

Production under Uncertainty and Choice under Uncertainty in the Emergence of Generalized Expected Utility Theory

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Abstract

This paper presents a personal view of the interaction between the analysis of choice under uncertainty and the analysis of production under uncertainty. Interest in the foundations of the theory of choice under uncertainty were stimulated by applications of expected utility theory such as the Sandmo model of production under uncertainty. This interest led to the development of generalized models including rank-dependent expected utility theory. In turn, the development of generalized expected utility models raised the question of whether such models could be used in the analysis of applied problems such as those involving production under uncertainty. Finally, the revival of the state-contingent approach led to the recognition of a fundamental duality between choice problems and production problems.

The field of generalized expected utility theory had its beginnings with the classic paper of Allais (1953). Although this paper was well-known, and frequently cited, over the next three decades, it was not until the late 1970s that Allais' critique of expected utility (EU) theory became the basis of a substantial research program. The state-contingent approach to production under uncertainty has had a rather similar history.

When economists were first embarking on the study of problems involving uncertainty, Arrow (1953) and Debreu (1952) developed the elegantly simple idea of state-contingent commodities. The project of developing a rigorous general equilibrium theory had already led to the notion of differentiating commodities by their time and place of delivery. It was a relatively

small step to deal with uncertainty through the notion of state-contingent commodities; that is, commodities whose delivery is contingent on the occurrence of a particular state of nature. Once this connection was made, all the tools developed for a non-stochastic world could be applied almost effortlessly to decision-making under uncertainty. In Debreu's words, the notion of a state-contingent commodity "... allows one to obtain a theory of uncertainty free from any probability concept and formally identical with the theory of certainty ..." (p. 98). Yaari (1969) developed this point further showing how notions of comparative risk aversion could be developed in a purely state-contingent framework, without any necessary reliance on probability distributions.

This should have been welcome news to economists. It means that tools honed in other areas can be used to analyze decision-making under uncertainty. Even more force was added to the argument by Hirshleifer's (1965) demonstration of the analogy between the insights obtained from a state-contingent interpretation of uncertainty and the way in which Fisher's (1930) treatment of time-preference had demystified the concept of production and consumption over time. Unfortunately, except in fairly restricted areas of economic theory, this pathbreaking insight was ignored in the analysis of production decisions under uncertainty.

Instead, initial analyses of the firm under uncertainty were largely undertaken using what was, in effect, a stochastic production function approach. This approach appeared to be simpler than the general state-contingent approach and seemed to allow exploitation of the analogy between production problems involving uncertainty and portfolio allocation problems. In fact, however, the apparent simplicity of the approach can be preserved only if attention is restricted to scalar choice sets.

Moreover, the analytical tools applied to the problem of the firm under uncertainty were derived under the assumption of expected-utility maximization.. A partial extension to the case of rank-dependent expected utility was possible (Quiggin 1991), but attempts to undertake comparative static analysis in any more general context encountered significant difficulties (Machina 1989). However, in recent work (Chambers and Quiggin 2000; Quiggin and Chambers 2000) it has been shown that the expected utility hypothesis is largely redundant in the analysis of state-contingent production.

This paper presents a personal view of the interaction between the analysis of choice under uncertainty and the analysis of production under uncer-

tainty. The paper is organised as follows. The starting point is successful application of the expected utility model to the analysis of the firm under uncertainty by Sandmo (1971) and others, which led to renewed concern about the criticisms of expected utility put forward by Allais in the 1980s.. The next section describes the development of generalized models including rank-dependent expected utility (RDEU) theory, from the viewpoint of a participant. The development of generalized expected utility models raised the question of whether such models could be used in the analysis of applied problems such as those involving production under uncertainty. The final section of the paper describes how the revival of the state-contingent approach of Arrow and Debreu permitted advances in this field. In particular, adoption of the state-contingent approach led to the recognition of a fundamental duality between choice problems and production problems.

1 The Sandmo model

The first formal treatment of production under uncertainty was the general equilibrium analysis put forward independently by Arrow (1953) and Debreu (1952). Using the notion of state-contingent commodities, Arrow and Debreu showed that the proof of the existence of competitive equilibrium, based on the Kakutani fixed-point theorem carried over, in a formal sense, to the case of uncertainty. The Arrow–Debreu analysis forms the basis of modern finance theory. However, as far as analysis of the behavior of firms under uncertainty is concerned, little progress was made. Tobin’s (1969) assessment that state-preference theory was ‘graceful but empty’ remained apposite three decades later

Microeconomic analysis of production under uncertainty began with Sandmo (1971), who drew on another multiple discovery, that of the Arrow–Pratt measure of risk aversion (Pratt 1964; Arrow 1965). Sandmo considered the simple case of a firm producing a single good, here denoted z , at a cost $C(z)$ and facing a random, competitively determined price p . The owner of the firm is a risk-averse expected utility maximizer who seeks to maximize the objective function

$$\max_z W = E[u(pz - C(z))].$$

It is easy to see that, in the linear case where $C(z) = cz$, this problem is exactly analogous to the two-asset portfolio problem, where the safe asset

returns zero and the risky asset yields the random return $(p - c)$. Sandmo extended a number of the results known to hold for the two-asset portfolio problem to the more general case of the firm under price uncertainty. In particular, he showed that given decreasing absolute risk aversion, an increase in mean price would result in an increase in output.

