (CRRA) if, for all $\mathbf{y}$, t, $v(t\mathbf{y}) = v(\mathbf{y})$.

To make the idea of risk-aversion more precise we introduce the notion of generalized Schur-concavity. As in Rothschild and Stiglitz (1970), we say that $\mathbf{y}'$ is a mean-preserving spread of $\mathbf{y}$ (denoted by $\mathbf{y} \preceq_{\pi} \mathbf{y}'$) if for all $y \in \Re$

$$\int_{-\infty}^{y} F_{\mathbf{y}}(t) dF_{\mathbf{y}}(t) \geq \int_{-\infty}^{y} F_{\mathbf{y'}}(t) dF_{\mathbf{y'}}(t)$$

where

$$F_{\mathbf{y}}(t) = Pr\{\mathbf{y} \leq t\}.$$

Define a certainty-equivalent function e to be *generalized Schur-concave* for π if and only if e: $\Re^S \to \Re$ satisfies:

$$\mathbf{y} \preceq_{\boldsymbol{\pi}} \mathbf{y}' \Rightarrow e(\mathbf{y}) \geq e(\mathbf{y}').$$

If e satisfies this condition we say that the individual has generalized Schur-concave preferences. Generalized Schur concavity thus encompasses all forms of preferences (including expected utility) which are risk-averse in our sense for the probabilities $\boldsymbol{\pi}$. The crucial property of generalized Schur-concavity, which is proven in Chambers and Quiggin (1997), for our purposes is:

Lemma 1 A smoothly differentiable certainty-equivalent function e is generalized Schur-concave for π only if

$$y_s \geq y_{s'} \Leftrightarrow e_s(\mathbf{y})/\pi_s \leq e_{s'}(\mathbf{y})/\pi_{s'} \quad \forall \mathbf{y}, s, s'$$

3. State contingent production

The idea that production under uncertainty may be represented simply as a special sort of multi-output production was first developed by Arrow and Debreu. To make this explicit, let $\mathbf{x} \in \Re_+^N$ be a vector of inputs committed prior to the resolution of uncertainty and let $\mathbf{z} \in \Re_+^{M \times S}$ be a vector of state-contingent outputs. So, if state $s \in \Omega$ is realized (picked by 'Nature'), the observed output is an M-dimensional vector $\mathbf{z}^s$, obtained as the projection of $\mathbf{z}$ onto $\Re_+^{M \times \{s\}}$.

Denote by $\mathbf{p} \in \Re_{++}^{M \times S}$ the matrix of state-contingent output prices and by $\mathbf{w} \in \Re_+^N$ the vector of input prices. When state s occurs the vector of s-contingent prices is denoted $\mathbf{p}^s$. In this paper, the price vector will be interpreted in an *ex post* sense, so that $\mathbf{p}^s$ is the set of spot prices that will prevail in the event that state s occurs. The state-contingent revenue vector $\mathbf{r} = \mathbf{pz} \in \Re_+^S$ has elements of the form $\mathbf{p}^s \bullet \mathbf{z}^s$. In all cases we consider, producers will be concerned with state-contingent revenue rather than output *per se*, and it is useful to consider the *revenue-cost function*

$$C\left(\mathbf{w}, \mathbf{r}, \mathbf{p}\right) = \min \left\{ \mathbf{w} \bullet \mathbf{x} : (\mathbf{x}, \mathbf{z}) \text{ is feasible}, \sum_m p_{ms} z_{ms} \geq r_s, s \in \Omega \right\}$$

if there exists a feasible state-contingent output array capable of producing $\mathbf{r}$ and ∞ otherwise. $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$ is homogeneous of degree 1 in $\mathbf{w}$ and homogeneous of degree 0 in $\mathbf{r}$ and $\mathbf{p}$. For analytic simplicity, we shall typically assume that $C\left(\mathbf{w}, \mathbf{r}, \mathbf{p}\right)$ is smoothly differentiable in all state-contingent revenues.

3.1. The certainty-equivalent revenue and the production-risk premium

Just as a risk-averse individual will pay a premium in each state to ensure the certainty outcome, achieving certainty in production may prove costly. The analogy between the theory of choice under uncertainty and the theory of production under uncertainty has not been developed much in the past, partly because of the restrictive focus on expected-utility preferences and stochastic production function technologies. As shown above, concepts of relative and absolute risk aversion apply for preferences represented by mappings from a space of state-contingent outcome vectors to the real line. Exactly analogous representations are available for the representation of the production technology by the revenue-cost function.

For the revenue-cost function, $C(\mathbf{w},\mathbf{r},\mathbf{p})$, and $\mathbf{r} \in \Re^S_+$, we define the *(cost) certainty equivalent revenue,* denoted by $e^c(\mathbf{r},\mathbf{p},\mathbf{w}) \in \Re_+$, as the maximum non-stochastic revenue that can be produced at cost $C(\mathbf{w},\mathbf{r},\mathbf{p})$, that is,

$$e^c(\mathbf{r},\mathbf{p},\mathbf{w}) = \sup\{e : C(\mathbf{w}, e\mathbf{1}^S, \mathbf{p}) \leq C(\mathbf{w},\mathbf{r},\mathbf{p})\},$$

where $\mathbf{1}^S$ is the S-dimensional unit vector. The certainty equivalent revenue satisfies:

$$\begin{aligned} e^c(\mathbf{r},\mathbf{p},\mu\mathbf{w}) &= e^c(\mathbf{r},\mathbf{p},\mathbf{w}) \\ e^c(\mu\mathbf{r},\mu\mathbf{p},\mathbf{w}) &= \mu e^c(\mathbf{r},\mathbf{p},\mathbf{w}), \mu > 0. \end{aligned}$$

By analogy with the risk premium used in the theory of choice under uncertainty, we define the *production-risk premium* as the difference between mean revenue and the certainty equivalent revenue. Notationally, letting $\bar{\mathbf{r}} \in \Re^S_+$ denote the vector with the mean of $\mathbf{r}$,

$$\bar{r} = \sum_s \pi_s r_s,$$

occurring in each state, then the production risk premium is defined by

$$p\left(\mathbf{r},\mathbf{p},\mathbf{w}\right)=\bar{r}-e^{c}\left(\mathbf{r},\mathbf{p},\mathbf{w}\right)$$

and satisfies:

$$\begin{aligned} C(\mathbf{w},\mathbf{r},\mathbf{p}) &= C(\mathbf{w},\bar{\mathbf{r}}-p(\mathbf{r},\mathbf{p},\mathbf{w})\,\mathbf{1^S},\mathbf{p}) \\ &= C(\mathbf{w},e^{c}\left(\mathbf{r},\mathbf{p},\mathbf{w}\right)\mathbf{1}^{S},\mathbf{p}). \\ p\left(\mathbf{r},\mathbf{p},\mu\mathbf{w}\right) &= p\left(\mathbf{r},\mathbf{p},\mathbf{w}\right) \\ p\left(\mu\mathbf{r},\mu\mathbf{p},\mathbf{w}\right) &= \mu p\left(\mathbf{r},\mathbf{p},\mathbf{w}\right), \quad \mu>0 \end{aligned}$$

A measure of the relative riskiness of the technology is given by the *relative production risk premium*

$$r\left(\mathbf{r},\mathbf{p},\mathbf{w}\right)=\frac{\bar{r}}{e^{c}\left(\mathbf{r},\mathbf{p},\mathbf{w}\right)}.$$

The technology will be called *inherently risky* if producing $\bar{\mathbf{r}}$ is more costly than producing $\mathbf{r}$ and *not inherently risky* if producing $\bar{\mathbf{r}}$ is less costly than producing $\mathbf{r}$.

By analogy to the concepts of constant absolute risk aversion and constant relative risk aversion for preferences, we define a state-contingent technology as displaying *constant absolute riskiness* if for all $\mathbf{r}, t \in \Re$:

$$p\left(\mathbf{r}+t\mathbf{1}^{S}, \mathbf{p}, \mathbf{w}\right) = p\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right).$$

and *constant relative riskiness* if for all $\mathbf{r}, t \in \Re_{+}$:

$$r\left(t\mathbf{r}, \mathbf{p}, \mathbf{w}\right) = r\left(\mathbf{r}, \mathbf{p}, \mathbf{w}\right).$$