Equally importantly, Sandmo left a number of loose ends and conjectures to stimulate subsequent researchers. Coes (1977) confirmed Sandmo's conjecture that, assuming decreasing absolute risk aversion, a multiplicative increase in price risk would lead to a reduction in output. Quiggin (1982a) derived conditions for the existence of a finite optimum (Most of these conditions had been established earlier, for a more general formulation of the decision problem, by Bertsekas (1974).)

Other writers generalized Sandmo's analysis. An easy generalization is to replace p by a random yield variable θ and replace z by an input variable x , such as the area of land devoted to a crop, and reinterpret the model as one in which the firm faces technological uncertainty. A more substantial change is to allow for a general stochastic production function $f(x, \theta)$ where θ can be interpreted as an input contributed by 'Nature' such as rainfall. The objective function therefore becomes:

$$\max_x W = E[u(pf(x, \theta) - C(x))].$$

Feder (1977) analyzed the comparative static problems of choice problems involving such general objective functions.¹

2 Generalized expected utility theory

Although it yielded useful insights into the problem of production under uncertainty, the Sandmo model relied on very restrictive assumptions. Most obviously, whereas the Arrow–Debreu model merely required that preferences over state-contingent commodities be monotonic, convex and continuous, the Sandmo model relied crucially on the assumption of expected-utility maximization.. Over the course of the 1970s, the expected-utility hypothesis came under increasing attack. The long-neglected and much-misinterpreted criticisms of Allais (1953) were reinforced by new empirical

¹Similarly, Dardanoni (1988) extended Sandmo's analysis of the consumption–savings problem to the general case of a two-argument objective function.

evidence, which demonstrated the robustness of the ‘Allais paradox’. A number of attempts were made to develop generalizations of expected utility theory that could account for the Allais paradox and related phenomena such as the common ratio effect.

One approach which fell into the category of ‘fruitful error’ was that of Handa (1977), who proposed a model based on the idea of probability weighting. Unfortunately, Handa’s model implied violation of first-order stochastic dominance, whereas the proposed axioms included preservation of first-order stochastic dominance. The *Journal of Political Economy* (JPE) was deluged with comments pointing out this error, of which the most elegant, and the only one published, was that of Fishburn (1978). Many of those who followed this debate concluded that the search for an alternative to expected utility was futile. Certainly, the JPE has been noticeably reticent in publishing work in the field of generalized expected utility. On the other hand, Fishburn became a prominent contributor to the field and the author of one of the first monographs (Fishburn 1988).

Handa’s paper and Fishburn’s response are fairly well-known. Less well-known is the fact that among the unpublished refutations of Handa was one contributed by Mark Machina, then a graduate student at MIT. Even less well-known is the effect on the work of Quiggin (1979), then a civil servant with the Bureau of Agricultural Economics (BAE) in Canberra, Australia, engaged in part-time study towards an undergraduate economics degree, with an Honours thesis focusing on the Sandmo model. Having read the relevant issue of the JPE some weeks after its arrival in Australia, I recognized the error in Handa’s paper and set about preparing a comment (By this time, comments from North American readers were already arriving in Chicago). However, the comment had barely been typed up when the next issue of the JPE, containing Fishburn’s much more comprehensive and elegant critique, reached Australia and made its way to the BAE Library. Work on the comment was abandoned.

Hope revived, however, when I realized that Handa’s basic idea could be salvaged if overweighting was confined to low-probability extreme events, rather than to low-probability events in general. As was observed in Quiggin (1982b):

The following example illustrates further the notion that equally probable events should not necessarily receive the same weight. Suppose an individual’s normal wage income is uniformly dis-

tributed over a range from \$20 000.01 to \$21 000.00. There is also a 1/100 000 chance that the person will win a contest for which the prize is a job paying \$1m a year. The probability of receiving any specified income in the relevant range \$20 000.01 to \$21 000.00 (e.g. \$21 439.72) is also 1/100 000. Nevertheless, it seems reasonable that the extreme outcome will not be weighted in the same way as an intermediate outcome such as \$21 439.72.

This simple idea led to the anticipated utility (AU) or rank-dependent expected utility (RDEU) model². As the name implies, the RDEU model is applied to the increasing rearrangement \mathbf{y}_{\square} of \mathbf{y} which satisfies $y_{[1]} \leq y_{[2]} \leq \dots \leq y_{[s]}$.

$$W(\mathbf{y}) = \sum_{s \in \Omega} h_{[s]}(\boldsymbol{\pi}) u(y_{[s]})$$

where $\boldsymbol{\pi} \in \Pi$, $u : \mathfrak{R} \rightarrow \mathfrak{R}$, and $h_{[s]}(\boldsymbol{\pi})$ is a probability weight such that

$$h_{[s]}(\boldsymbol{\pi}) = q\left(\sum_{t=1}^s \pi_{[t]}\right) - q\left(\sum_{t=1}^{s-1} \pi_{[t]}\right)$$

for a transformation function³ $q : [0, 1] \rightarrow [0, 1]$ with $q(0) = 0$, $q(1) = 1$. Note that

$$\sum_{s \in \Omega} h_{[s]}(\boldsymbol{\pi}) = q\left(\sum_{t=1}^S \pi_{[t]}\right) = q(1) = 1$$

²The term ‘anticipated utility’ was simply intended to mean ‘generalized expected utility’, since ‘anticipated’ is a more general synonym of ‘expected’. At the time the term ‘anticipated utility’ was coined, I assumed that no other generalized expected utility model, consistent with first-order stochastic dominance, was possible. In fact, this is only true if a functional separation between probabilities and utilities is required. When it became apparent that ‘anticipated utility’ was only one of many ‘generalized expected utility’ models, I adopted the term ‘rank-dependent expected utility’ proposed by Chew and Epstein (1989).

³Quiggin (1979,1982b) took the arguably mistaken view that a model of choice under uncertainty required an axiomatic foundation. The search for such a set of axioms that would generate the AU model led to the imposition of the requirement $q(1/2)=1/2$, dropped in all subsequent formulations of the rank-dependent model. Even with this restriction, the derivation of the AU model from the axioms was incorrect. A more accurate statement is given by Quiggin and Wakker (1994).

so that the decision weights sum to 1.

A derivation of this model and proof of its consistency with first-order stochastic dominance formed an appendix to my Honours thesis (Quiggin 1979). A naively optimistic view of the self-correcting nature of science was quickly revised when JPE rejected the resulting paper on the grounds that it was ‘purely theoretical’. The paper was finally published as Quiggin (1982b). An application to Australian data was written later, but published earlier, as Quiggin (1981).

This work attracted little attention for some years. As a result, the functional form associated with the rank-dependent model was independently rediscovered on several separate occasions, with quite different motivating arguments. Thus, the model represents an interesting example of the theory of multiple discoveries, put forward by Merton (1973) and discussed in an economic context by Stigler (1982).

First, there was the ‘dual model’ of Yaari (1987), developed further by Roell (1987). The motivation here was mainly axiomatic. Yaari exploited a number of formal symmetries between EU and a special case of RDEU in which the utility function is linear. The linearity of the utility function implied that, in the dual model, risk attitudes are entirely divorced from declining marginal utility of wealth.

A second rediscovery, also based on the notion of risk aversion as a phenomenon distinct from declining marginal utility of wealth, was made by Allais (1987, 1988) whose work had provided the starting point for the whole field. Most advocates of the EU approach have treated cardinal utility as a convenient instrument for modelling uncertain choice, but have denied that there is any association with the utilitarian assumption of decreasing marginal utility of wealth. By contrast, Allais has always insisted on the psychological reality of cardinal utility. It is the desire to incorporate an explicit cardinal utility function that led Allais to his formulation of RDEU.

Another closely related development is the work of Schmeidler (1989) and Gilboa (1987), based on the notion of non-additive measures or capacities, originally developed in the work of Choquet (1953–4). The problem here is taken to be one of ambiguity, in that the probabilities of the different states are unknown.

In addition to these developments in the theory of choice under uncertainty, the RDEU functional form (usually with a linear utility function, as in the work of Yaari 1987) has been applied to problems of social choice and income distribution, with distributions of welfare across individuals taking

the place of random variables over states. Examples include Weymark (1981) and Ebert (1988).

3 The state-contingent and parameterized distribution frameworks

The multiple discoveries of RDEU illustrate a more general point. Many different, but logically equivalent, formulations of a given model may be available and different formulations will lead in different directions. Most interestingly, consider the objects of choice in decision theory. Much analysis, including early formulations of the anticipated utility model, deals with choice over families of probability distributions or, more generally, cumulative distribution functions, that is, monotone increasing mappings from an ordered outcome space Y to the unit interval $[0,1]$. In the typical case where these families are indexed by a single parameter or a small set of parameters, this approach may be referred to as the parametrized distribution representation of choice under uncertainty. The stochastic production function model, in which each choice of input x is associated with the random output $f(x, \theta)$ is an example of the parametrized distribution representation.

In the parametrized distribution representations, the probabilities associated with the different states are treated as objectively given (although amenable to manipulation by altering the action or input vector). The critical question is: How can these objective probabilities be defined and observed?

Of the many answers that have been offered to this question, the most plausible, considered as a basis for an objective notion of probability, is based on relative frequency. The frequentist answer restricts the definition of probability to situations that may be regarded as draws from an infinite sample and defines the probability of any event as the limit to which its frequency of occurrence converges in an infinite series of draws. Apart from logical difficulties associated with this definition, it cannot be usefully applied to a wide range of economic problems involving uncertain choices in situations that are unique or will only be repeated a few times.

An alternative view, which may be traced back to the work of de Finetti (1931) and Savage (1954), is that probabilities used in economic decision-

making are inherently subjective. Consequently, probabilities can only be inferred from the observed behavior of decision-makers, and hence they are inseparably tied to beliefs and preferences. A logical consequence of this position is the recognition that any statement about subjective probabilities is ultimately a statement about the decision-maker's beliefs and preferences. Therefore, intermingling probabilities with the technology of production, as is done in the parametrized distribution formulation, represents an important logical confusion.