Constant absolute riskiness implies

$$p\left(\mathbf{r}+t\mathbf{1}^{S}, \mathbf{p}\right) = p\left(\mathbf{r}, \mathbf{p}\right),$$

which by the definition of the production risk premium requires

$$\bar{r} + t - e^{c}\left(\mathbf{r}+t\mathbf{1}^{\mathbf{S}}, \mathbf{p}\right) = \bar{r} - e^{c}\left(\mathbf{r}, \mathbf{p}\right)$$

whence

$$e^{c}\left(\mathbf{r}+t\mathbf{1}^{\mathbf{S}}, \mathbf{p}\right) = e^{c}\left(\mathbf{r}, \mathbf{p}\right) + t.$$

This demonstrates:

Result 1 The technology displays constant absolute riskiness if and only if the revenue-cost function can be expressed as

$$C(\mathbf{r},\mathbf{p}) = \hat{C}\left(\mathbf{T}\left(\mathbf{r},\mathbf{p}\right),\mathbf{p}\right)$$

where

$$T\left(\mathbf{r}+\delta\mathbf{1}^{S},\mathbf{p}\right) = T\left(\ \mathbf{r},\mathbf{p}\right)+\delta, \quad \delta \in \Re,$$

$$T\left(\lambda\mathbf{r},\lambda\mathbf{p}\right) = \lambda T\left(\mathbf{r},\mathbf{p}\right), \qquad \lambda > 0,$$

and $\hat{C}\left(\mathbf{T}\left(\mathbf{r},\mathbf{p}\right),\mathbf{p}\right)$ is homogeneous of degree zero in $T\left(\mathbf{r},\mathbf{p},\mathbf{w}\right)$ and $\mathbf{p}$, non-decreasing in $T\left(\mathbf{r},\mathbf{p}\right)$, and non-increasing in $\mathbf{p}$. $T\left(\mathbf{r},\mathbf{p},\mathbf{w}\right)$ can be chosen to be nondecreasing and convex in the state-contingent revenues, and $\hat{C}$ can be chosen to be convex in $T\left(\mathbf{r},\mathbf{p},\mathbf{w}\right)$.

Similarly, we may show

Result 2 The technology displays constant relative riskiness if and only if the revenue-cost function can be expressed as

$$C(\mathbf{r},\mathbf{p}) = \bar{C}\left(\bar{\mathbf{T}}\left(\mathbf{r},\mathbf{p}\right),\mathbf{p}\right)$$

where

$$\bar{T}(\lambda \mathbf{r}, \mathbf{p}) = \lambda \bar{T}(\mathbf{r}, \mathbf{p}),$$

$$\bar{T}(\mathbf{r}, \lambda \mathbf{p}) = \bar{T}(\mathbf{r}, \mathbf{p}) \quad \lambda > 0,$$

and $\bar{C}\left(\bar{\mathbf{T}}(\mathbf{r}, \mathbf{p}), \mathbf{p}\right)$ is homogeneous of degree zero in $\bar{T}(\mathbf{r}, \mathbf{p})$ and output prices, non-decreasing in $\bar{T}(\mathbf{r}, \mathbf{p})$, and non-increasing in output prices. $\bar{T}(\mathbf{r}, \mathbf{p})$ can be chosen to be non-decreasing in the state-contingent revenues.

3.2. The net returns objective function

We focus on the case when $\mathbf{y}$ is a vector of net returns. Net returns for state s are given by

$$\begin{aligned} y_s &= \mathbf{p}^s \bullet \mathbf{z}^s - \mathbf{w} \bullet \mathbf{x} \\ &= r_s - C(\mathbf{w}, \mathbf{r}, \mathbf{p}). \end{aligned}$$

Hence

$$\mathbf{y} = \mathbf{r} - C(\mathbf{w}, \mathbf{r}, \mathbf{p})\mathbf{1}_S.$$

Using this notation, then the producer's objective function can be expressed as

$$e(\mathbf{y}) = e(\mathbf{r} - C(\mathbf{w}, \mathbf{r}, \mathbf{p})\mathbf{1}_S).$$

where the function e represents risk-averse preferences for some given probability vector $\boldsymbol{\pi}$.

4. The producer's choice problem

The producer's choice problem is:

$$max_{\mathbf{r}}\{e(\mathbf{r} - C(\mathbf{w}, \mathbf{r}, \mathbf{p})\mathbf{1}_S)\}$$

Assuming e is generalized Schur-concave for π, the producer's maximization problem is well-behaved. The first-order condition on r_s becomes:

$$e_s(\mathbf{r} - C(\mathbf{w}, \mathbf{r}, \mathbf{p})\mathbf{1}_S) - C_s(\mathbf{w}, \mathbf{r}, \mathbf{p}) \sum_{t \in \Omega} e_t(\mathbf{y}) \leq 0, \quad r_s \geq 0,$$

with complementary slackness.