This view leads to the adoption of a state-contingent representation of choice under uncertainty, in which the primitive objects of choice are acts, regarded as mappings from a set of states of nature to a set of outcomes. More formally, we are concerned with preferences over state-contingent outcome vectors, represented as mappings from a state space Ω to an outcome space $Y \subseteq \mathfrak{R}$. Preferences over Y^Ω are given by a total ordering denoted notationally by \succ . A preference function is a mapping $W : Y^S \rightarrow \mathfrak{R}$ such that $W(\mathbf{y}) > W(\mathbf{y}')$ if and only if $\mathbf{y} \succ \mathbf{y}'$. W is assumed everywhere continuous and nondecreasing.

3.1 Probabilistic sophistication

Under appropriate conditions, observed choices will be consistent with preferences that are probabilistically sophisticated, that is, display first-order stochastic dominance with respect to a given set of subjective probabilities. For example, given two partitions of the state space into events $\{E_{11}, E_{12}\}$ and $\{E_{21}, E_{22}\}$ and two payoffs $H > L$ (that is, H with certainty is preferred to L with certainty), suppose that the act yielding payoff H under E_{11} and L under E_{12} is indifferent to the act yielding payoff H under E_{21} and L under E_{22} and that both these acts are preferred to the acts under which the outcomes are swapped, which in turn are indifferent to each other. These choices are consistent with subjective probabilities such that

$$p(E_{11}) = p(E_{21}) > p(E_{12}) = p(E_{22}) \tag{1}$$

Given a sufficiently rich set of observations, it is possible to identify a unique probability distribution consistent with observed choices or alternatively to determine that no such distribution exists.

Savage (1954) presented axioms which simultaneously implied the existence of well-defined subjective probabilities and the existence of a utility

function (unique up to an affine transformation) such that the preferred element of any choice set was that which maximized expected utility. Machina and Schmeidler (1992) showed that the link with expected utility was not logically necessary. They derived conditions under which observed preferences are probabilistically sophisticated, whether or not they are consistent with expected-utility maximization..

Since the state-contingent framework is capable of incorporating preferences that may or may not be probabilistically sophisticated, it is more general than the parameterized distribution framework. Over time, it has become apparent that the state-contingent approach is analytically more tractable, and that the existence of well-defined subjective probabilities is most appropriately regarded as a restriction on the functional form of a preference representation, analogous to additive separability in preferences over non-stochastic consumption bundles.

3.2 Probabilities derived from risk aversion

The basic definition of risk aversion is a preference for certainty over risk, usually interpreted to mean that a random variable \mathbf{y} is less preferred than the certainty of receiving the expected value $E[\mathbf{y}]$. Thus far in this paper, however, the expected value has not been defined and it has not been assumed that the decision-maker is probabilistically sophisticated.

The most common approach to these issues is to derive probabilities from the Savage axioms, then define the expected value and risk aversion in terms of probabilities. In the present paper, following the approach developed by Yaari (1969), the corresponding concepts are defined simultaneously.

Definition 1: A decision-maker is weakly risk-averse if there exists a vector $\pi \in \mathfrak{R}_+^S$, with $\sum \pi_i = 1$ and

$$W(E_\pi[\mathbf{y}]\mathbf{1}) \geq W(\mathbf{y}), \quad \forall \mathbf{y},$$

where $E_\pi[\mathbf{y}] = \sum \pi_i y_i$, and $E_\pi[\mathbf{y}]\mathbf{1}$ is the state-contingent outcome vector with $E_\pi[\mathbf{y}]$ occurring in every state of nature. A decision-maker is risk-neutral if equality holds for every \mathbf{y} , and risk-averse otherwise. A decision-maker is strictly risk-averse if strict inequality holds whenever $\mathbf{y} \neq E_\pi[\mathbf{y}]\mathbf{1}$.

4 State-contingent production

In the state-contingent framework, a production decision may be regarded as an act which requires a non-stochastic input x , and yields an output z in every state of nature. It is straightforward to reinterpret the stochastic production function model in state-contingent terms. If the state space is S , the set of acts may be identified with the set of possible input choices x , each of which produces the outcome (x, \mathbf{z}) where $\mathbf{z} \in \mathfrak{R}^S$, and z_s is the output realized if state $s \in S$ occurs. It is easy to see that this is analogous to a multi-output production function. A deeper insight, due to Arrow (1953) and Debreu (1952), is that the modern approach to producer and consumer theory, based on sets and correspondences rather than functions, may be applied without modification, to problems involving uncertainty.

However, until the development of the modern axiomatic approach to production analysis (Shephard 1970; McFadden 1978), few tools were available to permit the use of set-theoretic representations of production technologies in the analysis of firm behavior. Consequently, most applied analysis of firm behavior was based on the older idea of a production function. And when issues relating to firm-level stochastic production began to be seriously considered by theorists, they naturally based their analysis on the related notion of a stochastic production function. The widespread success of the axiomatic approach in creating the superstructure of duality theory and its many applications, however, suggests the possibility of an extension of this analysis to a state-contingent production technology as in the work of Arrow and Debreu.

The groundwork for such an extension is presented by Chambers and Quiggin (2000). Chambers and Quiggin represent the production technology in the form of an input correspondence which maps matrices of state-contingent outputs into sets of inputs that are capable of producing that state-contingent output matrix. Formally, it is defined by:

$$X(\mathbf{z}) = \{\mathbf{x} \in \mathfrak{R}_+^N : \mathbf{x} \text{ can produce } \mathbf{z} \in \mathfrak{R}_+^{M \times S}\}.$$

Intuitively, $X(\mathbf{z})$, typically referred to as the input set, is identified with everything on or above an isoquant for the state-contingent technology.

Denote by $\mathbf{p} \in \mathfrak{R}_{++}^{M \times S}$ the matrix of state-contingent output prices. When s occurs the vector of s -contingent prices is denoted \mathbf{p}^s . The state-contingent revenue vector $\mathbf{r} = \mathbf{p}\mathbf{z} \in \mathfrak{R}_+^S$ has elements of the form $\mathbf{p}^s \bullet \mathbf{z}^s$. Producers

are normally concerned with state-contingent revenue rather than output *per se*, and it is useful to consider the revenue-cost function

$$C(\mathbf{r}; \mathbf{w}, \mathbf{p}) = \min \left\{ \mathbf{w} \bullet \mathbf{x} : \mathbf{x} \in X(\mathbf{z}), \sum_m p_{ms} z_{ms} \geq r_s, s \in \Omega \right\}$$

if there exists a feasible state-contingent output array capable of producing \mathbf{r} and ∞ otherwise.

A trivial but crucial observation is that, for given \mathbf{w} and \mathbf{p} , the revenue-cost function is, like the preference function W , a mapping from \mathfrak{R}^S to \mathfrak{R} . Hence, for any property of the preference function, an analogous property may be defined for the revenue-cost function, and *vice versa*. The analysis of risk premiums yields some important applications of this observation.

5 Risk premiums in choice and production

Once the analogy between state-contingent income or output vectors and multi-commodity consumption or production bundles is recognized, a wide range of analytical tools become available. In particular, the concept of a risk premium, which has proved valuable in analysis using the expected-utility model, may be generalized using Luenberger's (1992) benefit function for the preference structure. The benefit function, $B : \mathfrak{R} \times Y^S \rightarrow \mathfrak{R}$, is defined for $\mathbf{g} \in \mathfrak{R}^S$ by:

$$B(w, \mathbf{y}) = \max\{\beta \in \mathfrak{R} : W(\mathbf{y} - \beta \mathbf{1}) \geq w\}$$

if $W(\mathbf{y} - \beta \mathbf{1}) \geq w$ for some β , and $-\infty$ otherwise. Similarly, concepts of relative risk aversion may be analyzed using the Shephard (1953)–Malmquist (1953) distance function $D : \mathfrak{R}_+^S \times \mathfrak{R} \rightarrow \mathfrak{R}_+$ defined by:

$$D(\mathbf{y}, w) = \sup\{\lambda > 0 : W(\mathbf{y}/\lambda) \geq w\} \quad y \in \mathfrak{R}_+^S .$$

For an individual risk-averse with respect to π , define the absolute risk premium:

$$\begin{aligned} r_\pi(\mathbf{y}) &= \max\{c : W((E_\pi[\mathbf{y}] - c)\mathbf{1}) \geq W(\mathbf{y})\} \\ &= B(W(\mathbf{y}), E_\pi[\mathbf{y}]\mathbf{1}), \end{aligned}$$

and the relative risk premium:

$$\begin{aligned} v_\pi(\mathbf{y}) &= \sup\{\lambda > 0 : W(E_\pi[\mathbf{y}]\mathbf{1}/\lambda) \geq W(\mathbf{y})\} \\ &= D(E_\pi[\mathbf{y}]\mathbf{1}, W(\mathbf{y})). \end{aligned}$$

For any \mathbf{y} and W , define the certainty equivalent:

$$e(\mathbf{y}) = \inf\{c > 0 : W(c\mathbf{1}) \geq W(\mathbf{y})\},$$

and observe:

$$\begin{aligned} r_\pi(\mathbf{y}) &= E_\pi[\mathbf{y}] + B(W(\mathbf{y}), \mathbf{0}) \\ &= E_\pi[\mathbf{y}] - e(\mathbf{y}). \end{aligned}$$

$$\begin{aligned} v_\pi(\mathbf{y}) &= E_\pi[\mathbf{y}]D(\mathbf{1}, W(\mathbf{y})) \\ &= E_\pi[\mathbf{y}]/e(\mathbf{y}). \end{aligned}$$

5.1 Production risk

An exactly dual analysis applies in the case of production risk. Just as a risk-averse individual will pay a premium in each state to ensure the certainty outcome, achieving the certainty outcome may prove costly. That is, typically it should cost more to remove production uncertainty and produce the same non-stochastic output in each state than to allow for stochastic production.