Summing these first-order conditions yields an *arbitrage condition*

$$\sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p}) \geq 1 \tag{4.1}$$

To see why we refer to (4.1) as an arbitrage condition, notice that the left-hand side of the expression represents the directional derivative of the cost function in the direction of the equal-revenue ray , i.e.,

$$\sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p}) = \frac{\partial C(\mathbf{w},\mathbf{r}+\gamma\mathbf{1}^S,\mathbf{p})}{\partial\gamma} \,|_{\gamma=0} \, .$$

So, intuitively speaking, $\sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p})$ is the marginal cost of increasing all state-contingent revenues by the same small amount, and (4.1) simply requires that this cost be at least as large as the expected return. If it were not, it would obviously be profitable for the decision-maker to continue increasing each state-contingent revenue. For an interior solution, (4.1) must hold as an equality. A revenue vector $\mathbf{r}$ is optimal for some decision-maker if and only if (4.1) holds

We shall refer to the set of revenue vectors $\mathbf{r}$ satisfying (4.1) for given $\mathbf{w}$,$\mathbf{p}$ as the *efficient set* , denoted $\Xi(\mathbf{w},\mathbf{p})$,

$$\Xi(\mathbf{w},\mathbf{p}) = \left\{\mathbf{r} : \sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p}) \geq 1\right\}.$$

We call the boundary of $\Xi(\mathbf{w}, \mathbf{p})$ the *efficient frontier* and note that its elements are given by:

$$\bar{\Xi}(\mathbf{w}, \mathbf{p}) = \left\{ \mathbf{r} : \sum_{s \in \Omega} C_s(\mathbf{w}, \mathbf{r}, \mathbf{p}) = 1 \right\}.$$

Because the revenue-cost function is homogeneous of degree 1 in $\mathbf{w}$ and homogeneous of degree 0 in $\mathbf{r}$ and $\mathbf{p}$, it is homogeneous of degree 1 in $(\mathbf{w}, \mathbf{r}, \mathbf{p})$. Therefore differentiating both sides of

$$C(\theta\mathbf{w}, \theta\mathbf{r}, \theta\mathbf{p}) = \theta C(\mathbf{w}, \mathbf{r}, \mathbf{p}), \quad \theta > 0$$

with respect to r_s gives:

$$\theta C_s(\theta\mathbf{w}, \theta\mathbf{r}, \theta\mathbf{p}) = \theta C_s(\mathbf{w}, \mathbf{r}, \mathbf{p})$$

which implies that

$$C_s(\theta\mathbf{w}, \theta\mathbf{r}, \theta\mathbf{p}) = C_s(\mathbf{w}, \mathbf{r}, \mathbf{p}).$$

This homogeneity property of marginal cost allows us to establish the following property of the efficient set:

Lemma 1 $\Xi(\theta\mathbf{w}, \theta\mathbf{p}) = \theta\Xi(\mathbf{w}, \mathbf{p})$ and $\bar{\Xi}(\theta\mathbf{w}, \theta\mathbf{p}) = \theta\bar{\Xi}(\mathbf{w}, \mathbf{p})$, $\theta > 0$. That is, the efficient set and the efficient frontier are positively linearly homogeneous in input and output prices..

If (4.1) holds for an arbitrary revenue vector, $\hat{\mathbf{r}}$ say, then one can say that revenue vector is consistent with expected profit maximizing behavior for an individual with the subjective probabilities $\hat{\pi}_s = C_s(\mathbf{w}, \hat{\mathbf{r}}, \mathbf{p})$. The efficient frontier may be interpreted in terms of the *shadow probabilities* $\hat{\boldsymbol{\pi}}(\hat{\mathbf{r}})$, that is, the subjective probabilities a risk-neutral individual would have to hold in order for $\hat{\mathbf{r}}$ to be the optimal revenue vector. The correspondence of these shadow probabilities with these state-contingent marginal costs then determines the optimal point on the efficient set.