For the revenue-cost function, $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$, and $\mathbf{r} \in \mathfrak{R}_+^S$, define the (cost) certainty equivalent revenue, denoted by $e^c(\mathbf{r}, \mathbf{p}, \mathbf{w}) \in \mathfrak{R}_+$, as the maximum non-stochastic revenue that can be produced at cost $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$, that is,

$$e^c(\mathbf{r}, \mathbf{p}, \mathbf{w}) = \sup\{e : C(\mathbf{w}, e\mathbf{1}^S, \mathbf{p}) \leq C(\mathbf{w}, \mathbf{r}, \mathbf{p})\},$$

where $\mathbf{1}^S$ is the S -dimensional unit vector. By analogy with the risk premium used in the theory of consumer choice, define the production-risk premium as the difference between mean revenue and the certainty equivalent revenue. Notationally, letting $\bar{\mathbf{r}} \in \mathfrak{R}_+^S$ denote the vector with the mean of \mathbf{r} ,

$$\bar{r} = \sum_k \pi_k r_k,$$

occurring in each state, then the production risk premium is defined by

$$p(\mathbf{r}, \mathbf{p}, \mathbf{w}) = \bar{r} - e^c(\mathbf{r}, \mathbf{p}, \mathbf{w})$$

and satisfies:

$$\begin{aligned} C(\mathbf{w}, \mathbf{r}, \mathbf{p}) &= C(\mathbf{w}, \bar{r} - p(\mathbf{r}, \mathbf{p}, \mathbf{w}) \mathbf{1}^S, \mathbf{p}) \\ &= C(\mathbf{w}, e^c(\mathbf{r}, \mathbf{p}, \mathbf{w}) \mathbf{1}^S, \mathbf{p}). \end{aligned}$$

The technology will be called inherently risky if producing $\bar{\mathbf{r}}$ is more costly than producing \mathbf{r} and not inherently risky if producing $\bar{\mathbf{r}}$ is less costly than producing \mathbf{r} . The technology is inherently risky at \mathbf{r} if and only if $p(\mathbf{r}, \mathbf{p}, \mathbf{w})$ is positive, or equivalently if and only if the certainty equivalent revenue is no greater than the mean. Both imply that producing $\bar{\mathbf{r}}$ is more costly than producing the stochastic, \mathbf{r} , so that there are costs to removing uncertainty. This seems the natural state of affairs. However, $p(\mathbf{r}, \mathbf{p}, \mathbf{w})$ may be negative, implying that certainty is less costly than the stochastic output vector, and in this case the technology is not inherently risky at \mathbf{r} . The certainty equivalent revenue and the production risk premium are alternative characterizations of the technology. Formally, this can be verified by noting that the certainty equivalent revenue is a nondecreasing transformation of revenue-cost.

5.2 Constant risk aversion

The benefit and distance functions yield simple characterizations of constant absolute and relative risk aversion for general preferences. Chambers and Quiggin (2000) define:

Definition 2: W displays constant absolute risk aversion (CARA) if, for all \mathbf{y}, t , $r(\mathbf{y} + t\mathbf{1}) = r(\mathbf{y})$.

Definition 3: W displays constant relative risk aversion (CRRA) if, for all \mathbf{y}, t , $v(t\mathbf{y}) = v(\mathbf{y})$.

Dually, define a state-contingent technology as displaying constant absolute riskiness (CAR) if for all $\mathbf{r}, t \in \mathfrak{R}$:

$$p(\mathbf{r} + t\mathbf{1}^S, \mathbf{p}, \mathbf{w}) = p(\mathbf{r}, \mathbf{p}, \mathbf{w}).$$

and constant relative riskiness (CRR) if for all $\mathbf{r}, t \in \mathfrak{R}_+$:

$$r(t\mathbf{r}, \mathbf{p}, \mathbf{w}) = r(\mathbf{r}, \mathbf{p}, \mathbf{w}).$$

Geometrically, if a revenue-cost function displays constant absolute riskiness, rays parallel to the equal-revenue ray will cut successive isocost contours for the revenue-cost function at points of equal slope. If a revenue-cost function displays constant relative riskiness, rays from the origin will cut successive isocost contours for the revenue-cost function at points of equal slope.

A preference function is defined as displaying constant risk aversion if it displays both CARA and CRRA. Similarly a technology displays constant riskiness if it displays both CAR and CRR.

6 The production problem with constant risk aversion

Suppose that the states of nature are given by the set $\Omega = \{1, 2, \dots, S\}$. Let $\mathbf{x} \in \mathfrak{R}_+^N$ be a vector of inputs committed prior to the resolution of uncertainty, and let $\mathbf{z} \in \mathfrak{R}_+^{M \times S}$ be a vector of state-contingent outputs. So, if state $s \in \Omega$ is realized (picked by ‘Nature’), the observed output is an M -dimensional vector \mathbf{z}^s , obtained as the projection of \mathbf{z} onto $\mathfrak{R}_+^{M \times \{s\}}$.

Denote by $\mathbf{p} \in \mathfrak{R}_{++}^{M \times S}$ the matrix of state-contingent output prices and by $\mathbf{w} \in \mathfrak{R}_+^N$ the vector of input prices. When state s occurs the vector of s -contingent prices is denoted \mathbf{p}^s . In this paper, the price vector will be interpreted in an *ex post* sense, so that \mathbf{p}^s is the set of spot prices that will prevail in the event that state s occurs. The state-contingent revenue vector $\mathbf{r} = \mathbf{p}\mathbf{z} \in \mathfrak{R}_+^S$ has elements of the form $\mathbf{p}^s \bullet \mathbf{z}^s$. In general, producers will be concerned with state-contingent revenue rather than output *per se*, and it is useful to consider the *revenue-cost function*

$$C(\mathbf{w}, \mathbf{r}, \mathbf{p}) = \min \left\{ \mathbf{w} \bullet \mathbf{x} : (\mathbf{x}, \mathbf{z}) \text{ is feasible, } \sum_m p_{m,s} z_{m,s} \geq r_s, s \in \Omega \right\}$$

if there exists a feasible state-contingent output array capable of producing \mathbf{r} and ∞ otherwise. $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$ is homogeneous of degree 1 in \mathbf{w} and homogeneous of degree 0 in \mathbf{r} and \mathbf{p} . For analytic simplicity, it is assumed that $C(\mathbf{w}, \mathbf{r}, \mathbf{p})$ is smoothly differentiable in all state-contingent revenues.

Quiggin and Chambers (1998) analyze general preferences of the form $W : Y^S \rightarrow \mathfrak{R}$, where $Y \subseteq \mathfrak{R}_+$. Thus the analysis is concerned with preferences over state-contingent income vectors $\mathbf{y} \in \mathfrak{R}_+^S$. We focus on the case when \mathbf{y} is a vector of net returns. Net returns for state s are given by:

$$\begin{aligned}
y_s &= \mathbf{p}^s \bullet \mathbf{z}^s - \mathbf{w} \bullet \mathbf{x} \\
&= r_s - C(\mathbf{w}, \mathbf{r}, \mathbf{p}).
\end{aligned}$$

Hence,

$$\mathbf{y} = \mathbf{r} - C(\mathbf{w}, \mathbf{r}, \mathbf{p})\mathbf{1}_S.$$

Using this notation, the producer's objective function can be expressed as:

$$W(\mathbf{y}) = W(\mathbf{r} - C(\mathbf{w}, \mathbf{r}, \mathbf{p})\mathbf{1}_S).$$

Quiggin and Chambers (2000) show

Result 1 If the revenue-cost function exhibits constant absolute riskiness, the expansion path for a producer with CRA preferences is homogeneous of degree zero in input prices, parallel to the equal-revenue vector, and all elements of the efficient frontier are equally costly.

Result 2 If the revenue-cost function displays constant relative riskiness, a proportional increase (reduction) in all state-contingent output prices leads to a proportional increase (reduction) in all state-contingent revenues.

7 Concluding comments

The interaction between production theory and consumer theory has been very important in the development of the theory of economics under certainty. Much less interaction has been evident in the theory of choice under uncertainty. However, both analysis of production problems involving uncertainty and the inherent logic of the rank-dependent expected utility model lead naturally to a focus on state-contingent representations of uncertainty.

The state-contingent approach provides the best way to think about all problems involving uncertainty, including problems of consumer choice, the theory of the firm, and principal-agent relationships. Proper exploitation of the properties of alternative primal representations of preferences, such as

the distance and benefit functions familiar from the dual approach, allows us to generalize and extend the results of the existing literature on preferences under uncertainty, such as those based on the concepts of absolute and relative risk aversion. Furthermore, the natural symmetry between production and consumption means that the same properties have a natural and meaningful interpretation in terms of the absolute and relative riskiness of production technologies. One of the clearest implications of the state-contingent approach is that the expected-utility model can be treated as a special case, involving an assumption of additive separability. This assumption increases tractability in some problems, but is superfluous in many others.

8 References

Allais, M. 1953, 'Le comportement de l'homme rationnel devant le risque: critique des axiomes et postulats de l'école Americaine', *Econometrica* vol. 21, pp.503–46.

Allais, M. 1987, The general theory of random choices in relation to the invariant cardinal utility function and the specific probability function: The (U, θ) model - a general overview, Centre National de la Recherche Scientifique, Paris.

Allais, M. (1988), 'The general theory of random choices in relation to the invariant cardinal utility function and the specific probability function', in Munier, B. (ed.), *Risk, Decision and Rationality*, Reidel, Dordrecht, pp. 233–289.

Arrow, K. 1953, Le rôle des valeurs boursières pour la repartition la meillure des risques, Cahiers du Seminaire d'Economie, CNRS, Paris.

Arrow, K. 1965, Aspects of the theory of risk-bearing, Yrjo Jahnsson Lecture, Helsinki.

Bertsekas, D. 1974, 'Necessary and sufficient conditions for the existence of an optimal portfolio', *Journal of Economic Theory* vol. 8, pp.235–47.

Chambers, R. G. and Quiggin, J. 2000, *Uncertainty, Production, Choice and Agency: The State-Contingent Approach*, Cambridge University Press, New York.

Chew, S. H. and Epstein, L. 1989, 'Axiomatic rank-dependent means', *Annals of Operations Research* vol. 19, pp.207–240.