The efficient frontier is easily characterized under the presumption that the revenue cost function displays constant absolute riskiness. By Result 1, revenue cost in this case can be written

$$C(\mathbf{w}, \mathbf{r}, \mathbf{p}) = \hat{C}(\mathbf{w}, \mathbf{T}(\mathbf{r}, \mathbf{p}, \mathbf{w}), \mathbf{p})$$

where $T(\mathbf{r}, \mathbf{p}, \mathbf{w})$ is now interpretable as a *revenue aggregate*, which satisfies

$$T\left(\mathbf{r} + \delta \mathbf{1}^S, \mathbf{p}, \mathbf{w}\right) = T(\mathbf{r}, \mathbf{p}, \mathbf{w}) + \delta.$$

Differentiating this last expression with respect to δ and evaluating the expression at $\delta = 0$ gives:

$$\sum_{s\in\Omega} T_s(\mathbf{r},\mathbf{p},\mathbf{w}) = 1, \tag{4.2}$$

while differentiating with respect to r_s gives:

$$T_s\left(\mathbf{r} + \delta\mathbf{1}^S, \mathbf{p}, \mathbf{w}\right) = T_s\left(\mathbf{r},\mathbf{p},\mathbf{w}\right), \quad \forall\delta. \tag{4.3}$$

Substituting (4.2) into the definition of the efficient set gives:

$$\begin{aligned}
\Xi\left(\mathbf{w},\mathbf{p}\right) &= \left\{\mathbf{r} : \frac{\partial\hat{C}\left(\mathbf{w},\mathbf{T}\left(\mathbf{r},\mathbf{p},\mathbf{w}\right),\mathbf{p}\right)}{\partial T}\sum_{s\in\Omega} T_s(\mathbf{r},\mathbf{p},\mathbf{w}) \geq 1\right\} \\
&= \left\{\mathbf{r} : \frac{\partial\hat{C}\left(\mathbf{w},\mathbf{T}\left(\ \mathbf{r},\mathbf{p},\mathbf{w}\right),\mathbf{p}\right)}{\partial T} \geq 1\right\}.
\end{aligned}$$

This last expression tells us that the efficient frontier uniquely determines the level of the revenue aggregate, and thereby the revenue-cost level, if the revenue-cost function exhibits constant absolute riskiness. Summarizing results, we have:

Result 3 If the revenue-cost function exhibits constant absolute riskiness, all elements of the efficient frontier are equally costly.

Hence we are led to conclude that when there are interior solutions to the expected profit maximizing problem, then for fixed input and output prices all individuals possessing a technology exhibiting constant absolute riskiness will incur the same level of revenue-cost regardless of their preferences towards state-contingent outcomes.

The reason that this happens is transparent. When the revenue-cost function exhibits constant absolute riskiness, it can be written in terms of a single revenue aggregate $T(\mathbf{r}, \mathbf{p}, \mathbf{w})$. This revenue aggregate is positively linearly homogeneous in revenues and state-contingent prices, homogeneous of degree zero in input prices, and has the desirable property that when all revenues increase by the same amount the aggregate goes up by that same amount. Since the arbitrage condition that determines the efficient set requires that the marginal cost of increasing all revenues by the same amount equal one, then for such a technology, the arbitrage condition boils down to requiring that the marginal cost of the aggregate equal one.

Observe that condition (**??**) holds with equality for an interior solution even in the absence of differentiability of e, since, for any $\mathbf{r}$ that does not satisfy the condition, there exists an $\mathbf{r}'$ such that $\mathbf{y}' - \mathbf{y} \geq \mathbf{0}$ with strict inequality in at least one state. Pictorially, the production equilibrium is illustrated by a tangency between one of the producer's indifference curve and one of her isocost curves.

An immediate implication of (**??**) and Result 3 is that an individual with generalized Schur concave preferences and a technology exhibiting constant absolute riskiness will incur the same level of costs as a risk-neutral producer. Hence, we have:

Result 4 If the producer has generalized Schur-concave preferences and uses a technology exhibiting constant absolute riskiness in state-contingent revenues, then for an interior solution the producer incurs the same level of cost as a risk-neutral producer.

Result 4 can also be illustrated with the use of Figure 1. A person with generalized Schur-concave preferences and a technology with constant absolute riskiness will produce on the isocost curve between the bisector and the point of risk-neutral production. In particular, since a risk-neutral producer maximizes expected profits at given $\boldsymbol{\pi}$, any other producer operating on the efficient set with the same level of cost must have lower expected profit and, therefore, lower expected revenue. Therefore, under these conditions, the expected-profit maximizing vector of net returns must be riskier (that is, have a higher risk premium in terms of the risk averter's preferences) than the vector of net returns chosen by the risk-averse producer. To confirm this statement, let the optimal state-contingent revenue vector for the risk-neutral individual be denoted $\mathbf{r}^N$ and the optimal state-contingent revenue vector for the risk-averse individuals be denoted $\mathbf{r}^A$. The risk premium associated with $\mathbf{r}^N$ from the risk-averter's perspective is:

$$\sum_{s\in\Omega}\pi_s\left(r_s^N - C\left(\mathbf{w},\mathbf{r}^A,\mathbf{p}\right)\right) - e\left(\mathbf{y}^A\right) = \sum_{s\in\Omega}\pi_s\left(r_s^N - r_S^A\right) + \sum_{s\in\Omega}\pi_s r_S^A - e\left(\mathbf{y}^A\right).$$

The term on the right-hand side of the equality represents the difference between the risk-neutral individual's expected revenue and that for the risk-averse individual plus the risk-averse individual's risk premium for his state-contingent revenue vector. In terms of Figure 1, the difference between the two risk premiums can be visualized as the difference (not drawn) between where the fair-odds line through the risk-neutral individual's choice intersects the bisector and where the fair-odds line through the risk-averter's choice intersects the bisector. Summarizing, we have:

Result 5 If the producer has generalized Schur-concave preferences and uses a technology exhibiting constant absolute riskiness, then a risk-neutral individual using the same technology adopts a riskier state-contingent revenue vector (from the risk-averter's perspective) than the risk averter.

We have seen that an individual with generalized Schur-concave preferences using a technology exhibiting constant absolute riskiness will adopt a state-contingent revenue vector that is less risky than that for a risk-neutral individual. It is also straightforward to show that the optimal revenue vector will be more risky than that chosen by an individual with maximin preferences of the form:

$$e(\mathbf{y}) = \min\{y_s\}$$

Combining Lemma 4 with the first-order conditions for an interior solution shows that an optimally chosen state-contingent revenue vector must be *risk-aversely efficient* (in the sense of Peleg and Yaari, 1978) with respect to π:

$$r_s \geq r_t \Leftrightarrow C_s(\mathbf{w}, \mathbf{r},\ \mathbf{p})/\pi_s \leq C_t(\mathbf{w}, \mathbf{r}, \mathbf{p})/\pi_t,$$

or

$$\left(\frac{C_s(\mathbf{w}, \mathbf{r}, \mathbf{p})}{\pi_s} - \frac{C_t(\mathbf{w}, \mathbf{r}, \mathbf{p})}{\pi_t}\right)(r_s - r_t) \leq 0. \tag{4.4}$$

The notion of risk-averse efficiency is due to Peleg and Yaari and can be heuristically identified with the notion that for any state-contingent revenue vector satisfying it there will be some risk-averse individual, with subjective probabilities $\boldsymbol{\pi}$, who would optimally adopt that vector if she incurred the same level of revenue-cost.

Once again, the result may be extended to the case where e is not differentiable but still generalized Schur concave by using the observation that for any $\mathbf{r}$ that does not satisfy the condition, there exists $\mathbf{r}'$ such that $\mathbf{y}' \preceq_{\pi} \mathbf{y}$. We define the *risk-aversely efficient set for* π as consisting of those elements of the efficient set satisfying (4.4).

By complementary slackness, so long as the preference function is differentiable, we have:

$$\frac{\sum_{s\in\Omega} e_s(\mathbf{y})\, r_s}{\sum\limits_{s\in\Omega} e_s(\mathbf{y})} = \sum_{s\in\Omega} C_s(\mathbf{w},\mathbf{r},\mathbf{p}) r_s. \tag{4.5}$$

¿From Lemma 1, generalized Schur-concave preferences e satisfy:

$$\sum_{s\in\Omega} e_s(\mathbf{y}) \left(y_s - \sum_{s\in\Omega} \pi_s y_s \right) \leq 0.$$

Substituting $y_s = r_s - C(\mathbf{w},\mathbf{r},\mathbf{p})$, this last inequality implies:

$$\sum_{s\in\Omega} \pi_s r_s \geq \frac{\sum_{s\in\Omega} e_s(\mathbf{y})\, r_s}{\sum\limits_{s\in\Omega} e_s(\mathbf{y})} \tag{4.6}$$

which when combined with (4.6) establishes that:

$$\sum_{s\in\Omega} \pi_s r_s - \sum_{s\in\Omega} C_s(\mathbf{w},\ \mathbf{r}, \mathbf{p}) r_s \geq 0.$$

Direct calculation establishes that this last expression equals the marginal change in expected profit associated with a small radial expansion of the revenue vector. Because it is non-negative, we have:

Result 6 If $\mathbf{r}$ is a risk-aversely efficient revenue vector, a small radial expansion in $\mathbf{r}$ leads to an increase in expected profits.