- Choquet, G. 1953-4, 'Theory of capacities', *Annales Institut Fourier* vol. 5, pp.131-295.
- Coes, D. 1977, 'Firm output and changes in uncertainty', *American Economic Review* vol. 67, pp.249-51.
- Dardanoni, V. 1988, 'Optimal choice under uncertainty: The case of two-argument utility functions', *Economic Journal* vol. 98, pp.429-50.
- de Finetti, B. 1931, 'Sul significato soggettivo della probabilita', *Fundamenta Mathematicae* vol. 17, pp.298-329.
- Debreu, G. 1952, 'A social equilibrium existence theorem', *Proceedings of the National Academy of Sciences* vol. 38, pp.886-93.
- Ebert, U. 1988, 'Rawls and Bentham reconciled', *Theory and Decision* vol. 24, pp.215-223.
- Feder, G. 1977, 'The impact of uncertainty in a class of objective functions', *Journal of Economic Theory* vol. 16, pp.504-12.
- Fisher, I. (1930), *The Theory of Interest*, Yale University Press, New Haven.
- Fishburn, P. 1978, 'On Handa's 'New Theory of Cardinal Utility' and the maximization of expected return', *Journal of Political Economy* vol. 86, pp.321-4.
- Fishburn, P. 1988, *Nonlinear Preference and Utility Theory*, Johns Hopkins, Baltimore.
- Gilboa, I. 1987, 'Expected utility with purely subjective non-additive probabilities', *Journal of Mathematical Economics* vol. 16, pp.65-88.
- Handa, J. 1977, 'Risk, probabilities and a new theory of cardinal utility', *Journal of Political Economy* vol. 85, pp.97-122.
- Hirshleifer, J. (1965), 'Investment decisions under uncertainty: choice-theoretic approaches', *Quarterly Journal of Economics* 79(4), 509-36.
- Luenberger, D. G. 1992, 'Benefit functions and duality', *Journal of Mathematical Economics* vol. 21, pp.461-481.
- Machina, M. 1982, 'A stronger characterization of declining risk aversion', *Econometrica* vol. 50, pp.1069-79.
- Machina, M. 1989, 'Comparative statics and non-expected utility preferences', *Journal of Economic Theory* vol. 47, pp.393-405.
- Machina, M. J. and Schmeidler, D. 1992, 'A more robust definition of subjective probability', *Econometrica* vol. 60, pp.745-80.
- Malmquist, S. 1953, 'Index numbers and indifference surfaces', *Trabajos de Estadística* vol. 4, pp.209-242.

- McFadden, D. (1978), 'Cost, revenue, and profit functions', in Fuss, M. and McFadden, D. (ed.), *Production Economics: A Dual Approach to Theory and Applications*, North Holland, Amsterdam, pp.
- Merton, R. 1973, *The Sociology of Science*, University of Chicago Press, Chicago.
- Pratt, J. 1964, 'Risk aversion in the small and in the large', *Econometrica* vol. 32, pp.122–36.
- Quiggin, J. 1979, The multi-product firm under price uncertainty, unpublished Honours thesis, Australian National University.
- Quiggin, J. 1981, 'Risk perception and risk aversion among Australian farmers', *Australian Journal of Agricultural Economics* vol. 25, pp.160–9.
- Quiggin, J. 1982a, 'A note on the existence of a competitive optimum', *Economic Record* vol. 55, pp.174–6.
- Quiggin, J. 1982b, 'A theory of anticipated utility', *Journal of Economic Behavior and Organization* vol. 3, pp.323–43.
- Quiggin, J. 1991, 'Comparative statics for Rank-Dependent Expected Utility theory', *Journal of Risk and Uncertainty* vol. 4, pp.339–50.
- Quiggin, J. and Chambers, R. G. 1998, 'Risk premiums and benefit measures for generalized expected utility theories', *Journal of Risk and Uncertainty* vol. 17, pp.121–37.
- Quiggin, J. and Chambers, R. G. 2000, 'The firm under price and output uncertainty: the state-theoretic approach', *Journal of Risk and Uncertainty*. forthcoming.
- Quiggin, J. and Wakker, P. (1994), 'The axiomatic basis of anticipated utility: a clarification', *Journal of Economic Theory* 64(2), 486–99.
- Roell, A. 1987, 'Risk-aversion in Quiggin and Yaari's rank-order model of choice under uncertainty', *Economic Journal* vol. 97, pp.143–59.
- Sandmo, A. 1971, 'On the Theory of the Competitive Firm under Price Uncertainty', *American Economic Review* vol. 61, pp.65-73.
- Savage, L. 1954, *Foundations of Statistics*, Wiley, New York.
- Schmeidler, D. 1989, 'Subjective probability and expected utility without additivity', *Econometrica* vol. 57, pp.571–87.
- Shephard, R. W. 1953, *Cost and Production Functions*, Princeton University Press, Princeton, New Jersey.
- Shephard, R. W. 1970, *Theory of Cost and Production Functions*, Princeton University Press, Princeton, New Jersey.
- Stigler, G. (1982), 'Merton on multiples, denied and affirmed', in Stigler, George (ed.), *The Economist as Preacher and other Essays*, University of

Chicago Press, Chicago.

Tobin, J. 1969, 'Comment on Borch and Feldstein', *Review of Economic Studies* vol. 36, pp.13–14.

Weymark, J. 1981, 'Generalized Gini inequality indices', *Mathematical Social Sciences* vol. 1, pp.409–30.

Yaari, M. 1969, 'Some remarks on measures of risk aversion and their uses', *Journal of Economic Theory* vol. 1, pp.315–29.

Yaari, M. 1987, 'The dual theory of choice under risk', *Econometrica* vol. 55, pp.95–115.