An early analogue of Result 6 was first proved by Sandmo (1971) for the expected-utility model with a non-stochastic technology and stochastic prices. Much later, this result was generalized to the expected-utility model with state-contingent production by Chambers and Quiggin (1997).

A comparison of Result 6 with Results 4 and 5, shows a crucial difference between the analysis of a general state-contingent production technology and the special case of a stochastic production function. Result 6, when applied to the case of a stochastic production function with a scalar input (effort), implies that a risk-averse producer will always commit less effort than a risk-neutral producer. Similarly, in the Sandmo model of non-stochastic technology and stochastic prices, price stabilization or price insurance always generate an increase in output and costs. For a general state-contingent production technology a risk-averse producer will typically produce less, and, therefore, incur smaller production costs, than a risk-neutral producer constrained to choose a point on the same output ray. However, a risk-averse producer will allocate resources to reducing risk at the expense of a reduction in expected net returns. Result 4 shows that for technology displaying constant absolute riskiness these effects will cancel out as far as costs are concerned, so that the level of costs is determined solely by the arbitrage condition.

Now consider what happens when an individual uses a technology displaying constant relative riskiness. By the first-order conditions,

$$\frac{\sum_{s\in\Omega} e_s(\mathbf{y})\, r_s}{\sum\limits_{s\in\Omega} e_s(\mathbf{y})} = \frac{\partial \bar{C}\left(\mathbf{w},\bar{T}(\mathbf{r},\mathbf{p},\mathbf{w}),\mathbf{p}\right)}{\partial \bar{T}}\bar{T}(\mathbf{r},\mathbf{p},\mathbf{w}). \tag{4.7}$$

Expression (**??**), on the other hand, indicated that a risk-neutral individual using the same technology would choose expected revenue so that:

$$\frac{\partial \bar{C}\left(\mathbf{w},\bar{T}(\mathbf{r},\mathbf{p},\mathbf{w}),\mathbf{p}\right)}{\partial \bar{T}}\bar{T}(\mathbf{r},\mathbf{p},\mathbf{w}) = \sum_s \pi_s r_s. \tag{4.8}$$

Because $\bar{C}\left(\mathbf{w},\bar{T}\left(\mathbf{r},\mathbf{p},\mathbf{w}\right),\mathbf{p}\right)$ is convex in $\bar{T}$, a risk-averter will incur more effort cost than risk-neutral individual if and only if their directional derivative of $\bar{C}$ in the direction of $\bar{T}$ (the right-hand side of (4.7)) is greater than the left-hand side of (4.8). When this fact is used in conjunction with (4.6), we see that:

Result 7 If the producer has generalized Schur concave preferences and uses a technology exhibiting constant relative riskiness in state-contingent revenues, then the producer incurs a greater level of revenue cost than a risk-neutral producer only if her mean revenue exceeds the risk-neutral individual's mean revenue.

5. Concluding comments

Despite its shortcomings as a description of choice under uncertainty, the expected-utility model has remained dominant in the analysis of problems involving choice under uncertainty, largely because it has appeared more tractable than alternative models. In this paper we have shown, on the contrary, that reliance on the expected-utility model obscures the fundamental symmetry between producer and consumer choices. The state-contingent concepts of absolute and relative risk aversion developed by Quiggin and Chambers (1998b) have as their natural analogs, measures of the inherent riskiness or otherwise of state-contingent production technology. The expected-utility model is a special case of limited interest, corresponding to the class of production technologies characterized by additively separable cost functions.

Rather than focusing on the additive separability assumption of expected utility, it seems more profitable to consider conditions of constant and absolute risk aversion such as those developed by Quiggin and Chambers (1998b) and possible generalizations such as the class of quasi-homothetic or affinely homothetic preferences.

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